國立成功大學
National Cheng Kung University

The 10th workshop on Boundary Element Methods, Integral Methods, & Related Methods in Taiwan

BEM course

Clifford algebra valued boundary integral equations and boundary element methods

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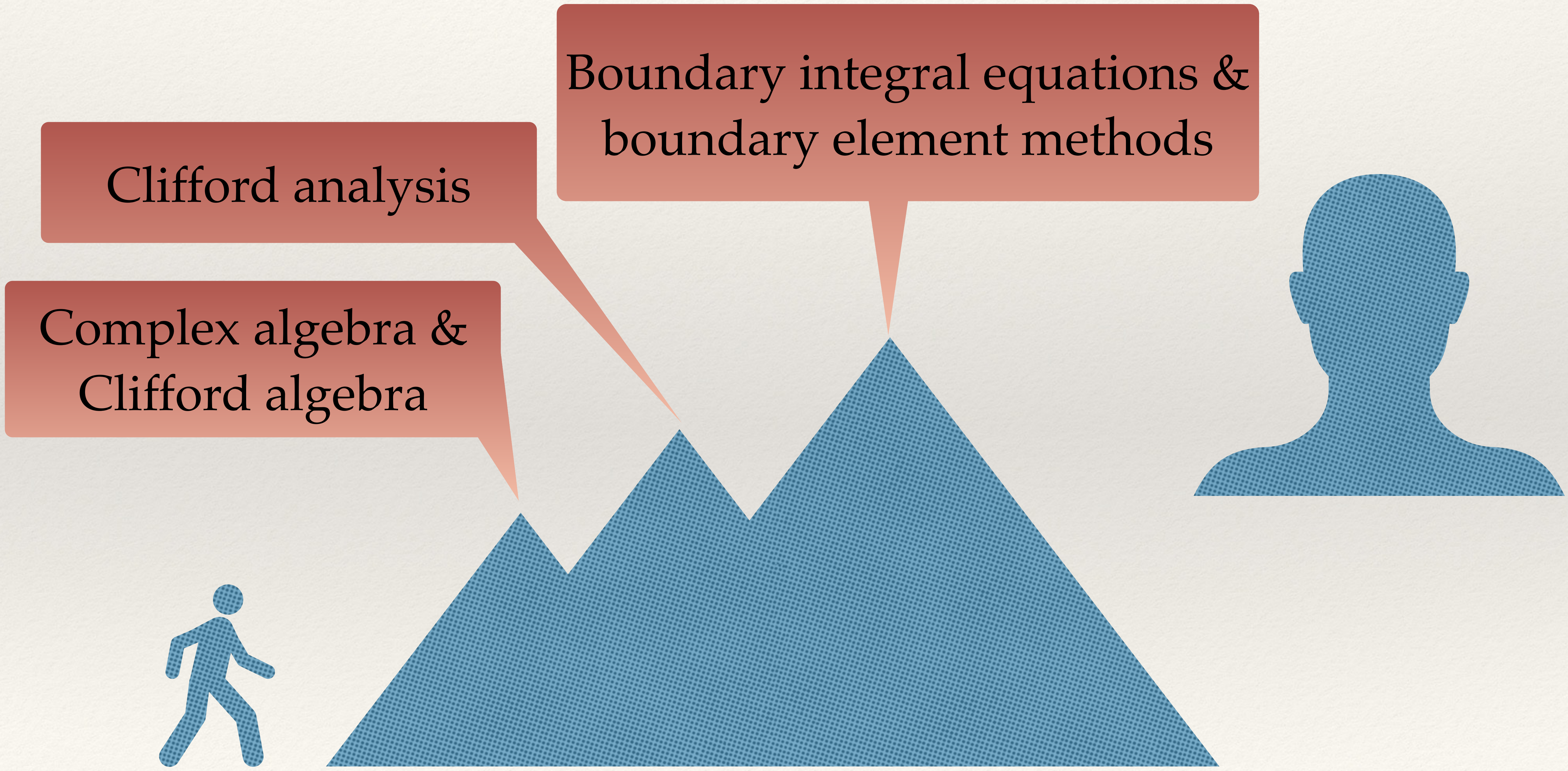
Outline


Boundary integral equations & boundary element methods
Clifford analysis
Complex algebra & Clifford algebra

People

William Rowan Hamilton
(1805-1865)

Quaternion algebra (1844)

$$1,\ \mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3$$

$$\mathbf{i}_k\mathbf{i}_l + \mathbf{i}_l\mathbf{i}_k = -2\delta_{kl},\ k, l = 1,2,3,$$

$$\mathbf{i}_1\mathbf{i}_2 = \mathbf{i}_3\,.$$

Hermann Gunther Grassmann
(1809-1877)

Grassmann algebra (1844)
Exterior algebra

$$\mathbf{a} \wedge \mathbf{b} = -\,\mathbf{b} \wedge \mathbf{a}$$

exterior (outer) product

William Kingdon Clifford
(1845-1879)

Clifford algebra (1878)
Geometric algebra

People

David Hestenes
(Arizona State University)

Wolfgang Sproessig
(TU Bergakademie Freiberg)

Chris Doran
(University of Cambridge)

Frank Sommen
(Ghent University)

Complex algebra & Clifford algebra

Complex numbers

- A complex number

$$a = a_1 + ia_2 \in \mathbb{C}, \quad a_1, a_2 \in \mathbb{R}$$

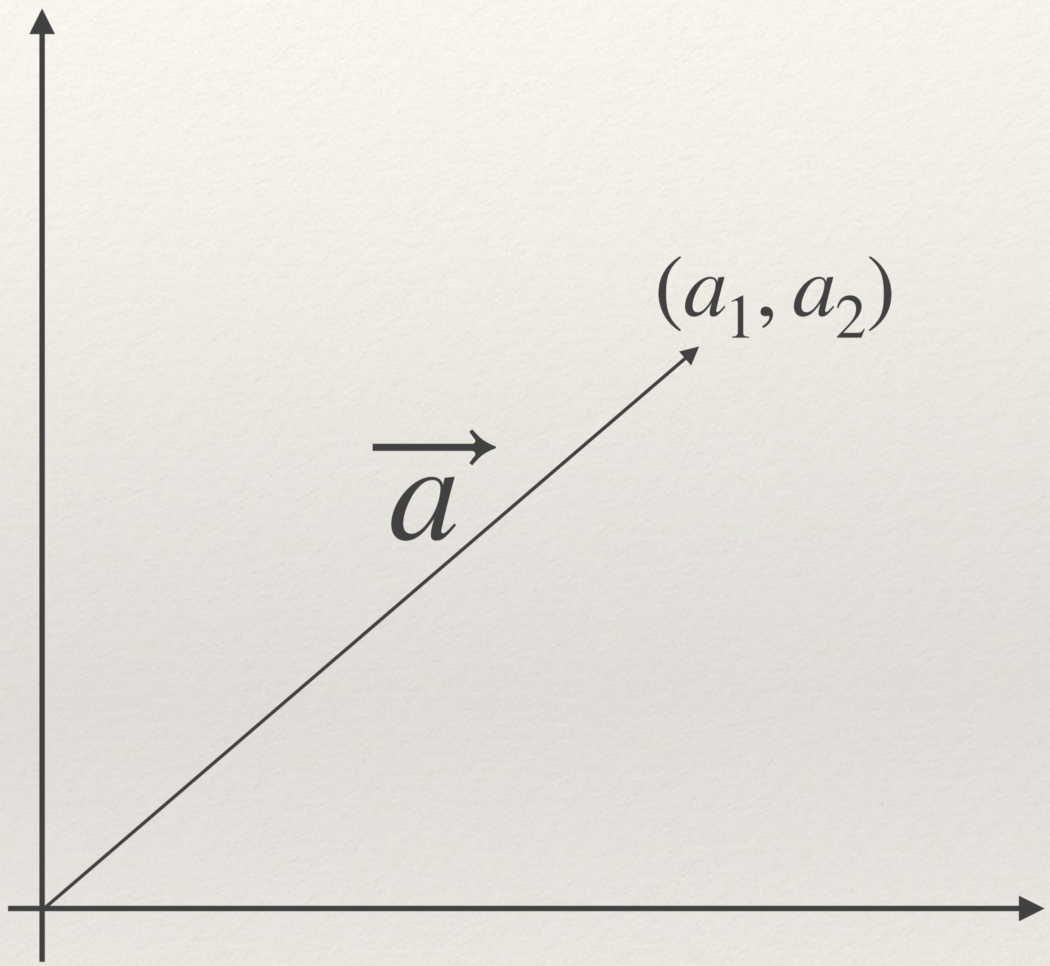
$$i^2 = ii = -1$$

Conjugate

$$\bar{a} = a_1 - ia_2$$

$$\bar{i} = -i,$$

$$\overline{uv} = \bar{u}\bar{v} = \bar{v}\bar{u}, \quad u, v \in \mathbb{C}$$



Length: $|a| = \sqrt{a\bar{a}} = \sqrt{a_1^2 + a_2^2}$.

Four operations

Addition

$$a + b = (a_1 + b_1) + i(a_2 + b_2)$$

Subtraction

$$a - b = (a_1 - b_1) + i(a_2 - b_2)$$

$$a = a_1 + ia_2$$

$$b = b_1 + ib_2$$

Multiplication

$$\begin{aligned} ab &= (a_1 + ia_2)(b_1 + ib_2) \\ &= a_1b_1 - a_2b_2 + i(a_1b_2 + a_2b_1) \end{aligned}$$

Division

$$\begin{aligned} \frac{a}{b} &= \frac{a}{b}1 = \frac{a}{b}\frac{\bar{b}}{\bar{b}} \\ &= \frac{a\bar{b}}{|b|^2} = \frac{a_1b_1 - a_2b_2 + i(a_2b_1 - a_1b_2)}{b_1^2 + b_2^2}. \end{aligned}$$

Matrix representations

- $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ ⟶ $\mathbf{i}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{1}$

  ⟶ $a = a_1\mathbf{1} + \mathbf{i}a_2 = \begin{bmatrix} a_1 + ia_2 & 0 \\ 0 & a_1 + ia_2 \end{bmatrix}$

- $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i}_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ ⟶ $\mathbf{i}_1^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{1}$

  ⟶ $a = a_1\mathbf{1} + \mathbf{i}_1a_2 = \begin{bmatrix} a_1 + ia_2 & 0 \\ 0 & a_1 - ia_2 \end{bmatrix}$

Matrix representations

- $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow \mathbf{i}_2^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{1}$

  $\longrightarrow a = a_1\mathbf{1} + \mathbf{i}_2 a_2 = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix}$

- $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i}_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \longrightarrow \mathbf{i}_3^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{1}$

  $\longrightarrow a = a_1\mathbf{1} + \mathbf{i}_3 a_2 = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$

Hand-in 1

	$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$		(1)		
$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$		(2)		
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$		(3)		
$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$		(4)		

5 min

Multiplication table

	$\mathbf{i}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{i}$	-1			
$\mathbf{i}_1$		-1	$\mathbf{i}_3$	$-\mathbf{i}_2$
$\mathbf{i}_2$		$-\mathbf{i}_3$	-1	$\mathbf{i}_1$
$\mathbf{i}_3$		$\mathbf{i}_2$	$-\mathbf{i}_1$	-1

Quaternion

- A quaternion

$$a = a_0 + \mathbf{i}_1 a_1 + \mathbf{i}_2 a_2 + \mathbf{i}_3 a_3 \in \mathbb{H},$$

$$\mathbf{i}_k \mathbf{i}_l + \mathbf{i}_l \mathbf{i}_k = -2\delta_{kl},\ k, l = 1,2,3,$$
$$\mathbf{i}_1 \mathbf{i}_2 = \mathbf{i}_3\,.$$

- Real quaternion: $a_0, a_1, a_2, a_3 \in \mathbb{R}$

Conjugate

$$\bar{a} = a_0 - \mathbf{i}_1 a_1 - \mathbf{i}_2 a_2 - \mathbf{i}_3 a_3$$
$$\bar{\mathbf{i}}_k = -\mathbf{i}_k,\ k = 1,2,3,$$
$$\overline{uv} = \bar{v}\bar{u}, \quad u, v \in \mathbb{H}$$

- Complex quaternion: $a_0, a_1, a_2, a_3 \in \mathbb{C}$

Conjugate

$$\bar{a} = a_0^H - \mathbf{i}_1 a_1^H - \mathbf{i}_2 a_2^H - \mathbf{i}_3 a_3^H$$
$$\bar{\mathbf{i}}_k = -\mathbf{i}_k,\ k = 1,2,3,$$
$$\overline{uv} = \bar{v}\bar{u}, \quad u, v \in \mathbb{H}$$
$$a_0 = a_0^R + i a_0^I, \quad a_0^H = a_0^R - i a_0^I$$

- Pure quaternion: $a_0 = 0$.

Hand-in 2

Please show the conjugate of a real quaternion $3 + \mathbf{i}_1 7 + \mathbf{i}_2 5 + \mathbf{i}_3 6$ and the conjugate of a complex quaternion $1 + i + \mathbf{i}_1(2 + i5) + \mathbf{i}_3(1 - i10)$.—(5), (6)

3 min

Four operations

Addition

$$a + b = (a_0 + b_0) + \mathbf{i}_1(a_1 + b_1) + \mathbf{i}_2(a_2 + b_2) + \mathbf{i}_3(a_3 + b_3)$$

Subtraction

$$a + b = (a_0 - b_0) + \mathbf{i}_1(a_1 - b_1) + \mathbf{i}_2(a_2 - b_2) + \mathbf{i}_3(a_3 - b_3)$$

$$a = a_0 + \mathbf{i}_1 a_1 + \mathbf{i}_2 a_2 + \mathbf{i}_3 a_3$$

$$b = b_0 + \mathbf{i}_1 b_1 + \mathbf{i}_2 b_2 + \mathbf{i}_3 b_3$$

Multiplication

$$ab \neq ba$$

Division

$$\frac{a}{b} = \frac{a}{b}1 = \frac{a}{b}\frac{\bar{b}}{\bar{b}} = \frac{a\bar{b}}{|b|^2}$$

Matrix representation

	$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$				
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$				
$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$				

	$\mathbf{i}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{i}$	-1			
$\mathbf{i}_1$		-1	$\mathbf{i}_3$	$-\mathbf{i}_2$
$\mathbf{i}_2$		$-\mathbf{i}_3$	-1	$\mathbf{i}_1$
$\mathbf{i}_3$		$\mathbf{i}_2$	$-\mathbf{i}_1$	-1

$$\mathbf{i}\,\mathbf{i}_1 = -e_1, \mathbf{i}\,\mathbf{i}_2 = -e_2, \mathbf{i}\,\mathbf{i}_3 = -e_3,$$

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hand-in 3

	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$		(1)	
$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$		(2)	
$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$		(3)	

3 min

Clifford algebra in 3D space

Multiplication rule

$$e_k e_l + e_l e_k = 2\delta_{kl}, \quad k, l = 1,2,3.$$

	e_1	e_2	e_3
e_1	1	e_1e_2	e_1e_3
e_2	e_2e_1	1	e_2e_3
e_3	e_3e_1	e_3e_2	1

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Clifford algebra in 3D space

Multiplication rule

$$e_k e_l + e_l e_k = 2\delta_{kl}, \quad k, l = 1,2,3.$$

Basis elements

$e_0 := e_\varnothing \equiv 1,$

$e_1,\ e_2,\ e_3,$

$e_{12} = e_1 e_2,\ e_{13} = e_1 e_3,\ e_{23} = e_2 e_3,$

$e_{123} = e_1 e_2 e_3$

A Clifford number

$$\begin{aligned} a = & e_0 a_0 \\ & + e_1 a_1 + e_2 a_2 + e_3 a_3 \\ & + e_{12} a_{12} + e_{13} a_{13} + e_{23} a_{23} \\ & + e_{123} a_{123} \end{aligned}$$

Four parts

scalar part: $\langle a \rangle_0 = a_0,$

vector part: $\langle a \rangle_1 = e_1 a_1 + e_2 a_2 + e_3 a_3,$

bivector part: $\langle a \rangle_2 = e_{12} a_{12} + e_{13} a_{13} + e_{23} a_{23},$

pseudo-scalar part: $\langle a \rangle_3 = e_{123} a_{123},$

$a = \langle a \rangle_0 + \langle a \rangle_1 + \langle a \rangle_2 + \langle a \rangle_3$

Clifford algebra in 3D space

$$a = e_0 a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3 + e_{12} a_{12} + e_{13} a_{13} + e_{23} a_{23} + e_{123} a_{123}$$

$a \in C\ell_3(\mathbb{R}) \Rightarrow a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, a_{123} \in \mathbb{R}$

$a \in C\ell_3(\mathbb{C}) \Rightarrow a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, a_{123} \in \mathbb{C}$

Norm of $a \in C\ell_3(\mathbb{R})$

$$|a| = (a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_{12}^2 + a_{13}^2 + a_{23}^2 + a_{123}^2)^{1/2}$$

Norm of $a \in C\ell_3(\mathbb{C})$

$$|a| = (\langle a^H a \rangle_0)^{1/2}$$

Conjugate

$$\bar{e}_k = -e_k, \quad k = 1,2,3,$$

$$\overline{ac} = \bar{c}\bar{a}, \quad a, c \in C\ell_3(\mathbb{R}),$$

$$\bar{a} = \langle a \rangle_0 - \langle a \rangle_1 - \langle a \rangle_2 + \langle a \rangle_3$$

Four operation

Addition

$$a + b = \sum_{I \in S} e_I (a_I + b_I)$$

Subtraction

$$a - b = \sum_{I \in S} e_I (a_I - b_I)$$

$$a = \sum_{I \in S} e_I a_I$$

$$S = \{0,1,2,3,12,13,23,123\}$$

$$b = \sum_{I \in S} e_I b_I$$

Multiplication

$$ab = (\sum_{I \in S} e_I a_I)(\sum_{J \in S} e_J b_j) \neq ba$$

Division

$$\frac{a}{b} = ab^{-1}$$

$$b^{-1}b = bb^{-1} = 1$$

Hand-in 4

- If $a = \langle a \rangle_1$ and $b = \langle b \rangle_1$, please show ab and ba. —(4)
- If $a = \langle a \rangle_2$ and $b = \langle b \rangle_2$, please show ab and ba.—(5)
- If $a = \langle a \rangle_1 + \langle a \rangle_3$ and $b = \langle b \rangle_1 + \langle b \rangle_3$, please show ab and ba.—(6)
- If $a = \langle a \rangle_0 + \langle a \rangle_2$ and $b = \langle b \rangle_0 + \langle b \rangle_2$, please show ab and ba.—(1)

5 min

Clifford product

Multiplication

If $a = \langle a \rangle_1$ and $b = \langle b \rangle_1$

then $ab = a \cdot b + a \times b$

or $ab = a \cdot b + a \wedge b$

Subalgebra

A subalgebra is a subset of an algebra, that closed under all its operations.

$$a = a_0 + e_{12}a_{12} \in C\ell_2^+(\mathbb{R})$$

	1	e_{12}
1		
e_{12}		

$$a = a_1 + ia_2 \in \mathbb{C}$$

	1	i
1	1	i
i	i	-1

Subalgebra

Hand-in 5

$$a = a_0 + e_{12}a_{12} + e_{23}a_{23} + e_{31}a_{31} \in C\ell_3^+(\mathbb{R})$$

	1	e_{32}	e_{13}	e_{21}
1			(2)	
e_{32}				
e_{13}			(3)	
e_{21}				

$$a = a_0 + \mathbf{i}_1 a_1 + \mathbf{i}_2 a_2 + \mathbf{i}_3 a_3 \in \mathbb{H}$$

	1	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
1	1	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{i}_1$	$\mathbf{i}_1$	-1	$\mathbf{i}_3$	$-\mathbf{i}_2$
$\mathbf{i}_2$	$\mathbf{i}_2$	$-\mathbf{i}_3$	-1	$\mathbf{i}_1$
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	-1

$$\mathbf{i}_1 = -e_{23}, \quad \mathbf{i}_2 = -e_{31}, \quad \mathbf{i}_3 = -e_{12}$$

3 min

Clifford analysis

Complex analysis

A complex function

$$f(x) = f_1(x) + if_2(x) : \mathbb{R}^2 \to \mathbb{C}$$

$$f_i : \mathbb{R}^2 \to \mathbb{R}$$

2D Laplace operator

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} =: \triangle_2 = D_{\mathbb{C}}\overline{D}_{\mathbb{C}} = \overline{D}_{\mathbb{C}}D_{\mathbb{C}}$$

Cauchy-Riemann operator

$$D_{\mathbb{C}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$$

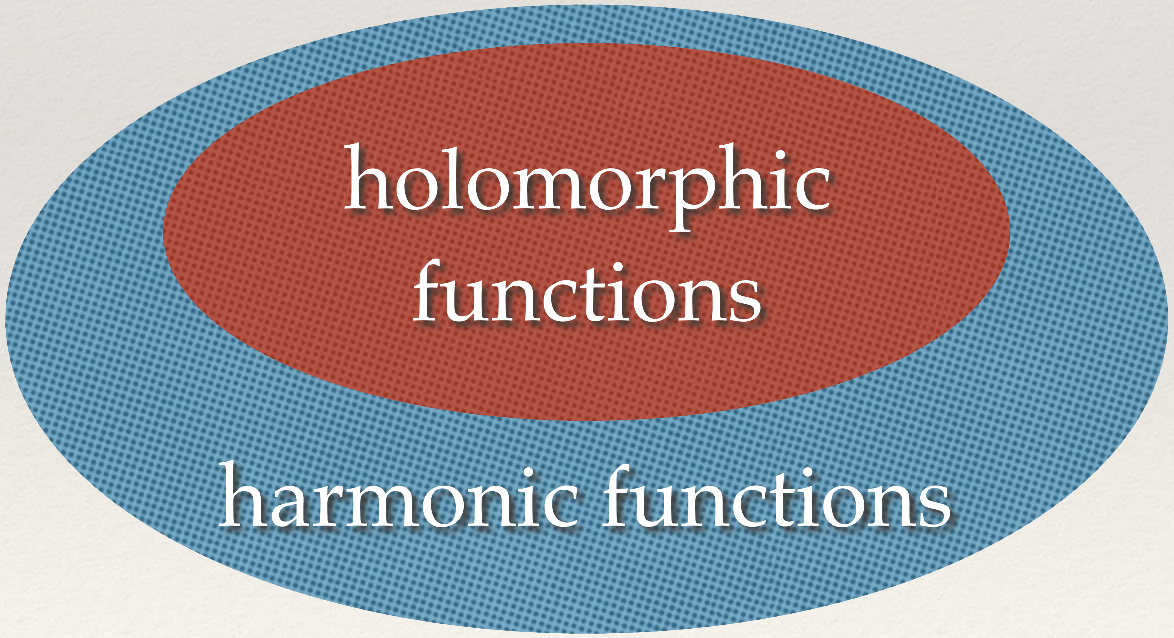
$$\overline{D}_{\mathbb{C}} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}$$

Cauchy-Riemann equation

$$D_{\mathbb{C}}f = 0 \Leftrightarrow \begin{cases} \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} = 0, \\ \dfrac{\partial f_2}{\partial x_1} + \dfrac{\partial f_1}{\partial x_2} = 0. \end{cases}$$

Complex analysis

- A function f satisfying the Cauchy-Riemann equation is called a holomorphic function.
- A function f satisfying the Laplace equation is called a harmonic function.



Quaternion analysis

A $\mathbb{H}$-valued function

$$f(x) = f_0(x) + \mathbf{i}_1 f_1(x) + \mathbf{i}_2 f_2(x) + \mathbf{i}_3 f_3(x) : \mathbb{R}^4 \to \mathbb{H}$$
$$f_i : \mathbb{R}^4 \to \mathbb{R}$$

Fueter operator

$$\partial_{\mathbb{H}} = \partial_{\mathbb{H}}(x) := \frac{\partial}{\partial x_0} + \mathbf{i}_1 \frac{\partial}{\partial x_1} + \mathbf{i}_2 \frac{\partial}{\partial x_2} + \mathbf{i}_3 \frac{\partial}{\partial x_3},$$
$$\bar{\partial}_{\mathbb{H}} = \bar{\partial}_{\mathbb{H}}(x) := \frac{\partial}{\partial x_0} - \mathbf{i}_1 \frac{\partial}{\partial x_1} - \mathbf{i}_2 \frac{\partial}{\partial x_2} - \mathbf{i}_3 \frac{\partial}{\partial x_3}.$$

4D Laplace operator

$$\sum_{k=0}^{3} \frac{\partial^2}{\partial x_k^2} =: \triangle_4 = \partial_{\mathbb{H}} \bar{\partial}_{\mathbb{H}} = \bar{\partial}_{\mathbb{H}} \partial_{\mathbb{H}}$$

Hand-in 6

Please show $\partial_{\mathbb{H}} f = 0$. —(4)

5 min

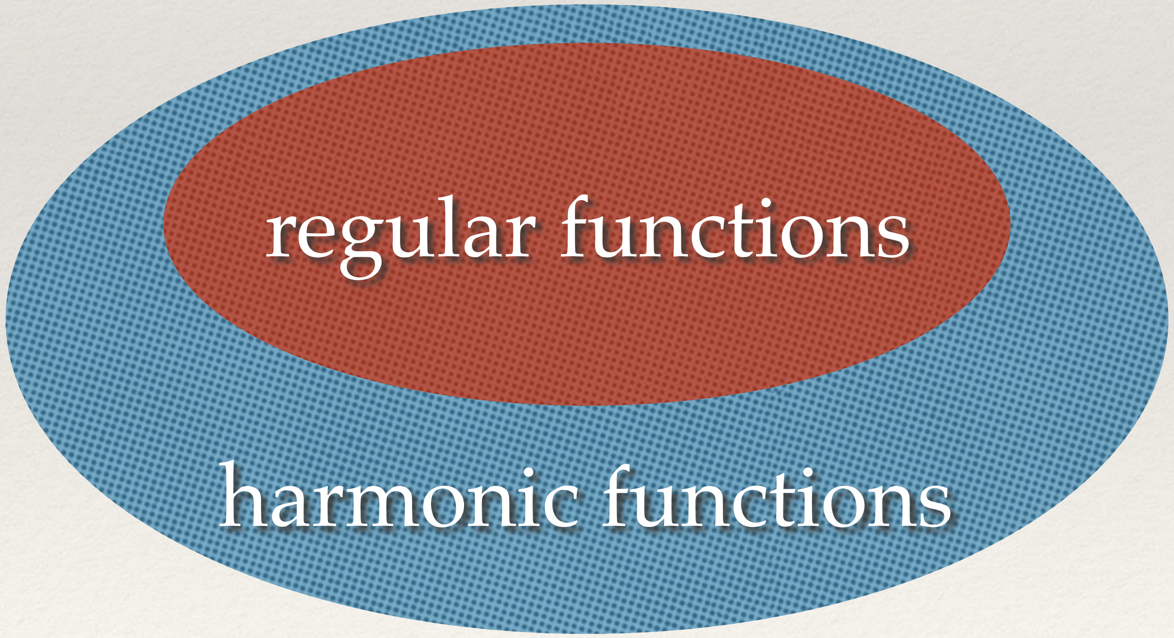
Quaternion analysis

The Fueter equation

$$\partial_{\mathbb{H}} f = 0 \Leftrightarrow \begin{cases} \dfrac{\partial f_4}{\partial x_4} - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} - \dfrac{\partial f_3}{\partial x_3} = 0, \\ \dfrac{\partial f_1}{\partial x_4} + \dfrac{\partial f_4}{\partial x_1} + \dfrac{\partial f_3}{\partial x_2} - \dfrac{\partial f_2}{\partial x_3} = 0, \\ \dfrac{\partial f_2}{\partial x_4} - \dfrac{\partial f_3}{\partial x_1} + \dfrac{\partial f_4}{\partial x_2} + \dfrac{\partial f_1}{\partial x_3} = 0, \\ \dfrac{\partial f_3}{\partial x_4} + \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2} + \dfrac{\partial f_4}{\partial x_3} = 0 \end{cases}$$

Quaternion analysis

- A $\mathbb{H}$-valued function f satisfying the Fueter equation is called a regular function.
- A function f satisfying the Laplace equation $\triangle_4 f = 0$ is called a harmonic function in 4D space.



Quaternion analysis

$$D_{\mathbb{H}} = D_{\mathbb{H}}(x) := \mathbf{i}_1 \frac{\partial}{\partial x_1} + \mathbf{i}_2 \frac{\partial}{\partial x_2} + \mathbf{i}_3 \frac{\partial}{\partial x_3}$$

3D Laplace operator

$$\triangle_3 = -D_{\mathbb{H}} D_{\mathbb{H}} = D_{\mathbb{H}} \overline{D}_{\mathbb{H}}$$

$$\triangle_3 := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

Clifford analysis—$C\ell_3$

A $C\ell_3$-valued function

$$f(x) = e_0 f_0(x) + e_1 f_1(x) + e_2 f_2(x) + e_3 f_3(x) + e_{12} f_{12}(x) + e_{13} f_{13}(x) + e_{23} f_{23}(x) + e_{123} f_{123}(x) : \mathbb{R}^3 \to C\ell_3(\mathbb{R})$$

$$f_i : \mathbb{R}^3 \to \mathbb{R}$$

Dirac operator

$$D = D(x) := e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}$$

3D Laplace operator

$$\triangle_3 = \triangle_3(x) = DD = D^2$$

Hand-in 7

Please show $Df = 0$. —(5)

6 min

Clifford analysis—$C\ell_3$

Dirac equation

$$Df = 0$$

Scalar part

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0$$

Vector part

$$\frac{\partial f_0}{\partial x_1} - \frac{\partial f_{12}}{\partial x_2} - \frac{\partial f_{13}}{\partial x_3} = 0, \quad \frac{\partial f_0}{\partial x_2} + \frac{\partial f_{12}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_3} = 0, \quad \frac{\partial f_0}{\partial x_3} + \frac{\partial f_{23}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_1} = 0,$$

Bivector part

$$\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_{123}}{\partial x_3} = 0, \quad \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_{123}}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} + \frac{\partial f_{123}}{\partial x_2} = 0$$

Pesudo-scalar part

$$\frac{\partial f_{12}}{\partial x_3} + \frac{\partial f_{23}}{\partial x_1} - \frac{\partial f_{13}}{\partial x_2} = 0$$

Clifford analysis—$C\ell_3$

The Laplace equation
$$\triangle_3 f = 0$$

Scalar part
$$\triangle_3 f_0 = 0$$

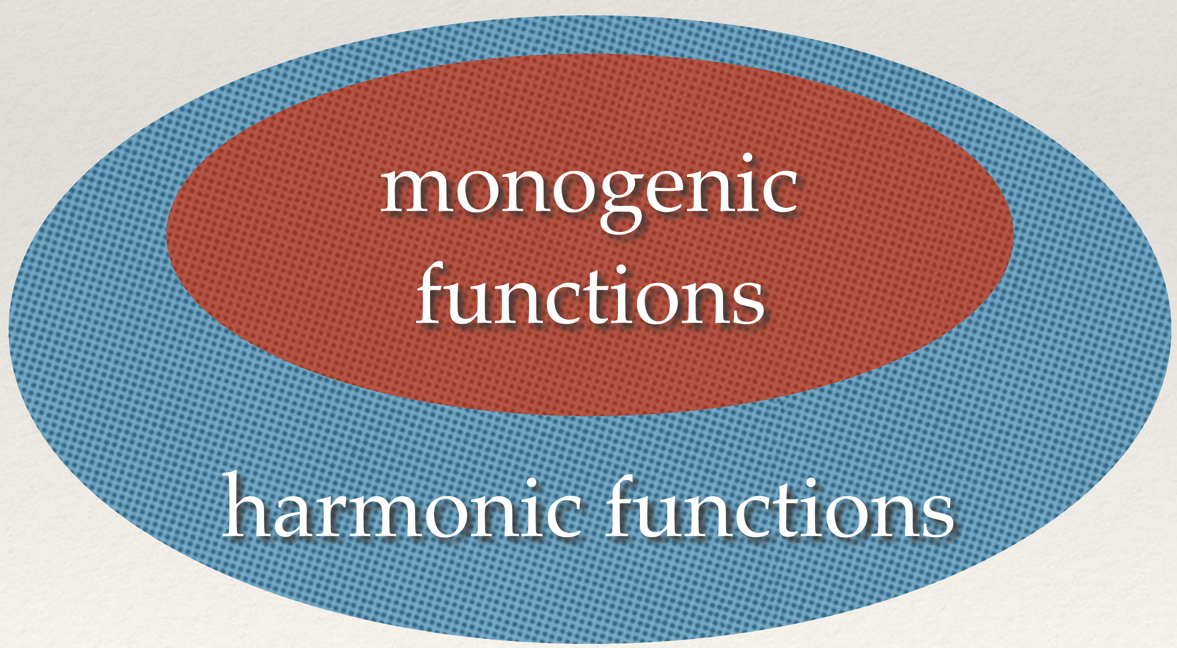
Vector part
$$\triangle_3 f_1 = 0,\ \triangle_3 f_2 = 0,\ \triangle_3 f_3 = 0$$

Bivector part
$$\triangle_3 f_{12} = 0,\ \triangle_3 f_{23} = 0,\ \triangle_3 f_{13} = 0$$

Pesudo-scalar part
$$\triangle_3 f_{123} = 0$$

Clifford analysis–$C\ell_3$

- A $C\ell_3$-valued function f satisfying the Dirac equation is called a left-monogenic function.
- A function f satisfying the Laplace equation $\triangle_3 f = 0$ is called a harmonic function in 3D space.



Fundamental solutions

Dirac equation

$$Df = 0$$

The Laplace equation

$$\triangle_3 f = 0$$

Fundamental solution

$$C(x - s)$$

$$DC(x - s) = C(x - s)D = \delta(x - s)$$

$$C(x - s) = \frac{1}{4\pi} \frac{x - s}{|x - s|^3}$$

Fundamental solution

$$U(x - s)$$

$$- \triangle U = \delta(x - s)$$

$$U(x - s) = \frac{1}{4\pi} \frac{1}{|x - s|}$$

$$DU(x - s) = U(x - s)D = - C(x - s)$$

Clifford analysis—$C\ell_3(\mathbb{C})$

k-Dirac operator

$$D_k = D_k(x) := e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3} + ik,$$

$$\overline{D}_k = -e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3} + ik$$

Helmholtz operator

$$\triangle_3 + k^2 = -D_k\overline{D}_k = -\overline{D}_kD_k$$

k-Dirac equation

$$D_kf = 0, \quad f \in C\ell_3(\mathbb{C})$$

- A $C\ell_3$-valued function satisfying the k-Dirac equation is called a k-monogenic function.

Fundamental solutions

Fundamental solution $U_k(x-s) \in C\ell_3(\mathbb{C})$

$$-\left(\triangle + k^2\right) U_k(x-s) = \overline{D}_k D_k U_k(x-s) = U_k(x-s) D_k \overline{D}_k = \delta(x-s)$$

$$U_k(x-s) = \frac{1}{4\pi}\frac{\exp(-ik|x-s|)}{|x-s|}$$

$$C_k(x-s) = D_k U_k(x-s) = U_k(x-s) D_k$$

Fundamental solution $C_k(x-s) \in C\ell_3(\mathbb{C})$

$$\overline{D}_k C_k(x-s) = C_k(x-s)\overline{D}_k = \delta(x-s),$$

$$C_k(x-s) = \frac{1}{4\pi}\left[\left(\frac{-ik}{|x-s|} - \frac{1}{|x-s|^2}\right)\frac{\exp(-k|x-s|)}{|x-s|}(x-s) + ik\frac{\exp(-ik|x-s|)}{|x-s|}\right]$$

Applications—electromagnetics

The Maxwell equation

$$\nabla \cdot \mathscr{D}(\mathbf{x}, t) = \hat{\rho}_f(\mathbf{x}, t),$$

$$\nabla \times \mathscr{H}(\mathbf{x}, t) = \mathscr{J}_f(\mathbf{x}, t) + \frac{\partial \mathscr{D}(\mathbf{x}, t)}{\partial t},$$

$$\nabla \times \mathscr{E}(\mathbf{x}, t) = -\frac{\partial \mathscr{B}(\mathbf{x}, t)}{\partial t},$$

$$\nabla \cdot \mathscr{B}(\mathbf{x}, t) = 0,$$

$$\mathscr{E}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{x})e^{i\omega t}\right],$$

$$\mathscr{D}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{D}(\mathbf{x})e^{i\omega t}\right],$$

$$\mathscr{B}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{B}(\mathbf{x})e^{i\omega t}\right],$$

$$\mathscr{H}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{H}(\mathbf{x})e^{i\omega t}\right],$$

$$\mathscr{J}_f(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{J}_f(\mathbf{x})e^{i\omega t}\right],$$

$$\hat{\rho}_f(\mathbf{x}, t) = \mathrm{Re}\left[\rho_f(\mathbf{x})e^{i\omega t}\right],$$

→

The Maxwell equation

$$\nabla \cdot \mathbf{D}(\mathbf{x}) = \rho_f(\mathbf{x}),$$

$$\nabla \times \mathbf{H}(\mathbf{x}) - i\omega \mathbf{D}(\mathbf{x}) = \mathbf{J}_f(\mathbf{x}),$$

$$\nabla \times \mathbf{E}(\mathbf{x}) + i\omega \mathbf{B}(\mathbf{x}) = \mathbf{0},$$

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0,$$

$\mathscr{E}(\mathbf{x}, t)$: the electric field
$\mathscr{D}(\mathbf{x}, t)$: the electric displacement field
$\mathscr{B}(\mathbf{x}, t)$: the magnetic flux density
$\mathscr{H}(\mathbf{x}, t)$: the magnetic field strength
$\mathscr{J}_f(\mathbf{x}, t)$: the free current density
$\hat{\rho}_f(\mathbf{x}, t)$: the free charge density

Applications—electromagnetics

The Maxwell equation

$$\nabla \cdot \mathbf{D}(\mathbf{x}) = \hat{\rho}_f(\mathbf{x}),$$

$$\nabla \times \mathbf{H}(\mathbf{x}) - i\omega \mathbf{D}(\mathbf{x}) = \mathbf{J}_f(\mathbf{x}),$$

$$\nabla \times \mathbf{E}(\mathbf{x}) + i\omega \mathbf{B}(\mathbf{x}) = \mathbf{0},$$

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0,$$

$$\mathbf{E}(x) = E_1(x)\mathbf{e}_1 + E_2(x)\mathbf{e}_2 + E_3(x)\mathbf{e}_3,$$

$$\mathbf{B}(x) = B_1(x)\mathbf{e}_1 + B_2(x)\mathbf{e}_2 + B_3(x)\mathbf{e}_3,$$

$$D = \nabla\,.$$

linear and the isotropic medium

$$\mathbf{D}(\mathbf{x}) = \varepsilon \mathbf{E}(\mathbf{x}),$$

$$\mathbf{H}(\mathbf{x}) = \frac{1}{\mu}\mathbf{B}(\mathbf{x}),$$

$$\frac{1}{\sqrt{\varepsilon\mu}} = \frac{\omega}{k} = c$$

k-Dirac equation

$$D_k(E(x) + cB(x)) = \frac{\hat{\rho}_f}{\varepsilon},$$

$$E(x) = E_1(x)e_1 + E_2(x)e_2 + E_3(x)e_3,$$

$$B(x) = B_1(x)e_{23} + B_2(x)e_{31} + B_3(x)e_{12}\,.$$

Applications—Elasticity

The Navier equation

$$(\lambda+\mu)\frac{\partial^2}{\partial x_k \partial x_j}u_j + \mu\frac{\partial^2}{\partial x_j \partial x_j}u_k + b_k = 0,$$

$$(\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \mu\triangle_3\mathbf{u} + \mathbf{b} = \mathbf{0}$$

$$\mathbf{u} = \mathbf{e}_1 u_1 + \mathbf{e}_2 u_2 + \mathbf{e}_3 u_3$$

$$\mathbf{b} = \mathbf{e}_1 b_1 + \mathbf{e}_2 b_2 + \mathbf{e}_3 b_3$$

$$\nabla = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}$$

$$\triangle_3 = D^2,$$

$$\nabla\cdot\mathbf{u} = \frac{1}{2}(Du + \overline{Du}),$$

$$\nabla = D,$$

$$u = e_1 u_1 + e_2 u_2 + e_3 u_3,$$

$$u = e_1 b_1 + e_2 b_2 + e_3 b_3 .$$

Dirac equation with forcing term

$$\frac{1}{2}(\lambda+\mu)D(Du + \overline{Du}) + \mu DDu = b,$$

$$D\left[\frac{1}{2}(\lambda+\mu)(Du + \overline{Du}) + \mu Du\right] = b .$$

Applications

The Maxwell equation

k-Dirac equation with forcing term

$$D_k(E(x) + cB(x)) = \frac{\hat{\rho}_f}{\varepsilon},$$

The Navier equation

Dirac equation with forcing term

$$D\left[\frac{1}{2}(\lambda + \mu)(Du + \overline{Du}) + \mu Du\right] = b\,.$$

Boundary integral equations & boundary element methods

BIEs for complex variable functions

Cauchy-Riemann equation

$$D_{\mathbb{C}} f = 0$$

complex Gauss theorem

$$\int_{\Omega} D_{\mathbb{C}}(x) f(x) d\xi d\eta = \frac{1}{i} \int_{\partial\Omega} f(x) dx$$

Boundary integral equations for a holomorphic function f

$$c(z)f(z) = \frac{1}{2\pi i} \int_{\partial\Omega}^{(z)} f(z) \frac{dz}{z-\zeta}, \quad \begin{array}{ll} z \in \Omega & \text{(Cauchy integral formulae)}, \\ z \in \partial\Omega & \text{(Plemelj-Sokhotzki formula)}, \\ z \in \mathbb{C}\backslash\overline{\Omega} & \text{(Cauchy integral formulae)}. \end{array}$$

BIEs for $C\ell_3$-valued functions

The Stokes theorem

$$\int_\Omega \left[(gD)h + g(Dh)\right] d^3x = \int_{\partial\Omega} gnhdS = \int_{\partial\Omega} gd\sigma h$$

- g, h: $C\ell_3$-valued functions
- $d^3x = dx_1dx_2dx_3$: the oriented volume element (a real valued 3-form)
- $n = e_1n_1 + e_2n_2 + e_3n_3$: normal vector
- dS: the "classical" surface element (a real valued 2-form)
- $d\sigma = ndS = e_1dx_2dx_3 + e_2dx_3dx_1 + e_3dx_1dx_2$: the oriented surface element (a vector-valued 2-form)

BIEs for $C\ell_3$-valued functions

The Stokes theorem

$$\int_{\Omega} \left[(gD)h + g(Dh)\right] d^3x = \int_{\partial\Omega} gnhdS = \int_{\partial\Omega} gd\sigma h$$

$$g = C(x-y), h = f(x)$$

+

$$g = U(x-y), h = Df(x)$$

$$C(x-y)D = \delta(x-y),$$
$$U(x-y)D = -C(x-y),$$
$$DD = \triangle_3$$

$$\int_{\Omega} \delta(x-y)f(x)d^3x + \int_{\Omega} U(x-y)(\triangle f(x))d^3x = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

BIEs for $C\ell_3$-valued functions

$$\int_\Omega \delta(x-y)f(x)d^3x + \int_\Omega U(x-y)(\triangle f(x))d^3x = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

For $f(x)$ is harmonic

$$\int_\Omega \delta(x-y)f(x)d^3x = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

$C\ell_3$-valued Cauchy integral formulae ($y \in \Omega$)

$$f(y) = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

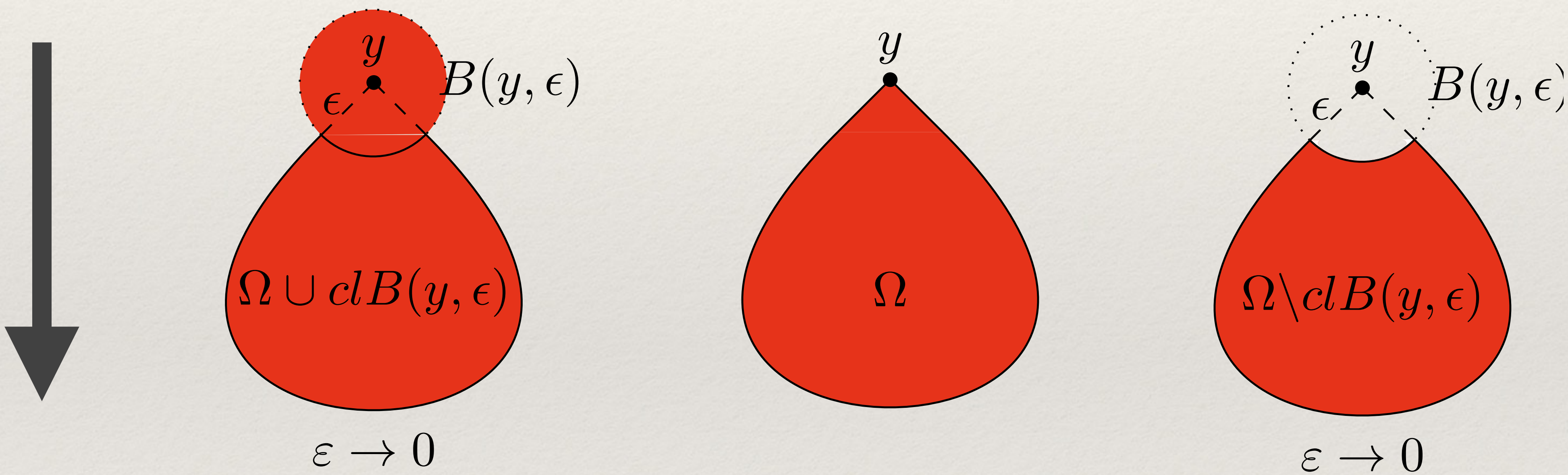
$C\ell_3$-valued Cauchy integral formulae ($y \in \mathbb{R}^3 \backslash cl\Omega$)

$$0 = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

BIEs for $C\ell_3$-valued functions

For $f(x)$ is harmonic

$$\int_{\Omega} \delta(x-y)f(x)d^3x = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$



$C\ell_3$-valued Plemelj-Sokhotzki formula ($y \in \partial\Omega$)

$$\frac{a_3^i(y)}{4\pi} f(y) = \int_{\partial\Omega} U(x-y)n(x)D(x)f(x)dS(x) + \text{C.P.V.} \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

C.P.V.—Cauchy principal value

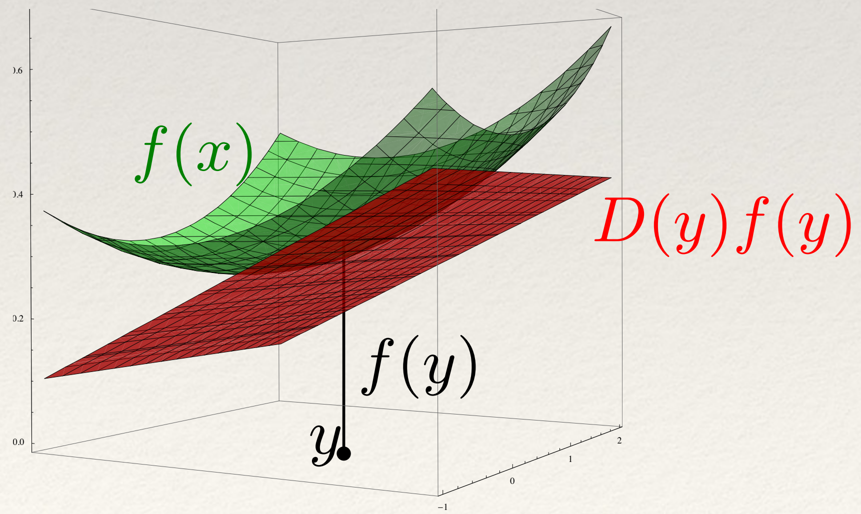
Singularity-free BIEs

$C\ell_3$-valued Plemelj-Sokhotzki formula ($y \in \partial\Omega$)

$$\frac{a_3^i(y)}{4\pi} f(y) = \int_{\partial\Omega} U(x-y)n(x)D(x)f(x)dS(x) + \text{C.P.V.} \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

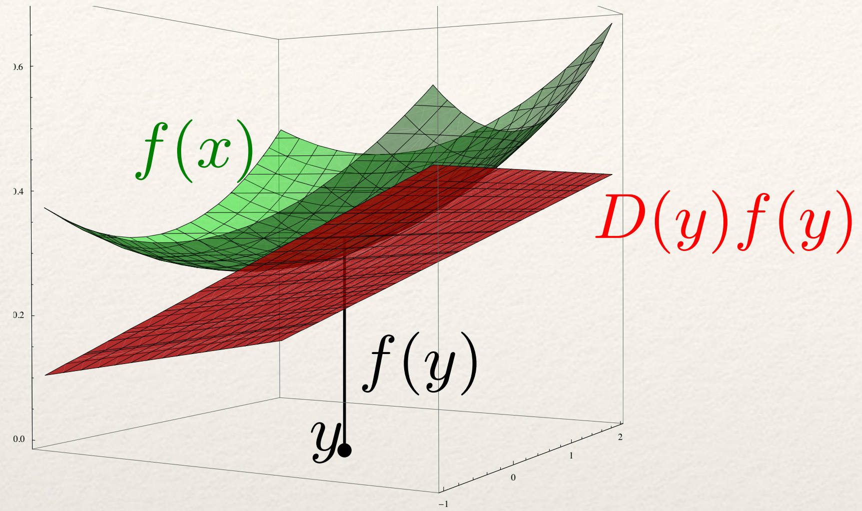
Exact evaluation of singularity ($y \in \partial\Omega$)

$$\text{C.P.V.} \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x) = \int_{\partial\Omega} C(x-y)n(x)[f(x)-w(x)]dS(x) + \text{C.P.V.} \int_{\partial\Omega} C(x-y)n(x)w(x)dS(x)$$



$$w(x) = f(y) + \frac{x-y}{3} D(y)f(y)$$

Singularity-free BIEs



$$w(x) = f(y) + \frac{x-y}{3} D(y) f(y)$$

No solid angle

Singularity-free BIEs ($y \in \partial\Omega$)

$$0 = \int_{\partial\Omega} U(x-y)n(x)\left[D(x)f(x) - D(y)f(y)\right] dS(x) - \int_{\partial\Omega} C(x-y)n(x)\frac{(x-y)}{3}D(y)f(y)dS(x)$$
$$+ \int_{\partial\Omega} C(x-y)n(x)\left[f(x) - f(y)\right] dS(x)$$

Hand-in 8

Please finish the flowchart to obtain the singularity-free BIEs from the Stokes theorem.—(6)

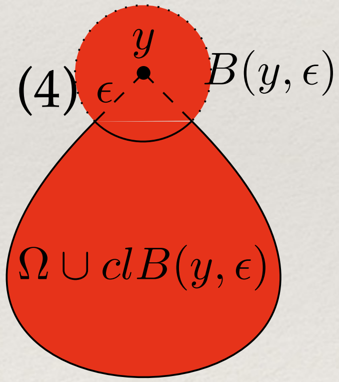
(1)
For $f(x)$ is harmonic

$$\int_{\Omega} \delta(x-y)f(x)d^3x = \int_{\partial\Omega} U(x-y)n(x)Df(x)dS(x) + \int_{\partial\Omega} C(x-y)n(x)f(x)dS(x)$$

(2)
$C\ell_3$-valued Cauchy integral formulae ($y \in \Omega$)

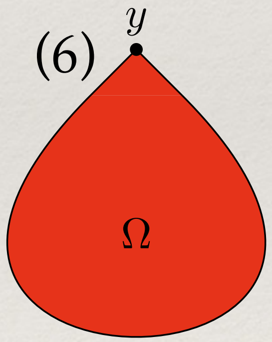
(3)
Singularity-free BIEs ($y \in \partial\Omega$)

(4)



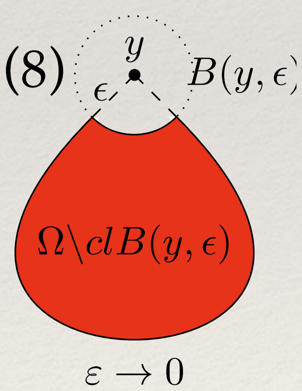
(5)
$C\ell_3$-valued Plemelj-Sokhotzki formula ($y \in \partial\Omega$)

(6)



(7)
$C\ell_3$-valued Cauchy integral formulae ($y \in \mathbb{R}^3 \backslash cl\Omega$)

(8)



(9)
The Stokes theorem

(10)
Exact evaluation of singularity ($y \in \partial\Omega$)

3 min

BIEs for $C\ell_3(\mathbb{C})$-valued functions

The Stokes theorem

$$\int_{\Omega} \left[(gD)h + g(Dh)\right] d^3x = \int_{\partial\Omega} gnhdS = \int_{\partial\Omega} gd\sigma h$$

$g = C_k(x-y), h = f(x)$ + $g = U_k(x-y), h = Df(x)$

$$C_k(x-y)_kD = \delta(x-y),$$
$$U_k(x-y)D_k = -C_k(x-y),$$
$$D_k\overline{D}_k = -(\triangle_3 + k^2)$$

$$\int_{\Omega} \delta(x-y)f(x)d^3x - \int_{\Omega} U_k(x-y)((\triangle + k^2)f(x))d^3x = \int_{\partial\Omega} U_k(x-y)n(x)D_kf(x)dS(x) - \int_{\partial\Omega} C_k(x-y)n(x)f(x)dS(x)$$

BIEs for $C\ell_3(\mathbb{C})$-valued functions

BIEs for k-harmonic function $f(x)$

$$c(y)f(y) = \int_{\partial\Omega} U_k(x-y)n(x)D_k f(x)dS(x) - \int_{\partial\Omega}^{(y)} C_k(x-y)n(x)f(x)dS(x)$$

$$\int_{\partial\Omega}^{(y)} = \begin{cases} \int_{\partial\Omega} & y \in \Omega \\ \text{C.P.V} \int_{\partial\Omega} & y \in \partial\Omega \\ \int_{\partial\Omega} & y \in \mathbb{R}^3 \backslash cl\Omega \end{cases}$$

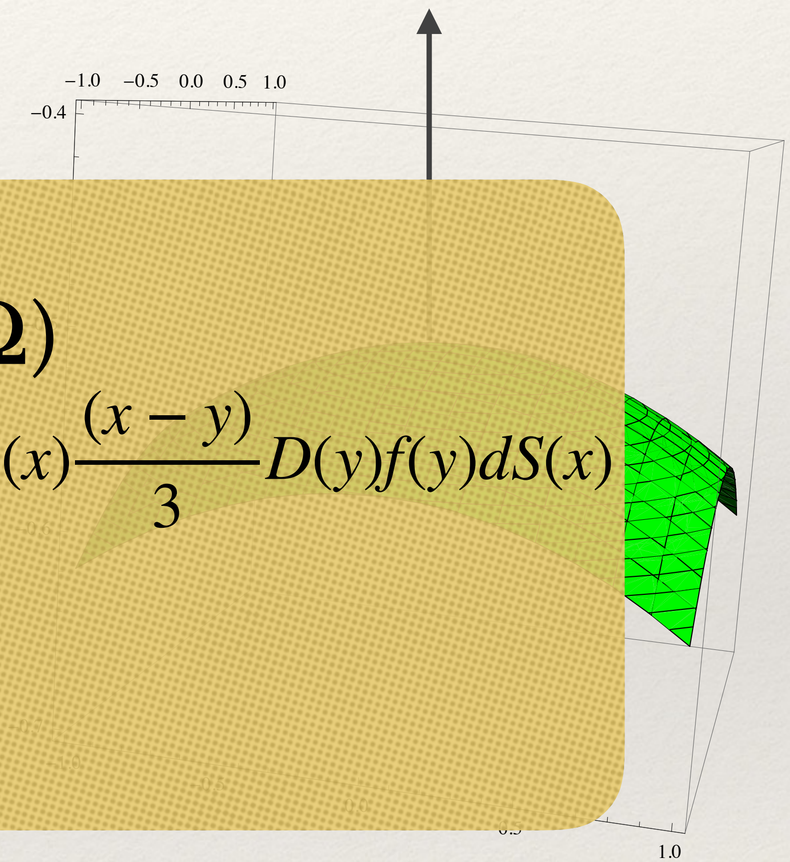
$C\ell_3$-valued BIE in terms of oriented surface element

Singularity-free BIEs for a harmonic function $f\,(y \in \partial\Omega)$

$$0 = \int_{\partial\Omega} U(x-y)d\sigma(x)\left[D(x)f(x) - D(y)f(y)\right] - \int_{\partial\Omega} C(x-y)d\sigma(x)\frac{(x-y)}{3}D(y)f(y)$$

$$+\int_{\partial\Omega} C(x-y)d\sigma(x)\left[f(x)-f(y)\right]$$

oriented surface element $d\sigma$



Singularity-free BIEs $(y \in \partial\Omega)$

$$0 = \int_{\partial\Omega} U(x-y)n(x)\left[D(x)f(x) - D(y)f(y)\right] dS(x) - \int_{\partial\Omega} C(x-y)n(x)\frac{(x-y)}{3}D(y)f(y)dS(x)$$

$$+\int_{\partial\Omega} C(x-y)n(x)\left[f(x)-f(y)\right] dS(x)$$

Singularity-free BIEs for a harmonic function $f\,(y \in \partial\Omega)$

$$0 = \int_{\partial\Omega} C(x-y)d\sigma(x)\left[f(x)-f(y)\right]$$

Boundary separation

Singularity-free BIEs for a harmonic function $f\,(y \in \partial\Omega)$

$$0 = \int_{\partial\Omega} U(x-y)d\sigma(x)\big[D(x)f(x) - D(y)f(y)\big] - \int_{\partial\Omega} C(x-y)d\sigma(x)\frac{(x-y)}{3}D(y)f(y)$$

$$+\int_{\partial\Omega} C(x-y)d\sigma(x)\big[f(x) - f(y)\big]$$

Separation of N pieces boundary

$$\partial\Omega = \sum_{j=1}^{N} \partial\Omega_j$$

Singularity-free BIEs for a harmonic function $f\,(y \in \partial\Omega)$

$$0 = \sum_{j=1}^{N}\int_{\partial\Omega_j} U(x-y)d\sigma(x)\big[D(x)f(x) - D(y)f(y)\big] - \sum_{j=1}^{N}\int_{\partial\Omega_i} C(x-y)d\sigma(x)\frac{(x-y)}{3}D(y)f(y)$$

$$+\sum_{j=1}^{N}\int_{\partial\Omega_i} C(x-y)d\sigma(x)\big[f(x) - f(y)\big]$$

Discretization—field variables interpolation

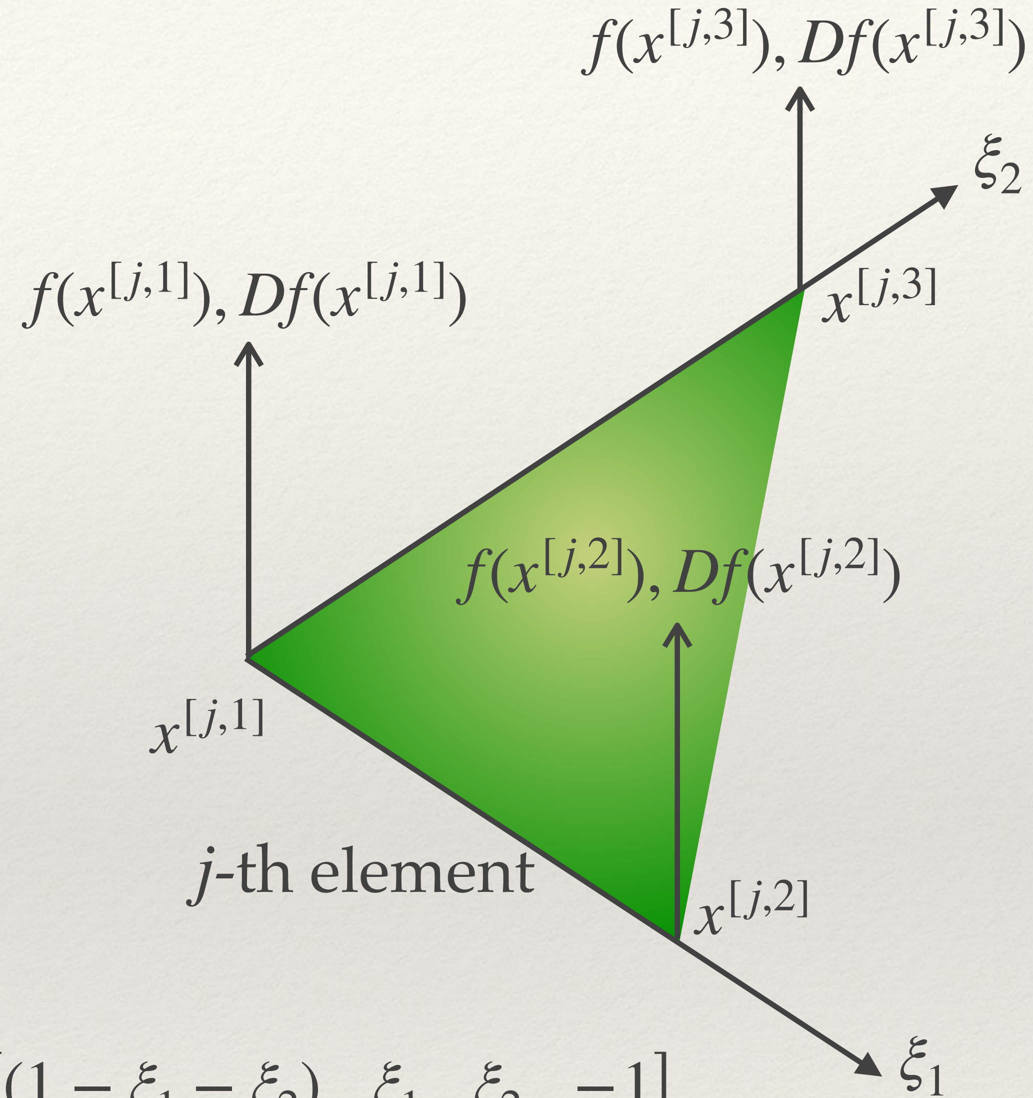
Field variables f and Df

$$f(x) - f(y) = \mathbf{Nt}^{[j]}(\xi_1, \xi_2)\mathbf{f}^{[j]}, x \in \partial\Omega_j$$

$$\mathbf{f}^{[j]} = \left[f(x^{[j,1]}) \quad f(x^{[j,2]}) \quad \cdots \quad f(x^{[j,m]}) \quad f(y) \right]^T$$

$$D(x)f(x) - D(y)f(y) = \mathbf{Nt}^{[j]}(\xi_1, \xi_2)\mathbf{g}^{[j]}$$

$$\mathbf{g}^{[j]} = \left[Df(x^{[j,1]}) \quad \cdots \quad Df(x^{[j,m]}) \quad D(y)f(y) \right]^T$$



$$\mathbf{f}^{[j]} = \begin{bmatrix} f(x^{[j,1]}) \\ f(x^{[j,2]}) \\ f(x^{[j,3]}) \\ f(y) \end{bmatrix}, \mathbf{g}^{[j]} = \begin{bmatrix} Df(x^{[j,1]}) \\ Df(x^{[j,2]}) \\ Df(x^{[j,3]}) \\ Df(y) \end{bmatrix}, \mathbf{Nt}^{[j]}(\xi_1, \xi_2) = \left[(1 - \xi_1 - \xi_2) \quad \xi_1 \quad \xi_2 \quad -1 \right]$$

$$0 \le \xi_1 \le 1,$$
$$0 \le \xi_2 \le 1,$$
$$\xi_1 + \xi_2 \le 1.$$

Discretization—geometry parametrization

Positions

$$x(\xi_1, \xi_2),\ y(\xi_1, \xi_2)$$

Fundamental solutions

$$U(x(\xi_1, \xi_2) - y(\xi_1, \xi_2)) =: \mathscr{U}(\xi_1, \xi_2),\ C(x(\xi_1, \xi_2) - y(\xi_1, \xi_2)) =: \mathscr{C}(\xi_1, \xi_2)$$

oriented surface elements

$$d\sigma(x(\xi_1, \xi_2))$$

planar element

$$\xi_1 \in (0,1), \xi_2 \in (0,1), \xi_1 + \xi_2 \leq 1$$

$$x(\xi_1, \xi_2) = (1 - \xi_1 - \xi_2)x^{[a]} + \xi_1 x^{[b]} + \xi_2 x^{[c]}$$

$$d\sigma(x(\xi_1, \xi_2))$$

$$= e_1 dx_2 dx_3 - e_2 dx_1 dx_3 + e_3 dx_1 dx_2$$

$$= J(\xi_1, \xi_2) d\xi_1 d\xi_2$$

curved element

$$\xi_1 \in (0,\pi), \xi_2 \in (0,2\pi]$$

$$x(\xi_1, \xi_2) = e_1(r \sin \xi_1 \cos \xi_2) + e_2(r \sin \xi_2 \sin \xi_2) + e_3(r \cos \xi_1)$$

$$d\sigma(x(\xi_1, \xi_2))$$

$$= e_1 dx_2 dx_3 - e_2 dx_1 dx_3 + e_3 dx_1 dx_2$$

$$= J(\xi_1, \xi_2) d\xi_1 d\xi_2$$

Collocation method

Singularity-free BIEs for a harmonic function $f\,(y = x^{[i]} \in \partial\Omega)$

$$0 = \sum_{j=1}^{N} \int_{\partial\Omega_j} \Bigg\{ \mathscr{U}(\xi_1, \xi_2, x^{[i]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]}$$

$$- \mathscr{C}(\xi_1, \xi_2, x^{[i]}) J(\xi_1, \xi_2) \frac{(x(\xi_1, \xi_2) - x^{[i]})}{3} \mathbf{B}^{[i]} \Bigg\} d\xi_1 d\xi_2 \mathbf{g}$$

$$+ \sum_{j=1}^{N} \int_{\partial\Omega_j} \mathscr{E}(\xi_1, \xi_2, x^{[i]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} dS(x) \mathbf{f}\,.$$

$$\mathbf{f}^{[j]} = \mathbf{Bt}^{[j]}\mathbf{f}, \quad \mathbf{g}^{[j]} = \mathbf{Bt}^{[j]}\mathbf{g}, \quad Df(x^{[i]}) = \mathbf{B}^{[i]}\mathbf{g}$$

Collocation method

$$\mathbf{f}^{[j]} = \mathbf{Bt}^{[j]}\mathbf{f}, \quad \mathbf{g}^{[j]} = \mathbf{Bt}^{[j]}\mathbf{g}, \quad Df(x^{[i]}) = \mathbf{B}^{[i]}\mathbf{g}$$

Singularity-free BIEs for a harmonic function f
$(y = x^{[i]} \in \partial\Omega, i = 1,2,\cdots, M\,.)$

$$0 = \sum_{j=1}^{N} \int_{\partial\Omega_j} \left\{ \mathscr{U}(\xi_1, \xi_2, x^{[1]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} - \mathscr{C}(\xi_1, \xi_2, x^{[1]}) J(\xi_1, \xi_2) \frac{(x(\xi_1, \xi_2) - x^{[i]})}{3} \mathbf{B}^{[i]} \right\} d\xi_1 d\xi_2 \mathbf{g} + \sum_{j=1}^{N} \int_{\partial\Omega_j} \mathscr{E}(\xi_1, \xi_2, x^{[1]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} dS(x) \mathbf{f}$$

$$0 = \sum_{j=1}^{N} \int_{\partial\Omega_j} \left\{ \mathscr{U}(\xi_1, \xi_2, x^{[2]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} - \mathscr{C}(\xi_1, \xi_2, x^{[2]}) J(\xi_1, \xi_2) \frac{(x(\xi_1, \xi_2) - x^{[i]})}{3} \mathbf{B}^{[i]} \right\} d\xi_1 d\xi_2 \mathbf{g} + \sum_{j=1}^{N} \int_{\partial\Omega_j} \mathscr{E}(\xi_1, \xi_2, x^{[2]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} dS(x) \mathbf{f}$$

$$\vdots$$

$$0 = \sum_{j=1}^{N} \int_{\partial\Omega_j} \left\{ \mathscr{U}(\xi_1, \xi_2, x^{[M]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} - \mathscr{C}(\xi_1, \xi_2, x^{[M]}) J(\xi_1, \xi_2) \frac{(x(\xi_1, \xi_2) - x^{[i]})}{3} \mathbf{B}^{[i]} \right\} d\xi_1 d\xi_2 \mathbf{g} + \sum_{j=1}^{N} \int_{\partial\Omega_j} \mathscr{E}(\xi_1, \xi_2, x^{[M]}) J(\xi_1, \xi_2) \mathbf{Nt}^{[j]}(\xi_1, \xi_2) \mathbf{Bt}^{[j]} dS(x) \mathbf{f}$$

Collocation method

Without normal or tangential derivative

A $C\ell_3$-valued linear system

$$\mathbf{U}\mathbf{g} + \mathbf{C}\mathbf{f} = \mathbf{0},$$

$$\mathbf{U}, \mathbf{C} \in \mathscr{M}(M, C\ell_3(\mathbf{R})),$$

$$\mathbf{f} = \begin{bmatrix} f(x^{[1]}) \\ f(x^{[2]}) \\ \vdots \\ f(x^{[M]}) \end{bmatrix} \text{ and } \mathbf{g} = \begin{bmatrix} Df(x^{[1]}) \\ Df(x^{[2]}) \\ \vdots \\ Df(x^{[M]}) \end{bmatrix}.$$

Hand-in 9

Please finish the flowchart to obtain the $C\ell_3$-valued linear system from the singularity-free BIEs..—(1)

(1)
Field variables f and Df interpolation

(2)
Singularity-free BIEs (oriented surface element) ($y \in \partial\Omega$)

(3)
Geometry parametrization position, FS, OSE

(4)
Singularity-free BIEs ($y \in \partial\Omega$)

(5)
A $C\ell_3$-valued linear system

(6)
Separation of N pieces boundary

$$\partial\Omega = \sum_{j=1}^{N} \partial\Omega_j$$

(7)
Collocation method

3 min

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Thanks for your attention

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