Are operators commutative?

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Abstract: In mathematics, we will sometimes employ operators to simplify the calculation process. However, are operators commutative in all situations? In fact, operators aren't commutative in some situations. We need to consider the properties of operators and functions, or we may lead to the wrong result. In this poster, we listed eight cases. Although the operators in most of the cases are not commutative, the operators are commutative in the specific conditions. Our interest is to study the commutative properties of two operators. Eight cases are shown here.

Problem description, Result and Discussion

1. Leibniz integral rule

$$\int_{b(x)}^{a(x)} \frac{\partial f(x,s)}{\partial x} ds \overset{?}{=} \frac{d}{dx}\int_{b(x)}^{a(x)} f(x,s)ds$$

If either $a(x)$ and $b(x)$ aren't constants, or interval between $a(x)$ and $b(x)$ isn't differentiable.

$$\int_{b(x)}^{a(x)} \frac{\partial f(x,s)}{\partial x} ds + \frac{df(x,a(x))}{da(x)}\frac{da(x)}{dx} - \frac{df(x,b(x))}{db(x)}\frac{db(x)}{dx}$$

$$= \int_{b(x)}^{a(x)} \frac{\partial f(x,s)}{\partial x} ds + f(x,a(x))a'(x) - f(x,b(x))b'(x)$$

If $a(x)$ and $b(x)$ are constants, and interval between $a(x)$ and $b(x)$ is differentiable

$$\frac{d}{dx}\int_{b(x)}^{a(x)} f(x,s)ds = \int_{b(x)}^{a(x)} \frac{\partial f(x,s)}{\partial x} ds \quad f(x,a(x))a'(x) - f(x,b(x))b'(x)$$

missing term

If $a(x)$ and $b(x)$ are constants, and interval between $a(x)$ and $b(x)$ is differentiable, then the operators are commutative.

2. Cauchy principle value & Hadamard principle value

$$\frac{d}{dx}\{C.P.V.\int_{-1}^{1}\frac{f(s)}{(s-x)}ds\} \overset{?}{=} \lim_{\varepsilon\to 0}[(\int_{-1}^{x-\varepsilon}+\int_{x+\varepsilon}^{1})\frac{\partial}{\partial x}\frac{f(s)}{(s-x)}ds]$$

$$-1\le x\le 1$$

$$\frac{d}{dx}\{C.P.V.\int_{-1}^{1}\frac{f(s)}{(s-x)}ds\} = \lim_{\varepsilon\to 0}[(\int_{-1}^{x-\varepsilon}+\int_{x+\varepsilon}^{1})\frac{f(s)}{(s-x)^2}ds - \frac{2f(x)}{\varepsilon}]$$

$$\lim_{\varepsilon\to 0}(\int_{-1}^{x-\varepsilon}+\int_{x+\varepsilon}^{1})\frac{\partial}{\partial x}\frac{f(s)}{(s-x)}ds = \lim_{\varepsilon\to 0}[(\int_{-1}^{x-\varepsilon}+\int_{x+\varepsilon}^{1})\frac{f(s)}{(s-x)^2}ds] \quad -\lim_{\varepsilon\to 0}\frac{2f(x)}{\varepsilon}$$

$$\{H.P.V\int_{-1}^{1}\frac{f(s)}{(s-x)^2}ds\} = \lim_{\varepsilon\to 0}[(\int_{-1}^{x-\varepsilon}+\int_{x+\varepsilon}^{1})\frac{f(s)}{(s-x)^2}ds - \frac{2f(x)}{\varepsilon}]$$

missing term

The operators aren't commutative. It is interesting that missing term relate to the terms, $f(x,a(x))a'(x)-f(x,b(x))b'(x)$, in first cases.

3. Stokes transformation

$f(t)$ → Express as Fourier series → $f(t)=a_0+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi t}{p})+\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi t}{p})$

Differentiate (Path 1) → $f'(t)$ → Express as Fourier Series → $f'(t)=q_0+\sum_{n=1}^{\infty}q_n\cos(\frac{n\pi t}{p})+\sum_{n=1}^{\infty}r_n\sin(\frac{n\pi t}{p})$

Differentiate (Path 2) → $f_d(t) \overset{?}{=} f'(t)$

Discontinuous function $f(t+2)=f(t),\ f(t)=t, 0\le t\le 2$

$$f(t)=\begin{cases}a_0=1\\ a_n=0\\ b_n=\frac{-2}{n\pi}\end{cases} \Rightarrow f'(t)=1$$

Path 1: $f_d(t)=f_d(t)=\sum_{n=1}^{\infty}-2\cos(n\pi t) \Rightarrow \begin{cases}s_0=0 & +1\\ s_n=-2 & +2\\ t_n=0\end{cases}$ missing term

Path 2: $f'(t)=1 \Rightarrow \begin{cases}q_0=1\\ q_n=\frac{-2n\pi}{n\pi}+2=0\\ r_n=0\end{cases}$

Continuous function $f(t+2)=f(t),\ f(t)=t^2, 0\le t\le 2$

$$f(t)=\begin{cases}a_0=\frac{\pi^2}{3}\\ a_n=\frac{4(-1)^n}{n^2}\\ b_n=0\end{cases} \Rightarrow f(t)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nt)$$

Path 1: $f_d(t)=\begin{cases}s_0=0\\ s_n=0\\ t_n=\frac{-4(-1)^n}{n}\end{cases} \Rightarrow f'(t)=\sum_{n=1}^{\infty}\frac{-4(-1)^n}{n}\sin(nt)$ correct

Path 2: $f'(t)=\begin{cases}q_0=0\\ q_n=0\\ r_n=\frac{-4(-1)^n}{n}\end{cases} \Rightarrow f'(t)=\sum_{n=1}^{\infty}\frac{-4(-1)^n}{n}\sin(nt)$

If function is continuous, then operators are commutative.

4. Matrix Commutation law

$$AB \overset{?}{=} BA$$

$A=\begin{bmatrix}1&1\\-1&2\end{bmatrix}$, $B=\begin{bmatrix}6&2\\1&5\end{bmatrix}$, $AB=\begin{bmatrix}7&7\\-4&8\end{bmatrix}$, $BA=\begin{bmatrix}4&10\\-4&11\end{bmatrix}$, $AB\ne BA$

$A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}9&6\\3&4\end{bmatrix}$, $AB=\begin{bmatrix}9&6\\3&4\end{bmatrix}$, $BA=\begin{bmatrix}9&6\\3&4\end{bmatrix}$, $AB=BA$

For matrix multiplication, most of the cases aren't commutative, but sometimes are commutative.

5. Fourier Transform

$f(t)$ → Fourier Transform → $F(\omega)$ → Differentiate → $F'(\omega)$

$f(t)$ → Differentiate → $f'(t)$ → Fourier Transform → $i\omega F(\omega)$

$F'(\omega) \overset{?}{=} i\omega F(\omega)$

The operators aren't commutative . Unless ,there is a function its expression of Fourier transform which satisfies $F(\omega)=e^{0.5i\omega^2}$.

6. Laplace Transform

$f(t)$ → Laplace Transform → $F(s)$ → Differentiate → $F'(s)$

$f(t)$ → Differentiate → $f'(t)$ → Laplace Transform → $sF(s)-f(0)$

$F'(s) \overset{?}{=} sF(s)-f(0)$

The operators aren't commutative. Unless , there is a function its expression of Laplace transform which satisfies $F(s)=e^{0.5s^2}-sf(0)$.

7. Multiplication & Fourier Transform

$f(t)$、$g(t)$ → Multiplication → $j(t)=f(t)\cdot g(t)$ → Fourier Transform → $J_{MF}(\omega)$

$f(t)$、$g(t)$ → Fourier Transform → $F(\omega)$、$G(\omega)$ → Multiplication → $J_{FM}(\omega)=F(\omega)\cdot G(\omega)=\int f(t)*g(t)$

The operators aren't commutative.

8. Fourier Transform & Odd, Even Function Expansion

$f(t)$ → Fourier Transform → $F(\omega)$

$f(t)$ → Odd, even function Expansion → Odd function: $\frac{f(t)-f(-t)}{2}$, Even function: $\frac{f(t)+f(-t)}{2}$ → Fourier Transform

$F(\omega)$ → Odd, even function Expansion → Odd function: $\frac{F(\omega)-F(-\omega)}{2}$ (Image), Even function: $\frac{F(\omega)+F(-\omega)}{2}$ (Real)

The operators are commutative.

Conclusions: For the first three cases, the operators are not commutative if the function is discontinuous. After exchanging the order of the operators for the discontinuous function, the result may lose boundary terms and lead to the wrong result. Moreover, the missing terms are related to the boundary terms of the discontinuous function in the first two cases. In this poster, operators are not commutative in most of the cases, except under the specific conditions. It indicates that checking the properties of operators and functions before commutative process is needed to avoid the wrong result.

References:

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