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Boundary knot method for Poisson equations

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Abstract

The boundary knot method is a recent truly meshfree boundary-type radial basis function (RBF) collocation scheme, where the nonsingular general solution is used instead of the singular fundamental solution to evaluate the homogeneous solution, while the dual reciprocity method is employed to the approximation of particular solution. Despite the fact that there are not nonsingular RBF general solutions available for Laplace-type problems, this study shows that the method can successfully be applied to these problems.

Keywords: Boundary knot method; Method of fundamental solution; General solution; Poisson equation; Radial basis function

1. Introduction

Among typical meshfree boundary-type numerical schemes are the local boundary integral equation (MLBIE) method [1], the boundary node method (BNM) [2], boundary point interpolation method [3], and the method of fundamental solutions (MFS) [4]. The meshfree MLBIE and BNM are in fact a combination of the moving least-square (MLS) technique with the boundary element scheme, whereas the MFS is a boundary-type radial basis function (RBF) collocation scheme [5]. Both the MLBIE and the BNM involve singular integration and hence are mathematically more complicated in comparing with the commonly used finite element method (FEM). In addition, their low-order approximations also downgrade computational efficiency and are not easily used for engineers. On the other hand, the MFS possesses integration-free, spectral convergence, easy-to-use, and inherently meshfree merits. In recent years, the MFS, also known as the regular boundary element method, revives partly thanks to its combination with the dual reprocity method (DRM) for handling inhomogeneous problems [4]. In the use of a singular fundamental solution, which can be considered a RBF, the MFS, however, requires a controversial fictitious boundary outside physical domains, which largely impedes its practical use for complex geometry problems.

As an alternative RBF approach, Chen and Tanaka [6] recently developed the boundary knot method (BKM), where the perplexing artificial boundary in the MFS is eliminated via the nonsingular general solution instead of the singular fundamental solution. In the meshfree collocation fashion, Chen et al. [7–9] also used the general solution to calculate the eigenvalue problems. Just like the MFS and the dual reciprocity BEM (DR-BEM) [10], the BKM also uses the DRM to evaluate the particular solution. The method is essentially symmetric, spectral convergence, integration-free, meshfree, easy to learn and implement, and has successfully been applied to the Helmholtz, diffusion, and convection–diffusion problems under complex-shaped two- and three-dimension domains. The method can be considered a new type of the Trefftz method, which combines the DRM, RBF, and nonsingular general solution. It is noted that the BKM is free of the domain dependence and quite robust for complex-shaped surface problems.

Unfortunately, the nonsingular RBF general solution of Laplace equations, however, is a constant rather than a RBF, which impedes the direct BKM solution of the problems of this kind. In this study, by using the nonsingular general solution of the Helmholtz-like equations, we develop a simple strategy to overcome this difficulty in applying the BKM solution of Poisson equation problems.

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2. BKM scheme for Poisson equation

Here, we introduce the BKM with a Poisson equation problem

$$\nabla^2 u = f(x), \quad x \in \Omega, \tag{1}$$

$$u(x) = R(x), \quad x \subset S_u, \tag{2a}$$

$$\frac{\partial u(x)}{\partial n} = N(x), \quad x \subset S_T, \tag{2b}$$

where x means multi-dimensional independent variable, and n is the unit outward normal. The governing equation (1) can be restated as

$$\nabla^2 u + \delta^2 u = f(x) + \delta^2 u \tag{3a}$$

or

$$\nabla^2 u - \delta^2 u = f(x) - \delta^2 u, \tag{3b}$$

where δ is an artificial parameter. Eqs. (3a) and (3b) are, respectively, Helmholtz and modified Helmholtz equations. This strategy can be understood that the use of nonsingular general solutions of Helmholtz-like equation with a small characteristic parameter δ approximates the constant general solution of the Laplace equation. For example, the general solution of the 2D Helmholtz operator (3a) is the Bessel function of the first kind of the zero-order and can be expanded as

$$J_0(\delta r) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} k! k!} (\delta r)^{2k}, \tag{4}$$

where r denotes the Euclidean distance. As the parameter δ goes to zero, the $J_0(\delta r)$ approaches constant 1. In the limiting process, the general solution of the Helmholtz operator is the general solution of the Laplacian.

The solution of the Poisson problem can be split as the homogeneous and particular solutions

$$u = u_{\text{h}} + u_{\text{p}}, \tag{5}$$

The latter satisfies the governing equation but not the boundary conditions. To evaluate the particular solution, the inhomogeneous term is approximated by

$$f(x) \cong \sum_{j=1}^{N+L} \beta_j \varphi(r_j), \tag{6}$$

where β_j are the unknown coefficients. N and L are, respectively, the numbers of knots on the domain and boundary. The use of interior points is usually necessary to guarantee the accuracy and convergence of the BKM solution of inhomogeneous problems. $r_j = \|x - x_j\|$ represents the Euclidean distance norm, and φ is the radial basis function to be specified later on. By forcing approximation representation (6) to exactly satisfy governing equations at all nodes, we can uniquely determine

$$\beta = A_\varphi^{-1}\{f(x_i)\}, \tag{7}$$

where A_φ is the nonsingular RBF interpolation matrix. Then we have

$$u_{\text{p}} = \sum_{j=1}^{N+L} \beta_j \phi(\|x - x_j\|), \tag{8}$$

where the RBF ϕ is related to the RBF ϕ through governing equations. In this study, we choose the first- and second-order general solutions of the Helmholtz equation or the modified Helmholtz equation as the RBFs ϕ and φ, which can be calculated with Eqs. (3a) and (3b), respectively, by [11]

$$u_m^{\#}(r) = Q_m (\gamma r)^{-n/2+1+m} J_{n/2-1+m}(\gamma r), \tag{9a}$$

and

$$u_m^{\#}(r) = Q_m (\tau r)^{-n/2+1+m} I_{n/2-1+m}(\tau r), \quad n \geq 2, \tag{9b}$$

where n is the dimension of the problem; $Q_m = Q_{m-1}/(2 \times m \times \gamma^2)$, $Q_0 = 1$; m denotes the order of general solution; J and I represent the Bessel and modified Bessel function of the first kind.

On the other hand, the homogeneous solution u_{h} has to satisfy both governing equation and boundary conditions. By means of the nonsingular general solution, the unsymmetric and symmetric BKM [12] expressions are given, respectively, by

$$u_{\text{h}}(x) = \sum_{k=1}^{L} \alpha_k u_0^{\#}(r_k), \tag{10a}$$

$$u_{\text{h}}(x) = \sum_{s=1}^{L_{\text{d}}} a_s u_0^{\#}(r_s) - \sum_{s=L_{\text{d}}+1}^{L_{\text{d}}+L_{\text{N}}} a_s \frac{\partial u_0^{\#}(r_s)}{\partial n}, \tag{10b}$$

where k is the index of source points on boundary, α_k are the desired coefficients; n is the unit outward normal as in boundary condition (2b), and L_{d} and L_{N} are, respectively, the numbers of knots on the Dirichlet and Neumann boundary surfaces. The minus sign associated with the second-term is due to the fact that the Neumann condition of the first-order derivative is not self-adjoint. In terms of representation (10b), the collocation analogue equations (3a) (or (3b)) and (2a) and (2b) are written as

$$\sum_{s=1}^{L_{\text{d}}} a_s u_0^{\#}(r_{is}) - \sum_{s=L_{\text{d}}+1}^{L_s+L_{\text{N}}} a_s \frac{\partial u_0^{\#}(r_{is})}{\partial n} = R(x_i) - u_{\text{p}}(x_i), \tag{11}$$

$$\sum_{s=1}^{L_{\text{d}}} a_s \frac{\partial u_0^{\#}(r_{js})}{\partial n} - \sum_{s=L_{\text{d}}+1}^{L_{\text{d}}+L_{\text{N}}} a_s \frac{\partial^2 u_0^{\#}(r_{js})}{\partial n^2} = N(x_j) - \frac{\partial u_{\text{p}}(x_j)}{\partial n}, \tag{12}$$

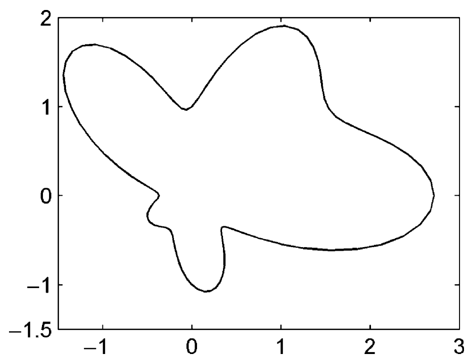

Fig. 1. A 2D irregular geometry.

$$\sum_{s=1}^{L_d} a_s u_0^{\#}(r_{ls}) - \sum_{s=L_d+1}^{L_d+L_N} a_s \frac{\partial u_0^{\#}(r_{ls})}{\partial n} = u_l - u_p(x_l). \quad (13)$$

Note that i, s and j are reciprocal indices of Dirichlet (S_u) and Neumann boundary (S_Γ) nodes. l indicates response knots inside domain Ω. Then we can employ the obtained expansion coefficients α and inner knot solutions u_l to calculate the BKM solution at any other knots.

3. Numerical results and discussions

The tested 2D and 3D Poisson equation examples have accurate solutions

$$u = x^3y - 2xy^3 + x + 10, \quad (14a)$$

$$u = x^3yz - xy^3z + xyz^3 + x + 10. \quad (14b)$$

Figs. 1 and 2 show the tested 2D and 3D irregular geometries, where the 3D ellipsoid cavity locates at the center of the cube with the characteristic lengths 3/8, 1/8, and 1/8. Except Neumann boundary conditions on $x=0$ surface of the 3D case, the otherwise boundary are all Dirichlet type. The 2D ellipse has the characteristic lenghts 1 and 2 with three inner nodes located in (0,0), (−0.5,0), and (0.5,0).

We note that the unsymmetric (Eq. (9a)) and symmetric (Eq. (9b)) BKM formulations produce insignificant differences of accuracy for all cases. Therefore, Tables 1–3 only displays the unsymmetric BKM L_2 norms of relative errors, which were calculated at 492 sample nodes for 2D and 1000 sample nodes for 3D. Note that the abbreviations Helm and MHelm in Tables 1 and 2 mean that the general solutions of Helmholtz and modified Helmholtz equations (see Eqs. (3a) and (3b)) are, respectively, used. The first and second numbers in the bracket of tables represents, respectively, the numbers of boundary and inner nodes used in the BKM solution. Here, the absolute error is taken as the relative error if the absolute value of the solution is less than 0.001.

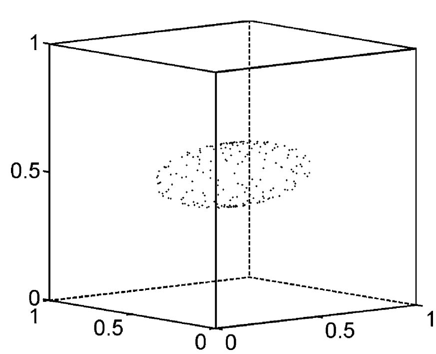

Fig. 2. A cube with an ellipsoid cavity.

Table 1
L_2 relative errors of 2D Poisson equation under a domain shown in Fig. 1

	Helm (9+3)	MHelm (9+3)	Helm (13+3)	MHelm (13+3)
$\delta=0.1$	9.58×10^{-4}	4.28×10^{-4}	4.64×10^{-4}	3.50×10^{-4}
$\delta=0.2$	5.10×10^{-3}	5.10×10^{-3}	2.24×10^{-6}	2.27×10^{-6}

It is found that a few inner nodes are usually necessary to significantly improve the solution accuracy and stability compared without inner nodes as discussed in [13]. Without the inner nodes, the BKM solutions were found not stable for irregular geometry since the poor accuracy appears at very few nodes. For regular geometries, it is, however, noted that the BKM can produce very accurate solutions without using inner nodes. For instance, the L_2 relative error norm at 495 nodes of an ellipse for the 2D problem by the BKM using only nine boundary nodes is 5.3×10^{-3}.

It can be observed from Tables 1–3 that the accuracy of our numerical experiments is quite accurate, and the convergence is also stable. The artificial parameter δ was chosen by numerical experiments. To our experiences, the BKM produces more accurate results when the value of δ ranges from 0.1 to 0.3 for the 2D cases and 0.3–0.6 for the 3D cases, which depends on the dimensionality and the size of geometry contour. Our observations also find that the BKM solutions are stable and insensitive to the artificial parameter δ. But nevertheless we note that the value of δ has something to do with the solution accuracy. The proper choice of δ value is still an open issue under the study. It is also noted that the performances of the general solutions of the Helmholtz and modified Helmholtz operators are close.

The programming is particularly easy in this study thanks to the simplicity of the BKM algorithm. The 2D and 3D programs are almost the same except for the change of the general solution and the definition of the distance. We have also tested the present BKM schemes to some other Poisson equation cases and observe the similar good performances.

Table 2
L_2 relative errors of 2D Poisson equation under an elliptic domain

	Helm (9+3)	MHelm (9+3)	Helm (13+3)	MHelm (13+3)
$\delta=0.1$	1.23×10^{-5}	1.42×10^{-5}	4.22×10^{-4}	1.20×10^{-3}
$\delta=0.2$	4.85×10^{-5}	4.83×10^{-5}	4.06×10^{-5}	9.08×10^{-5}

Table 3
L_2 relative errors of 3D Poisson equation under a domain shown in Fig. 2 with the general solution of modified Helmholtz operator (Eq. (3b))

	Helm (66+8)	MHelm (96+8)	Helm (138+8)	MHelm (192+8)
$\delta=0.3$	3.73×10^{-4}	2.43×10^{-4}	3.81×10^{-4}	2.41×10^{-4}
$\delta=0.5$	5.70×10^{-3}	3.00×10^{-3}	2.90×10^{-3}	4.80×10^{-5}

4. Completeness, convergence and conditioning number

Hon and Chen [13] discussed the completeness, convergence and conditioning number of the BKM in terms of the solution of the Helmholtz, diffusion, and convection–diffusion problems. For the Poisson equation, the completeness using the Helmholtz general solution is also an open issue as for the Helmholtz problem. In fact, a central issue is whether or not the singularity is essential to attain reliable solutions by the boundary-type discretization schemes. The MRM and BEM with the real part of the Helmholtz fundamental solution encounter the same incompleteness concerns as in the BKM [13].

As of the convergence, Fig. 3 displays the convergence curve of the Poisson equation under the elliptical domain versus the numbers of boundary nodes. It is seen from Fig. 3 that the solution converges very fast, and oscillates slightly after the accuracy peaks as in the BKM solution of the other PDE problems due to the severely ill-conditioned interpolation matrix. On the other hand, like the BEM and the MFS, the BKM discretization results in a full matrix, which tends to be ill-conditioned. Ref. [13] has a detailed discussion on this issue for the BKM. Some preconditioning techniques of recent origin such as the fast multi-pole approach will be useful to result in the better-conditioned sparse matrix and thus overcome this perplexing issue.

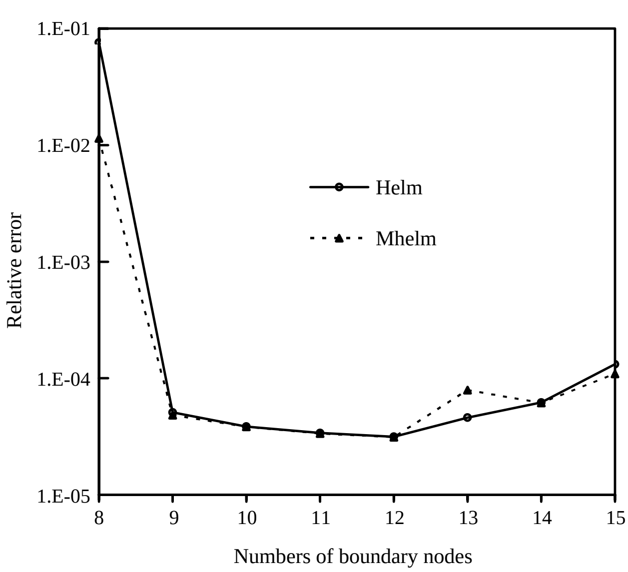

Fig. 3. Solution accuracy versus the numbers of boundary nodes (three inner nodes, $\delta=0.2$).

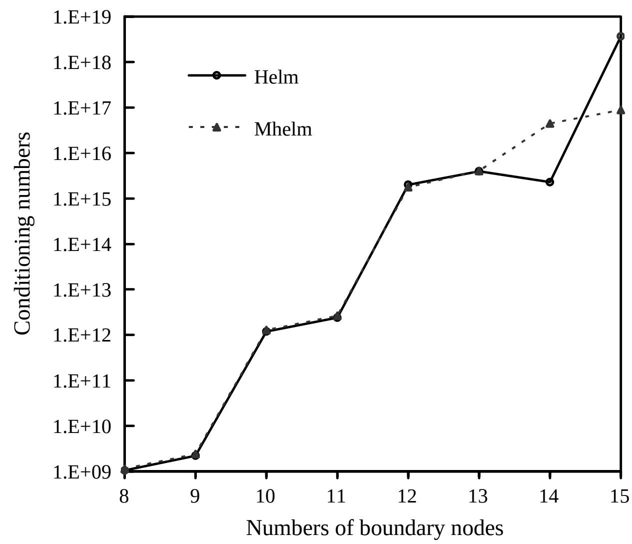

Fig. 4. Conditioning numbers versus the numbers of boundary nodes (three inner nodes, $\delta=0.2$).

Fig. 4 illustrated the conditioning numbers of the same case in terms of the numbers of the boundary nodes. We find that the conditioning number increases quickly as the number of the boundary nodes increases. In this case, we observe that the use of the modified Helmholtz general solution produces slightly smaller conditioning numbers and better accuracy than that of the Helmholtz general solution. In all our experiments, we find that the performances of both general solutions are similar.

5. Concluding remarks

For Helmholtz-like problems, the BKM outperforms the DR-BEM and MFS significantly in terms of accuracy, symmetricity, efficiency, stability, and mathematical simplicity [13]. The present study shows that the method is also impressive for Poisson equation problems. The major drawbacks of the BKM are severe ill-conditioning and costly full matrix for large system problems, which is a subject presently under investigation. The major concern of the BKM is the possible incompleteness in solving some types of problems due to the only use of nonsingular general solution.

In this study, we used the high-order general solutions of the Helmholtz and modified Helmholtz equations to evaluate the particular solution. It will be interesting to investigate the performances of the nonsingular high-order fundamental solution of the Laplacian equation in the evaluation of the particular solution.

The small parameter δ is somewhat arbitrary despite the fact that our numerical investigations found it is not sensitive to the geometry and node density. But nevertheless this parameter causes some concerns and needs further investigation for a variety of diverse problems. This study shows that the present BKM scheme has some obvious advantages over the MFS in that the BKM only requires adjusting one parameter δ, while the MFS has to arrange all artificial nodes outside physical domains, which can be quite tricky for problems having complex geometry. Compared with the BEM, the method has higher accuracy and does not require mesh and the evaluation of singular integration, and thus the overall computing cost of the BKM is dramatically lowered.

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