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Kernels of the Method of Fundamental Solutions for Kirchhoff Plates

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Abstract

This paper gives explicitly the kernels required for numerical implementations of the method of fundamental solutions for Kirchhoff plates in equilibrium as well as simply and multiply connected vibrational plates. The kernels associated to the boundary conditions of lateral displacement, slope, normal moment, and effective sheer force of the prescribed three cases are all derived. They are addressed in forms suitable for numerical implementations of the method of fundamental solutions. Also, the formulations of method of fundamental solutions for these three cases are briefly reviewed. And, the linear least squares method of fundamental solutions are supplied for obtaining eigenfunctions of vibrational plates.

Keywords: method of fundamental solutions, Kirchhoff plate, mixed potential method, plate vibration

1 Introduction

Meshless numerical methods have recently become alternatives to the classical mesh dependent numerical methods, such as the finite difference method (FDM), finite element method (FEM), and boundary element method (BEM). The method of fundamental solutions (MFS), first developed by Kupradze and Aleksidze [1] in 1964, has re-emerged as a promising meshless numerical scheme for solving various types of partial differential equations. The basic idea of the MFS is to decompose the solutions of the partial differential equations into a linear combination of the fundamental solutions, in which source points are located on a fictitious boundary outside the computational domain. Here, the intensities of the sources are the unknown parameters to be found. Excellent reviews of the MFS are available in the recent literature [2-5].

There are many numerical methods available in the analysis of plate problems. Leissa [6] has provided a comprehensive survey for the numerical solutions of plate vibration problems. Integral equations and BEM have also been utilized to solve plate vibration problems for a long time. Kitahara [7] employed the complex-valued boundary integral equation (BIE) method to solve the eigenvalues and eigenmodes for plate vibrations with various boundary conditions. On the other hand, Hutchinson [8] has carried out a series of studies by utilizing the BIE with real-part kernel to solve plate vibration problems. Recently, Chen et al. [9] further developed this method and utilized the SVD updating technique to avoid the occurrence of the spurious eigenvalues.

The MFS has also applied to plate problems. In 1989, Raamachandran and Bhaskar [10] first solved the problem of clamped circular plate in equilibrium by the MFS. It was later revisited by Poullikkas et al. [11] by considering the sources also as unknowns and solving the algebraic system by nonlinear optimization. For vibrational plates, Young et al. [12] and Tsai et al. [13] applied the MFS for simply and multiply connected domains, respectively. Inwhich the basic theories of avoiding the spurious eigenvalues was established by Tsai et al. [14] for Helmholtz equations. On the other hand, Tsai et al. [15] developed the linear least squares MFS for solving the acoustic modes. However, none of the previous studies expressed explicitly the kernels of boundary conditions of normal moment and effective sheer force.

In this paper, we first revisit the MFS formulations for the following three cases: Kirchhoff plates in equilibrium, simply connected vibrational plates, and multiply connected vibrational plates. The linear least squares MFS for obtaining eigenfunctions of vibrational plates are also supplied. Furthermore, the kernels associated to the boundary conditions of lateral displacement, slope, normal moment, and effective sheer force of the prescribed three cases are derived. They are addressed in a form suitable for numerical implementations of the method of fundamental solutions.

2 MFS formulation for Kirchhoff plates in equilibrium

The governing differential equation and boundary conditions for a Kirchhoff plate in equilibrium subjected to transverse loading $q(x_1, x_2)$ is given as

$$\begin{cases} \nabla^2\nabla^2 u(x_1,x_2) = \frac{q(x_1,x_2)}{D} \text{ in } \Omega \\ \mathbf{K}_u u(x_1,x_2) = \bar{u}(x_1,x_2) \text{ on } \Gamma_c + \Gamma_s \\ \mathbf{K}_\theta u(x_1,x_2) = \bar{\theta}(x_1,x_2) \text{ on } \Gamma_c \\ \mathbf{K}_m u(x_1,x_2) = \bar{m}(x_1,x_2) \text{ on } \Gamma_s + \Gamma_f \\ \mathbf{K}_v u(x_1,x_2) = \bar{v}(x_1,x_2) \text{ on } \Gamma_f \end{cases} \quad (1)$$

where $\bar{u}$, $\bar{\theta}$, $\bar{m}$ & $\bar{v}$ are given boundary data, $u(x_1, x_2)$ is the lateral deflection of the plate, $D = \frac{Eh}{12(1-\nu)}$ is the the flexural rigidity of the thin plate, ν is the Poisson ratio, E is the Young's modulus, Ω is the domain of the plate, and $\Gamma = \Gamma_c + \Gamma_s + \Gamma_f$ is the plate boundary, in which Γ_c is the clamped part, Γ_s is the simply-supported part, as well as Γ_f is the free part. Furthermore, the boundary operators of lateral displacement $\mathbf{K}_u(\bullet)$, slope $\mathbf{K}_\theta(\bullet)$, normal moment $\mathbf{K}_m(\bullet)$, and effective sheer force $\mathbf{K}_v(\bullet)$ are given by:

$$\mathbf{K}_u(\bullet) = 1 \quad (2a)$$

$$\mathbf{K}_\theta(\bullet) = \frac{\partial(\bullet)}{\partial n_{\mathbf{x}}} \quad (2b)$$

$$\mathbf{K}_m(\bullet) = \nu\nabla^2_{\mathbf{x}}(\bullet) + (1-\nu)\frac{\partial^2(\bullet)}{\partial n^2_{\mathbf{x}}} \quad (2c)$$

$$\mathbf{K}_v(\bullet) = \frac{\partial\nabla^2_{\mathbf{x}}(\bullet)}{\partial n_{\mathbf{x}}} + (1-\nu)\frac{\partial}{\partial t_{\mathbf{x}}}\frac{\partial^2(\bullet)}{\partial n_{\mathbf{x}}\partial t_{\mathbf{x}}} \quad (2d)$$

where $\frac{\partial}{\partial n_{\mathbf{x}}}$ and $\frac{\partial}{\partial t_{\mathbf{x}}}$ are the normal and tangential derivatives, respectively, on the boundary point $\mathbf{x} = (x_1, x_2)$. Therefore, (2a) and (2b) are selected for plate vibrations with clamped boundary condition, (2a) and (2c) for simply-supported boundary condition, and (2c) and (2d) for free boundary condition. Eqs. (2) can also be written more clearly as follows:

$$\mathbf{K}_u(\bullet) = 1 \quad (3a)$$

$$\mathbf{K}_\theta(\bullet) = \frac{\partial(\bullet)}{\partial x_1}n_1 + \frac{\partial(\bullet)}{\partial x_2}n_2 \quad (3b)$$

$$\mathbf{K}_m(\bullet) = f_1\frac{\partial^2(\bullet)}{\partial x_1^2} + f_2\frac{\partial^2(\bullet)}{\partial x_1\partial x_2} + f_3\frac{\partial^2(\bullet)}{\partial x_2^2} \quad (3c)$$

$$\mathbf{K}_v(\bullet) = g_1\frac{\partial^3(\bullet)}{\partial x_1^3} + g_2\frac{\partial^3(\bullet)}{\partial x_1^2\partial x_2} + g_3\frac{\partial^3(\bullet)}{\partial x_1\partial x_2^2} + g_4\frac{\partial^3(\bullet)}{\partial x_2^3} \quad (3d)$$

with

$$f_1 = Dn_1^2 + \nu Dn_2^2 \quad (4a)$$

$$f_2 = 2(1-\nu)Dn_1 n_2 \quad (4b)$$

$$f_3 = Dn_2^2 + \nu Dn_1^2 \quad (4c)$$

$$g_1 = Dn_1(1+n_2^2) - \nu Dn_1 n_2^2 \quad (4d)$$

$$g_2 = \nu Dn_2(1+n_1^2) + 2(1-\nu)Dn_2^3 - Dn_1^2 n_2 \quad (4e)$$

$$g_3 = \nu Dn_1(1+n_2^2) + 2(1-\nu)Dn_1^3 - Dn_2^2 n_1 \quad (4f)$$

$$g_4 = Dn_2(1+n_1^2) - \nu Dn_2 n_1^2 \quad (4g)$$

where $\mathbf{n}_x = (n_1, n_2)$ is the outward boundary normal vector.

Formally, Eq. (1) can be transformed to homogeneous problem by the dual reciprocity method (DRM) [3], whose details are omitted here.

$$\begin{cases} \nabla^2\nabla^2 u^c(x_1,x_2) = 0 \text{ in } \Omega \\ \mathbf{K}_u u^c(x_1,x_2) = \bar{u}(x_1,x_2) - \mathbf{K}_u u^p(x_1,x_2) \text{ on } \Gamma_c + \Gamma_s \\ \mathbf{K}_\theta u^c(x_1,x_2) = \bar{\theta}(x_1,x_2) - \mathbf{K}_\theta u^p(x_1,x_2) \text{ on } \Gamma_c \\ \mathbf{K}_m u^c(x_1,x_2) = \bar{m}(x_1,x_2) - \mathbf{K}_m u^p(x_1,x_2) \text{ on } \Gamma_s + \Gamma_f \\ \mathbf{K}_v u^c(x_1,x_2) = \bar{v}(x_1,x_2) - \mathbf{K}_v u^p(x_1,x_2) \text{ on } \Gamma_f \end{cases} \quad (5)$$

where $u^p(x_1,x_2)$ obtained by the DRM is the particular solution satisfying

$$\nabla^2\nabla^2 u^p(x_1,x_2) = \frac{q(x_1,x_2)}{D} \text{ in } \Omega \quad (6)$$

and the complementary solution $u^c(x_1,x_2)$ is defined by

$$u^c(x_1,x_2) = u(x_1,x_2) - u^p(x_1,x_2) \quad (7)$$

Then, Eq. (5) can formally be solved by the MFS for Kirchhoff plates in equilibrium [10] as

$$u^c(\mathbf{x}) \cong \sum_{j=1}^{L} \left\{ \alpha_j \left|\mathbf{x}-\mathbf{s}_j\right|^2 \log(|\mathbf{x}-\mathbf{s}_j|) + \beta_j \log(|\mathbf{x}-\mathbf{s}_j|) \right\} \quad (8)$$

where $\mathbf{s}_j$ are L prescribed sources located outside the computational domain as depicted in Figure 1. For simplicity, we define the following notations of kernels:

$$U_1(\mathbf{x},\mathbf{s}) = \mathbf{K}_u(|\mathbf{x}-\mathbf{s}|^2 \log(|\mathbf{x}-\mathbf{s}|)) \quad (9a)$$

$$U_2(\mathbf{x},\mathbf{s}) = \mathbf{K}_u(\log(|\mathbf{x}-\mathbf{s}|)) \quad (9b)$$

$$\Theta_1(\mathbf{x},\mathbf{s}) = \mathbf{K}_\theta(|\mathbf{x}-\mathbf{s}|^2 \log(|\mathbf{x}-\mathbf{s}|)) \quad (9c)$$

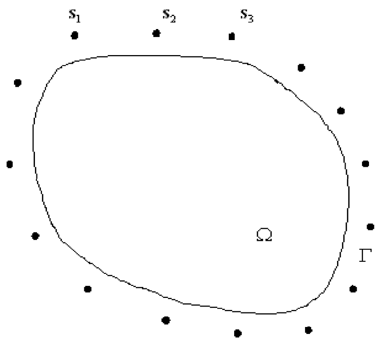

Figure 1: Geometry configuration for a simply connected plate

$$\Theta_2(\mathbf{x},\mathbf{s}) = \mathbf{K}_\theta(\log(|\mathbf{x}-\mathbf{s}|)) \quad (9d)$$
$$M_1(\mathbf{x},\mathbf{s}) = \mathbf{K}_m(|\mathbf{x}-\mathbf{s}|^2\log(|\mathbf{x}-\mathbf{s}|)) \quad (9e)$$
$$M_2(\mathbf{x},\mathbf{s}) = \mathbf{K}_m(\log(|\mathbf{x}-\mathbf{s}|)) \quad (9f)$$
$$V_1(\mathbf{x},\mathbf{s}) = \mathbf{K}_v(|\mathbf{x}-\mathbf{s}|^2\log(|\mathbf{x}-\mathbf{s}|)) \quad (9g)$$
$$V_2(\mathbf{x},\mathbf{s}) = \mathbf{K}_v(\log(|\mathbf{x}-\mathbf{s}|)) \quad (9h)$$

Eq. (8) can then be written as

$$u^c(\mathbf{x}) \cong \sum_{j=1}^{L}\{\alpha_j U_1(\mathbf{x},\mathbf{s}_j) + \beta_j U_2(\mathbf{x},\mathbf{s}_j)\} \quad (10)$$

To obtained the unknown intensities α_j and β_j, $2 \times L$ boundary conditions should be collocated as follows:

$$\sum_{j=1}^{L}\{\alpha_j U_1(\mathbf{x}_i,\mathbf{s}_j)+\beta_j U_2(\mathbf{x}_i,\mathbf{s}_j)\} = \bar{u}(\mathbf{x}_i)-\mathbf{K}_u(u^p(\mathbf{x}_i))\text{for }\{\mathbf{x}_i\}_1^L \in \Gamma_c+\Gamma_s \quad (11a)$$

$$\sum_{j=1}^{L}\{\alpha_j \Theta_1(\mathbf{x}_i,\mathbf{s}_j)+\beta_j \Theta_2(\mathbf{x}_i,\mathbf{s}_j)\} = \bar{\theta}(\mathbf{x}_i)-\mathbf{K}_\theta(u^p(\mathbf{x}_i))\text{for }\{\mathbf{x}_i\}_1^L \in \Gamma_c \quad (11b)$$

$$\sum_{j=1}^{L}\{\alpha_j M_1(\mathbf{x}_i,\mathbf{s}_j)+\beta_j M_2(\mathbf{x}_i,\mathbf{s}_j)\} = \bar{m}(\mathbf{x}_i)-\mathbf{K}_m(u^p(\mathbf{x}_i))\text{for }\{\mathbf{x}_i\}_1^L \in \Gamma_s+\Gamma_f \quad (11c)$$

$$\sum_{j=1}^{L}\{\alpha_j V_1(\mathbf{x}_i,\mathbf{s}_j)+\beta_j V_2(\mathbf{x}_i,\mathbf{s}_j)\} = \bar{v}(\mathbf{x}_i)-\mathbf{K}_v(u^p(\mathbf{x}_i))\text{for }\{\mathbf{x}_i\}_1^L \in \Gamma_f \quad (11d)$$

When α_j and β_j are solved, the desired solution $u(x_1, x_2)$ is able to be obtained by Eqs. (7) and (8). Before closing this section, we give the partial derivatives of kernels required in Eqs. (3) and (9) explicitly in forms suitable for numerical implementations:

$$\frac{\partial U_1}{\partial x_i} = d_i[1 + 2\log(r)] \quad (12a)$$

$$\frac{\partial^2 U_1}{\partial x_i \partial x_j} = \delta_{ij}[1 + 2\log(r)] + \frac{2d_i d_j}{r^2} \quad (12b)$$

$$\frac{\partial^3 U_1}{\partial x_i \partial x_j \partial x_k} = \frac{2(\delta_{ij} d_k + \delta_{jk} d_i + \delta_{ki} d_j)}{r^2} - \frac{4d_i d_j d_k}{r^4} \quad (12c)$$

$$\frac{\partial U_2}{\partial x_i} = \frac{d_i}{r^2} \quad (12d)$$

$$\frac{\partial^2 U_2}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r^2} - \frac{2d_i d_j}{r^4} \quad (12e)$$

$$\frac{\partial^3 U_2}{\partial x_i \partial x_j \partial x_k} = -\frac{2(\delta_{ij} d_k + \delta_{jk} d_i + \delta_{ki} d_j)}{r^4} + \frac{8d_i d_j d_k}{r^6} \quad (12f)$$

where $d_i = x_i - s_i$ and $r = |\mathbf{x} - \mathbf{s}|$ with $\mathbf{s} = (s_1, s_2)$. Also, δ_{ij} is the Kronecker delta.

3 MFS formulation for simply connected vibrational plates

For free flexural vibration of a uniform thin plate, the governing equation and the boundary conditions are [7]:

$$\begin{cases} \nabla^2\nabla^2 u(x_1, x_2) - \lambda^4 u(x_1, x_2) = 0 \; in \; \Omega \\ \mathbf{K}_u u = 0 \; on \; \Gamma_c + \Gamma_s \\ \mathbf{K}_\theta u = 0 \; on \; \Gamma_c \\ \mathbf{K}_m u = 0 \; on \; \Gamma_s + \Gamma_f \\ \mathbf{K}_v u = 0 \; on \; \Gamma_f \end{cases} \quad (13)$$

In order to obtain the eigenvalues λ for which the above equations have non-trivial solution, $u(x_1, x_2)$ is represented by the MFS approximation [12]:

$$u(\mathbf{x}) \cong \sum_{j=1}^{M} [\alpha_j^E H_0(\lambda |\mathbf{x} - \mathbf{s}_j|) + \beta_j^E K_0(\lambda |\mathbf{x} - \mathbf{s}_j|)] \quad (14)$$

where $2 \times M$ is the number of sources, α_j^E and β_j^E are the unknown source intensities (as depicted in Figure 1), $H_n(\lambda r) = H_n^{(1)}(\lambda r)$ is the Hankel function of the first kind of order n and $K_n(\lambda r)$ is the modified Bessel function of the second kind of order n. Similarly, we define the following kernels:

$$U_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_u(H_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15a)$$

$$U_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_u(K_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15b)$$

$$\Theta_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_\theta(H_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15c)$$

$$\Theta_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_\theta(K_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15d)$$

$$M_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_m(H_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15e)$$

$$M_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_m(K_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15f)$$

$$V_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_v(H_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15g)$$

$$V_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_v(K_0(\lambda\,|\mathbf{x}-\mathbf{s}|)) \quad (15h)$$

where the partial derivatives required are also given in explicit forms suitable for numerical implementations as follows:

$$\frac{\partial U_{1,\lambda}}{\partial x_i} = -\frac{d_i \lambda H_1(\lambda r)}{r} \quad (16a)$$

$$\frac{\partial^2 U_{1,\lambda}}{\partial x_i \partial x_j} = -\frac{\delta_{ij}\lambda H_1(\lambda r)}{r} + \frac{d_i d_j \lambda^2 H_2(\lambda r)}{r^2} \quad (16b)$$

$$\frac{\partial^3 U_{1,\lambda}}{\partial x_i \partial x_j \partial x_k} = \frac{(\delta_{ij}d_k + \delta_{jk}d_i + \delta_{ki}d_j)\lambda^2 H_2(\lambda r)}{r^2} - \frac{d_i d_j d_k \lambda^3 H_3(\lambda r)}{r^3} \quad (16c)$$

$$\frac{\partial U_{2,\lambda}}{\partial x_i} = -\frac{d_i \lambda K_1(\lambda r)}{r} \quad (16d)$$

$$\frac{\partial^2 U_{2,\lambda}}{\partial x_i \partial x_j} = -\frac{\delta_{ij}\lambda K_1(\lambda r)}{r} + \frac{d_i d_j \lambda^2 K_2(\lambda r)}{r^2} \quad (16e)$$

$$\frac{\partial^3 U_{2,\lambda}}{\partial x_i \partial x_j \partial x_k} = \frac{(\delta_{ij}d_k + \delta_{jk}d_i + \delta_{ki}d_j)\lambda^2 K_2(\lambda r)}{r^2} - \frac{d_i d_j d_k \lambda^3 K_3(\lambda r)}{r^3} \quad (16f)$$

Then, Eq. (14) can be rewritten as

$$u(\mathbf{x}) \cong \sum_{j=1}^{M} [\alpha_j^E U_{1,\lambda}(\mathbf{x},\mathbf{s}_j) + \beta_j^E U_{2,\lambda}(\mathbf{x},\mathbf{s}_j)] \quad (17)$$

In order to obtain α_j^E and β_j^E, $2\times M$ boundary should conditions be collocated, i.e.:

$$\sum_{j=1}^{M} \{\alpha_j^E U_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j) + \beta_j^E U_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\} = 0 \text{ for } \{\mathbf{x}_i\}_1^M \in \Gamma_c + \Gamma_s \quad (18a)$$

$$\sum_{j=1}^{M}\{\alpha_j^E\Theta_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j)+\beta_j^E\Theta_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\}=0 \text{ for } \{\mathbf{x}_i\}_1^M\in\Gamma_c \quad (18b)$$

$$\sum_{j=1}^{M}\{\alpha_j^E M_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j)+\beta_j^E M_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\}=0 \text{ for } \{\mathbf{x}_i\}_1^M\in\Gamma_s+\Gamma_f \quad (18c)$$

$$\sum_{j=1}^{M}\{\alpha_j^E V_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j)+\beta_j^E V_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\}=0 \text{ for } \{\mathbf{x}_i\}_1^M\in\Gamma_f \quad (18d)$$

Equation (18) is an eigenproblem for λ and we are searching for eigenvalues $\lambda_1<\lambda_2<\lambda_3<\cdots$ such that Eq. (18) has nontrivial solutions, which is usually solved by the direct determinant search method [12].

It seems straightforward that the eigenfunctions can be obtained by substituting the corresponding eigenvectors to Eq. (17). However, it is found that the linear least squares MFS [15] should be applied to obtain the eigenfunctions stably. In other words, the following condition should be imposed in addition to the given boundary conditions in Eq. (18) by substituting the obtained eigenvalues λ to the kernels.

$$\mathbf{K}(u(\mathbf{x}))=1 \ at\ \mathbf{x}=\mathbf{a} \quad (19)$$

where $\mathbf{K}(\bullet)$ is one of the operators defined in Eq. (2), and $\mathbf{a}$ is a prescribed point. Here, $\mathbf{K}(\bullet)$ and $\mathbf{a}$ are so selected that they do not contradict with the boundary conditions in Eq. (18). Eqs. (18) and (19) compose an over-determined linear equations system and can then be solved by linear least squares method to obtain α_j^E and β_j^E. As a result, the desired eigenfunctions can be obtained by utilizing Eq. (17).

4 MFS formulation for multiply connected vibrational plates

For plate vibrations in multiply connected domains, the MFS formulation should be modified as [13]

$$u(\mathbf{x})=\sum_{j=1}^{M}[\alpha_j^E U_{1,\lambda}(\mathbf{x},\mathbf{s}_j^E)+\beta_j^E U_{2,\lambda}(\mathbf{x},\mathbf{s}_j^E)]+\sum_{j=1}^{N}[\alpha_j^I U_{3,\lambda}(\mathbf{x},\mathbf{s}_j^I)+\beta_j^I U_{4,\lambda}(\mathbf{x},\mathbf{s}_j^I)] \quad (20)$$

where $\{\alpha_j^E,\beta_j^E\}$ and $\{\alpha_j^I,\beta_j^I\}$ are the intensities of the sources at M exterior sources $\mathbf{s}_j^E$ and N interior sources $\mathbf{s}_j^I$, respectively. Figure 2 shows the geometry configuration of sources. The kernels are defined by Eq. (15a), Eq. (15b) and the follows:

$$U_{3,\lambda}(\mathbf{x},\mathbf{s})=U_{1,\lambda}(\mathbf{x},\mathbf{s})+i\frac{\partial U_{1,\lambda}(\mathbf{x},\mathbf{s})}{\partial n_{\mathbf{s}}} \quad (21a)$$

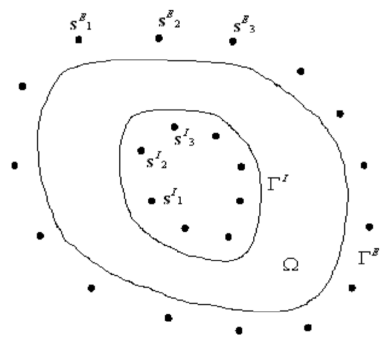

Figure 2: Geometry configuration for a multiply connected plate

$$U_{4,\lambda}(\mathbf{x},\mathbf{s}) = U_{2,\lambda}(\mathbf{x},\mathbf{s}) + i\frac{\partial U_{2,\lambda}(\mathbf{x},\mathbf{s})}{\partial n_{\mathbf{s}}} \quad (21b)$$

where $\frac{\partial}{\partial n_{\mathbf{s}}}$ is the normal derivative with respect to the outward normal vector, $\mathbf{n}_s = (\tilde{n}_1, \tilde{n}_2)$, associated to the fictitious curve of interior sources. The eigenvalues and eigenfunctions can also be obtained similarly by the direct determinant search method [13] and the linear least squares MFS [15], respectively. Before closing this section, we give the required partial derivatives of the kernels explicitly in forms suitable for numerical implementations:

$$\frac{\partial U_{3,\lambda}}{\partial x_i} = \frac{(-d_i + \mathbf{i}\tilde{n}_i)\lambda H_1(\lambda r)}{r} - \frac{\mathbf{i}d_i d_l \tilde{n}_l \lambda^2 H_2(\lambda r)}{r^2} \quad (22a)$$

$$\frac{\partial^2 U_{3,\lambda}}{\partial x_i \partial x_j} = -\frac{\delta_{ij}\lambda H_1(\lambda r)}{r} + \frac{(d_i d_j - \mathbf{i}\tilde{n}_i d_j - \mathbf{i}\tilde{n}_j d_i - \mathbf{i}\delta_{ij} d_l \tilde{n}_l)\lambda^2 H_2(\lambda r)}{r^2}$$
$$+\frac{\mathbf{i}d_i d_j d_l \tilde{n}_l \lambda^3 H_3(\lambda r)}{r^3} \quad (22b)$$

$$\frac{\partial^3 U_{3,\lambda}}{\partial x_i \partial x_j \partial x_k} = \frac{[\delta_{ij}(d_k - \mathbf{i}\tilde{n}_k) + \delta_{jk}(d_i - \mathbf{i}\tilde{n}_i) + \delta_{ki}(d_j - \mathbf{i}\tilde{n}_j)]\lambda^2 H_2(\lambda r)}{r^2}$$
$$-\frac{[d_i d_j d_k - \mathbf{i}d_l \tilde{n}_l(\delta_{ij}d_k + \delta_{jk}d_i + \delta_{ki}d_j) - \mathbf{i}(\tilde{n}_i d_j d_k + \tilde{n}_j d_i d_k + \tilde{n}_k d_i d_j)]\lambda^3 H_3(\lambda r)}{r^3}$$
$$-\frac{\mathbf{i}d_i d_j d_k d_l \tilde{n}_l \lambda^4 H_4(\lambda r)}{r^4} \quad (22c)$$

$$\frac{\partial U_{4,\lambda}}{\partial x_i} = \frac{(-d_i + \mathbf{i}\tilde{n}_i)\lambda K_1(\lambda r)}{r} - \frac{\mathbf{i}d_i d_l \tilde{n}_l \lambda^2 K_2(\lambda r)}{r^2} \quad (22d)$$

$$\frac{\partial^2 U_{4,\lambda}}{\partial x_i \partial x_j} = -\frac{\delta_{ij}\lambda K_1(\lambda r)}{r} + \frac{(d_i d_j - \mathbf{i}\tilde{n}_i d_j - \mathbf{i}\tilde{n}_j d_i - \mathbf{i}\delta_{ij} d_l \tilde{n}_l)\lambda^2 K_2(\lambda r)}{r^2} + \frac{\mathbf{i} d_i d_j d_l \tilde{n}_l \lambda^3 K_3(\lambda r)}{r^3} \quad (22e)$$

$$\frac{\partial^3 U_{4,\lambda}}{\partial x_i \partial x_j \partial x_k} = \frac{[\delta_{ij}(d_k - \mathbf{i}\tilde{n}_k) + \delta_{jk}(d_i - \mathbf{i}\tilde{n}_i) + \delta_{ki}(d_j - \mathbf{i}\tilde{n}_j)]\lambda^2 K_2(\lambda r)}{r^2}$$
$$-\frac{[d_i d_j d_k - \mathbf{i} d_l \tilde{n}_l(\delta_{ij} d_k + \delta_{jk} d_i + \delta_{ki} d_j) - \mathbf{i}(\tilde{n}_i d_j d_k + \tilde{n}_j d_i d_k + \tilde{n}_k d_i d_j)]\lambda^3 K_3(\lambda r)}{r^3}$$
$$-\frac{\mathbf{i} d_i d_j d_k d_l \tilde{n}_l \lambda^4 K_4(\lambda r)}{r^4} \quad (22f)$$

In the above equations, summations are applied with respect to the index l.

5 Main Results

The kernels required for numerical implementations of the method of fundamental solutions for Kirchhoff plate in equilibrium as well as simply and multiply connected vibrational plates are all given explicit. These kernels include the lateral displacement, slope, normal moment, and effective sheer force for all the three cases. They are addressed in forms suitable for numerical implementations. Also, the formulations of method of fundamental solutions for these three cases are briefly reviewed. And, the linear least squares method of fundamental solutions are supplied for obtaining eigenfunctions of vibrational plates.

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