The Effect of Internal Support Conditions to the Elastoplastic Transient Response of Reissner-Mindlin Plates

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Abstract: The method of Domain/Boundary Element is used to achieve a dynamic analysis of elastoplastic thick plates resting on internal supports. All possible boundary conditions on the edge of the plate with any interior support conditions such as isolated points (column), lines (walls) or regions (patches) can be treated without practical difficulties. The formulation presented includes the effects of shear deformation and rotatory inertia following Reissner-Mindlin's deformation theory assumptions. The method employs the elastostatic fundamental solution of the problem resulting in both boundary and domain integrals due to inertia, plasticity and interior support effect terms. By discretizing the integral equations and integrating the resulting matrix equation of motion by an explicit step-by-step time integration algorithm, the dynamic inelastic response of the plate can be obtained. Several complicated examples for impacted thick plates with different internal support conditions are presented to illustrate the efficiency of the method.

Keyword: Thick Plates, Domain/Boundary Element Method, Dynamic Analysis, Elastoplastic Plates, Internal Supports.

1 Introduction

It is recognized that in many engineering applications, particularly in civil engineering structures, there is a need for the ability to predict the response of internally supported plates. Internally supported plates such as columns point and/or load bearing walls with certain combination of classical boundary conditions are encountered much more frequently than others. Analytic solutions to this kind of plate problems are limited to simple plate geometries, boundary conditions and loadings. Both approximate (Narita (1986)) and numerical methods such as the Finite Difference Method (FDM) or the Finite Element Method (Hrabock and Hrudey (1983) and Utjes, Laura, Sanchez Sarmiento and Gelos (1986) have been used quite extensively to give a solution to this problem.

The Boundary Element Method (BEM) is being explored as a possible alternative to the FEM for solving problems of bending of plates which in addition to the boundary supports are also supported on internal supports. This method has been used by Bezine (1981) and Hartmann and Zotemantel (1986) to analyze the effect of internal supports in plate elastostatics. In these papers an integral representation for the deflection is obtained inside the domain which after discretization for line or surface supports yields additional collocation equations which are solved simultaneously with those involving unknowns defined on the boundary. Finally the formulation yields a system of equations involving unknowns defined inside the plate domain only.

During the last ten years or so, the BEM has been also successfully employed for the dynamic analysis of elastic and inelastic plates as it is evident, e.g. in the review articles of Beskos (1987, 1991, 1995, 1997), and Providakis and Beskos(1999). Zhang and Atluri (1986) presented a boundary/interior element method for the elastic quasi-static and transient response analysis of shallow shells by employing the elastostatic fundamental solution of plates. Providakis and Beskos (1994) and Providakis (1996) by using the elastostatic fundamental solution of a plate developed a general domain/boundary element method (D/BEM) for elastoplastic dynamic analysis of thin flexural plates. Fotiu (1992, 1993) and Fotiu,

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Irschik and Ziegler (1994) using the elastostatic or the elastodynamic fundamental Green's function in conjunction with modal analysis created a special BEM for dynamic analysis of certain thin elastoplastic and damaging plates. Recently, El –Zafrany (2001) presented a boundary element stress analysis of thick Reissner plates under generalized loading. Recently, Rashed (2006), Chen, Lin, Chen and Lee (2006) and Chen, Shen, Shen and Yuan (2005) attempted to analytically evaluate the required BEM Kernels for forced and free plate vibration problems under various mathematical analysis procedures.

Katsikadelis, Sapountzakis and Zorba (1988) were the first who proposed the use of the direct boundary element method to the dynamic analysis of thin elastic plates with internal supports. Their approach was mainly based on the capability to establish numerically the Green's function for the corresponding static problem of the plate subjected to the given boundary conditions without supports using the BEM. In earlier works of the present author (see Providakis (1998, 2000)) a D/BEM solution to the dynamic analysis problem of thin elastoplastic Kirchoff plates was presented, which besides, the boundary supports, takes into account supports within the domain of the plate. A further extension of this D/BEM solution was presented in Providakis(2000) to solve dynamic response problems of thick plates resting on elastic foundation. This work was limited to the study of the linear elastic interaction between Reissner plates and supporting soil medium by introducing the elastic medium in the boundary integral derivations as a uniformly distributed loading term. An efficient BEM method to analyze the domain integrals which introduced when solving nonlinear problems or problems with initial domain effects was also proposed by Ochiai and Sladek (2004). In this work, the domain discretization was completely eliminated by using arbitrary internal points instead of internal cells in combination with a conversion procedure of the domain integrals to boundary ones. More recently in the works of Moraru (2006), Pavlou (2004) and Mendonca and Paiva (2003) presented numerical approaches for the analysis of plates on elastic

foundations using BEM discontinuous solutions or combined FEM/BEM modelling of the static behaviour of elastic plates.

However, deficiencies of the classical thin plate theory of Kirchoff are well known and in many cases cannot be accepted. Among the numerous attempts to improve the classical plate theory, the method proposed by Reissner (1945) and later by Mindlin (1951) has become the standard compared to all other theories. In the so called Reissner-Mindlin plate theory, by taking into account the transverse shear deformation, the influence of the thickness of the plate can be analyzed more consistent. Elastic Reissner-Mindlin plates have been dynamically analyzed by the direct BEM in the frequency domain, the D/BEM in the frequency domain and by special BEMs such as the Green's function BEM and the boundary collocation method. For a review on the subject one can consult Antes (1991) and Providakis and Beskos (1994) or the earlier works of Katsikadelis et al (1990, 1993).

A direct D/BEM approach based on Reissner Mindlin theory is presented, for the first time, in the present paper to treat the dynamic response of thick elastoplastic plates which are supported on points, lines or regions (patches) within the domain of the plate, besides the boundary supports. It can be considered as an extension of the work of Providakis and Beskos(1994) in order to include internal supports which may yield elastically linearly or nonlinearly. Using the simpler form of the elastostatic fundamental solution of the problem, the computational difficulties of the formulation have been reduced adequately. This approach, even though requires an internal discretization, in addition to the boundary one, has certain advantages over a possible pure boundary element method based on a complex fundamental solution. The numerical procedure is then accomplished by an incremental and iterative algorithm based on the initial plastic moment procedure of Karam and Telles (1988, 1992) and in the time-marching scheme of Sori'c, Li, Jarak and Atluri(2004) and Fedelinski and Gorski (2006). The Praudl-Reuss stress strain law based on Von Mises' yield condition have been imple-

mented into the developed computer code in order to model the hardening elastoplastic material behavior. The descritized version of the equation of motion after using the boundary conditions are solved by the step-by-step time integration algorithm of the central predictor method. A number of numerical results are presented to illustrate the effectiveness and the applicability of the proposed method.

2 Integral formulation of the problem

Consider a homogeneous isotropic elastoplastic plate of uniform thickness h occupying a two dimensional domain S bounded by a boundary Γ and undergoing a lateral motion response. The plate in addition to the boundary supports, is also supported on point supports p_i, lines supports l_i or regions r_i (patches) inside the domain of the plate (Fig. 1).
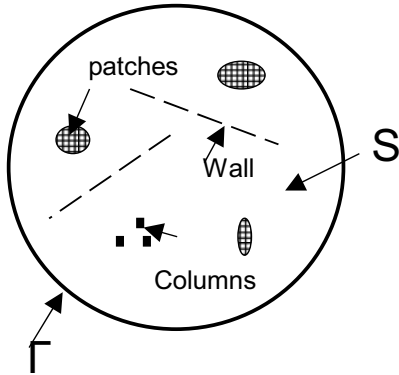

Figure 1: Geometry of the plate

The plane $x - y$ is assumed to coincide with the middle surface of the plate. Following Reissner-Mindlin's plate theory, the equations of dynamic equilibrium of an elastic plate in lateral motion can be reformulated in incremental form to include bending plastic strain increments as

$$\frac{\partial \delta M_x}{\partial x} + \frac{\partial \delta M_{xy}}{\partial y} - \delta Q_x - \frac{\rho h^3}{12} \delta \ddot{\varphi}_x = 0$$

$$\frac{\partial \delta M_{xy}}{\partial x} + \frac{\partial \delta M_y}{\partial y} - \delta Q_y - \frac{\rho h^3}{12} \delta \ddot{\varphi}_y = 0 \qquad (1)$$

$$\frac{\partial \delta Q_x}{\partial x} + \frac{\partial \delta Q_y}{\partial y} + \delta \overline{q}^* - \rho h \delta \ddot{w} = 0$$

where ρ, h and $\overline{q}^*$ are the mass density per unit area, the plate thickness and the transient dynamic loading per unit area, respectively. In addition, $\delta \ddot{\varphi}_x$, $\delta \ddot{\varphi}_y$ and $\delta \ddot{w}$ indicate increments of the accelerations of the two slopes ϕ_χ, ϕ_y and of the lateral deflection w, respectively, δM_x, δM_y and δM_{xy} represent increments of the bending and twisting moments, δQ_x and δQ_y represent increments of the shear forces and overdots denote time differentiation.

In the case of a plate resting on internal supports, the lateral load $\overline{q}^*$ is given by:

For a support at a point p_i:

$$\delta \overline{q}^* = -f [\delta w(\xi_i)] + \delta \overline{q} \quad \xi_i \in S \qquad (2)$$

For a support on a line l_i

$$\delta \overline{q}^* = -f [\delta w(\xi)] + \delta \overline{q} \quad \xi \in l_i \subset S \qquad (3)$$

For a support on a region (patch) r_i :

$$\delta \overline{q}^* = -f [\delta w(\xi)] + \delta \overline{q} \quad \xi \in r_i \subseteq S \qquad (4)$$

where $f = f(w)$ is, in general, a nonlinear function, describing the reacting forces at, say, interior point ξ and $\overline{q}$ is the dynamic lateral load applied on the plate. Consequently, the equations of dynamic equilibrium (1) in theirs incremental form is given by

$$\frac{\partial \delta M_x}{\partial x} + \frac{\partial \delta M_{xy}}{\partial y} - \delta Q_x - \frac{\rho h^3}{12} \delta \ddot{\varphi}_x = 0$$

$$\frac{\partial \delta M_{xy}}{\partial x} + \frac{\partial \delta M_y}{\partial y} - \delta Q_y - \frac{\rho h^3}{12} \delta \ddot{\varphi}_y = 0 \qquad (5)$$

$$\frac{\partial \delta Q_x}{\partial x} + \frac{\partial \delta Q_y}{\partial y} + \delta q - p[\delta w] - \rho h \delta \ddot{w} = 0$$

The increments of the total bending and shear strains can be given as

$$\delta \varepsilon_x = \delta \varepsilon_x^e + \delta \varepsilon_x^p \qquad \delta \psi_x = \delta \psi_x^e$$
$$\delta \varepsilon_y = \delta \varepsilon_y^e + \delta \varepsilon_y^p \qquad \delta \psi_y = \delta \psi_y^e \qquad (6)$$
$$\delta \varepsilon_{xy} = \delta \varepsilon_{xy}^e + \delta \varepsilon_{xy}^p$$

where ε_χ, ε_y and ε_{xy} represent bending strains, ψ_x and ψ_y represent shear strains and the superscripts e and p indicate the elastic and plastic part of the strains, respectively. Within the small strain theory the increments of the total bending and shear

strains can also be expressed in terms of the increments of the generalized displacements as

$$\delta\varepsilon_x = \frac{\partial\delta\varphi_x}{\partial x} \qquad \delta\psi_x = \delta\varphi_x + \frac{\partial\delta w}{\partial x}$$

$$\delta\varepsilon_y = \frac{\partial\delta\varphi_y}{\partial y} \qquad \delta\psi_y = \delta\varphi_y + \frac{\partial\delta w}{\partial y} \tag{7}$$

$$\delta\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial\delta\varphi_x}{\partial y} + \frac{\partial\delta\varphi_y}{\partial x}\right)$$

Following the initial plastic moment procedure of Karam and Telles (1988, 1992), the increments of the bending moments and shear forces can be expressed as

$$\delta M_x = D\left[\frac{\partial\delta\varphi_x}{\partial x} + v\frac{\partial\delta\varphi_y}{\partial y}\right] + \frac{v\delta q}{(1-v^2)\lambda^2} - \delta M_x^p$$

$$\delta Q_x = \frac{D(1-v)}{2}\lambda^2\left(\delta\varphi_x + \frac{\partial\delta w}{\partial x}\right)$$

$$\delta M_y = D\left[\frac{\partial\delta\varphi_y}{\partial y} + v\frac{\partial\delta\varphi_x}{\partial x}\right] + \frac{v\delta q}{(1-v^2)\lambda^2} - \delta M_y$$

$$\delta Q_y = \frac{D(1-v)}{2}\lambda^2\left(\delta\varphi_y + \frac{\partial\delta w}{\partial y}\right)$$

$$\delta M_{xy} = \frac{D(1-v)}{2}\left[\frac{\partial\delta\varphi_x}{\partial y} + \frac{\partial\delta\varphi_y}{\partial x}\right] - \delta M_{xy}^p$$

$$\tag{8}$$

where δM_x^p, δM_y^p and δM_{xy}^p are the increments of the plastic moments which can be defined in initial stress form by expressions derived in Providakis (2000).

In the above, $D = Eh^3/12(1-v^2)$ is the plate flexural rigidity with E and v being the elastic modulus and Poisson's ratio, respectively and $\lambda^2 = 10/h^2$ is the shear correction factor of Reissner's theory. The shear correction factor κ^2 of Mindlin's theory is usually taken as 5 / 6 in order for the two theories to coincide provided that $\lambda^2 = 12\kappa^2/h^2$. Substituting Equations (8) into (5) one can obtain the following incremental equations of motion in terms of the increments of the generalized displacements and plastic moments

$$\frac{D}{2}\left[(1-v)\nabla^2\delta\varphi_x + (1+v)\frac{\partial}{\partial x}\left(\frac{\partial\delta\varphi_x}{\partial x} + \frac{\partial\delta\varphi_y}{\partial y}\right)\right]$$

$$-\frac{D(1-v)}{2}\lambda^2\left(\delta\varphi_x - \frac{\partial\delta w}{\partial x}\right)$$

$$= \frac{\rho h^3}{12}\delta\ddot{\varphi}_x + \frac{v}{(1-v)\lambda^2}\frac{\partial\delta q}{\partial x} - \frac{\partial M_x^p}{\partial x} - \frac{\partial M_{xy}^p}{\partial y}$$

$$\frac{D}{2}\left[(1-v)\nabla^2\delta\varphi_y + (1+v)\frac{\partial}{\partial y}\left(\frac{\partial\delta\varphi_x}{\partial x} + \frac{\partial\delta\varphi_y}{\partial y}\right)\right]$$

$$-\frac{D(1-v)}{2}\lambda^2(\delta\varphi_y - \frac{\partial\delta w}{\partial y})$$

$$= \frac{\rho h^3}{12}\delta\ddot{\varphi}_y + \frac{v}{(1-v)\lambda^2}\frac{\partial\delta q}{\partial y} - \frac{\partial M_{xy}^p}{\partial x} - \frac{\partial M_y^p}{\partial y}$$

$$\frac{D(1-v)}{2}\lambda^2\left(\nabla^2\delta w - \frac{\partial\delta\varphi_x}{\partial x} - \frac{\partial\delta\varphi_y}{\partial y}\right)$$

$$= \rho h\delta\ddot{w} - \delta q + f[\delta w]$$

$$\tag{9}$$

The static-like form of Equations (**??**) with the inertial terms being in their right hand sides, suggests using known integral identities as described in Karam and Telles (1988, 1992) for the elasto-plasto-static Reissner plate problem. By using the elastostatic fundamental solution of the problem and by replacing the reactive forces on the internal supports by the load applied at each node of the mesh used to descritize the internal plate domain, one can obtain for a point $\underset{\sim}{\xi}$ inside the domain S of the plate the integral equations

$$C_{ij}\left(\underset{\sim}{\xi}\right)\delta u_j\left(\underset{\sim}{\xi}\right) =$$

$$\int_{\Gamma}\left\{u_{ij}^*\left(\underset{\sim}{\xi},\underset{\sim}{X}\right)\delta p_j\left(\underset{\sim}{X}\right) - p_{ij}^*\left(\underset{\sim}{\xi},\underset{\sim}{X}\right)\delta u_j\left(\underset{\sim}{X}\right)\right\}$$

$$\cdot d\Gamma\left(\underset{\sim}{X}\right)$$

$$+\int_S\left[u_{i3}^*\left(\underset{\sim}{\xi},\underset{\sim}{x}\right) - \frac{v}{(1-v)\lambda^2}u_{i\alpha,\alpha}^*\right]$$

$$\cdot\delta q\left(\underset{\sim}{x}\right)dS\left(\underset{\sim}{x}\right)$$

$$+\frac{1}{12}\rho h^3\int_S u_{i\alpha}^*\left(\underset{\sim}{\xi},\underset{\sim}{x}\right)\delta\ddot{u}_\alpha\left(\underset{\sim}{x}\right)dS\left(\underset{\sim}{x}\right)$$

$$-\rho h\int_S u_{i3}^*\delta\ddot{u}_3\left(\underset{\sim}{x}\right)dS\left(\underset{\sim}{x}\right)$$

$$+\int_S E_{\alpha\beta i}^*\left(\underset{\sim}{\xi},\underset{\sim}{x}\right)\delta M_{\alpha\beta}^p dS\left(\underset{\sim}{x}\right)$$

$$-\sum_i\int_{S_{r_i}} u_{i3}^*\left(\underset{\sim}{\xi},\underset{\sim}{x}\right)f\left[\delta u_3\left(\underset{\sim}{x}\right)\right]dS_r\left(\underset{\sim}{x}\right)$$

$$-\sum_i \int_{l_i} u_{i3}^* \left(\underset{\sim}{\xi}, \underset{\sim}{x}\right) f\left[\delta u_3 \left(\underset{\sim}{x}\right)\right] dl\left(\underset{\sim}{x}\right)$$

$$-\sum_i u_{i3}^* \left(\underset{\sim}{\xi}, \underset{\sim}{x_i}\right) f\left[\delta u_3 \left(\underset{\sim}{x_i}\right)\right] \tag{10}$$

where $i, j = 1, 2, 3$, $\alpha, \beta = 1, 2$ and $\underline{X}$ and $\underset{\sim}{x}$ represents the field point at the boundary and in the interior of the plate, respectively. The matrix C_{ij}, in general, depends upon the geometry of the boundary at point $\underset{\sim}{\xi}$ and in the case of corners is defined indirectly, by using the rigid body motion concept, as it is elaborated in the following. However, it should be observed that equation (10) can be made valid for internal points (with $C_{ij} = \delta ij$) and for boundary points with C_{ij} having the value $\delta_{ij}/2$ in the case of smooth boundaries. The tensors u_{ij}^*, p_{ij}^* and $E_{\alpha\beta i}^*$ represent the fundamental solution at the field point $\underset{\sim}{x}$ of an infinite plate when a unit couple (for i=1 and 2) or a unit force (for i=3) is applied at the source point $\underset{\sim}{\xi}$. Thus the generalized displacements, the corresponding surface tractions, the expressions for $u_{ia,a}^*$ and the one for $E_{\alpha\beta i}^*$ are given explicitly in Providakis and Beskos (1994).

3 Matrix formulation and numerical implementation

The integral equations (10) can be expressed in discrete form by dividing the boundary Γ and line supports li into a number of three noded boundary elements. The interior of the plate domain Ω and the domains of the internal surface supports (patches) can be divided into a number eight noded quadrilateral interior elements, respectively. The discretization of boundary integrals and internal support line integrals is accomplished by expressing the coordinates, the generalized displacements, the tractions and internal support reaction forces of an arbitrary point within a boundary element Γ_b and internal line support as

$$\tilde{X}_j = N^a(\zeta)\tilde{X}_j^a$$
$$\delta u_j = N^a(\zeta)\delta U_j^a$$
$$\delta p_j = N^a(\zeta)\delta P_j^a \tag{11}$$
$$\delta f_j = N^a(\zeta)\delta F_j^a$$

where $N^a(\zeta)$ is a set of polynomial shape functions defined on boundary element element Γ_b and line element l_i, ζ is an intrinsic coordinate on Γ_b which varies between -1 and +1 and the superscript α is summed from 1 to $\overline{n}$, which is the number of nodes on Γ_b while $\tilde{X}_j^a$, δU_j^a, δP_j^a, and δF_j^a are vectors containing the nodal values of coordinates, generalized displacement increments, boundary traction increments and nodal values of line supports reactions increments, respectively.

For the discretization of the inertial, transverse loading, plastic moments and internal supports surface integrals, it is assumed that the coordinates of an arbitrary interior point $\underset{\sim}{\xi}$ within any interior element can be calculated by the equation

$$\tilde{x}_j = N^b(\zeta_1, \zeta_2)\tilde{x}_j^b \tag{12}$$

where $\tilde{x}_j$ is the vector that contains the Cartesian coordinates of an arbitrary interior point $\underset{\sim}{\xi}$ within an element S_n, and S_{ri}, $N^b(\zeta_1, \zeta_2)$ is a set of polynomial shape functions and $\tilde{X}_j^b$ is the vector of the Cartesian coordinates related to the nodal point of the elements S_n and S_{ri}, ζ_1 and ζ_2 are intrinsic coordinates on any interior element S_n and S_{ri} and the superscript b is summed from 1 to $\overline{m}$, which is the number of nodes on element S_n and S_{ri}. The transverse loading, the inertial, the plastic moments and the internal surface supports effect terms at an arbitrary point within the element S_n and S_{ri} can be expressed by using the equations

$$\delta q = N^b(\zeta_1, \zeta_2)\delta Q^b$$
$$\delta \ddot{U}_j = N^b(\zeta_1, \zeta_2)\delta U_j^b$$
$$\delta M_{\alpha\beta}^p = N^b(\zeta_1, \zeta_2)\delta M_{\alpha\beta}^{p\ b} \tag{13}$$
$$\delta U_3 = N^b(\zeta_1, \zeta_2)\delta U_3^b$$
$$\delta f_j = N^b(\zeta_1, \zeta_2)\delta F_j^b$$

where δq, $\delta \ddot{u}_j$, $\delta M_{\alpha\beta}^p$ and δF_j^b represent the vectors of the increments of the transverse loading, the acceleration, the plastic moment and the surface supports reaction forces terms, respectively at an arbitrary point $\underset{\sim}{\xi}$ inside the interior element S_n. and S_{ri} Thus, integral equations (10) utilizing the function expansions (11)-(13) can be written

in discretized form as

$$C_{ij}\delta U_j(\xi) =$$

$$\sum_{m=1}^{M}\left(\int_{\Gamma_m} N^a(\zeta)U_{ij}^*\left(\xi,\tilde{X}(\zeta)\right)d\Gamma(\tilde{X})\right)\delta P_j^a$$

$$-\sum_{m=1}^{M}\left(\int_{\Gamma_m} N^a(\zeta)P_{ij}^*\left(\xi,\tilde{X}(\zeta)\right)d\Gamma(\tilde{X})\right)\delta U_j^a$$

$$+\sum_{n=1}^{N}\left(\int_{S_n} N^b(\zeta_1,\zeta_2)S_{ijk}^*(\xi,\tilde{x}(\zeta_1,\zeta_2))d\Omega(\tilde{x})\right)\delta Q_j^b$$

$$-\sum_{n=1}^{N}\left(\int_{S_n} N^b(\zeta_1,\zeta_2)M_{ij}^*(\xi,\tilde{x}(\zeta_1,\zeta_2))d\Gamma(\tilde{x})\right)\delta \ddot{U}_j^b$$

$$+\sum_{n=1}^{N}\left(\int_{S_n} N^b(\zeta_1,\zeta_2)E_{\alpha\beta i}^*(\xi,\tilde{x}(\zeta_1,\zeta_2))d\Omega(\tilde{x})\right)$$
$$\cdot\delta M_{\alpha\beta}^{p\ b}$$

$$-\sum_{r=1}^{N_r}\left(\int_{S_r} N^b(\zeta_1,\zeta_2)U_{i3}^*(\xi,\tilde{x}(\zeta_1,\zeta_2))dS_r(\tilde{x})\right)\delta F_j^b$$

$$-\sum_{l=1}^{N_l}\left(\int_{\Gamma_l} N^a(\zeta)U_{i3}^*(\xi,\tilde{x}(\zeta))d\Gamma_l(\tilde{x})\right)\delta F_j^a$$

$$-\sum_{k=1}^{K}U_{i3}^*(\xi_k,\tilde{x}_k)\delta F_k$$

$$(14)$$

where M is the number of boundary elements, N is the number of interior elements, M_l is the number of the interior line support elements, K is the number of interior point supports, N_r is the number of interior support surface elements, Γ_m is the mth boundary element ($\Gamma = \sum_{m=1}^{m}\Gamma_m$), S_n is the nth surface element ($S = \sum_{n=1}^{N}S_n$)) and U_{ij}^*, P_{ij}^*, S_{ijk}^*, M_{ij}^*, K_{ij}^* and $E_{\alpha\beta}^*$ are the corresponding tensors of the boundary integrals of equations (14). By applying a boundary nodal point collocation procedure to equation (14) one can obtain for any point ξ on the boundary the following system of equations in matrix form

$$[C]\delta\{U\}_b = [P^*]\delta\{U\}_b + [U^*]\delta\{P\}_b + [S^*]\delta\{Q\}_i$$
$$+[M^*]\delta\{\ddot{U}\}_i + [E^*]\delta\{M^p\}_i + [F^*]\delta\{f(\delta U_3)_i\}$$
$$(15)$$

where $\delta\{U\}_b$, $\delta\{P\}_b$, are vectors of the increments of the nodal boundary values of the generalized displacements and tractions, $\delta\{Q\}_i$, $\delta\{\ddot{U}\}_i$, $\delta\{M\}_i^p$, and $\delta\{f(\delta(U_3))\}_i$ are vectors of the increments of the interior domain nodal

values, respectively. $[P^*]$ and $[U^*]$ are boundary element integral matrices, while $[S^*]$, $[M^*]$, $[E^*]$ and $[F^*]$ are domain element integral matrices related to the transverse loading, inertial, plastic moment and internal support conditions effect terms. By using the boundary conditions and eliminating and rearranging one can obtain from (15) the matrix equation

$$[A^*]\delta\{Y\}_b + [B^*]\delta\{J\}_b = [S^*]\delta\{Q\}_i + [M^*]\delta\{\ddot{U}\}_i$$
$$+[E^*]\delta\{M^p\}_i + [F^*]\delta\{f(\delta U_3)\}_i \quad (16)$$

where $\delta\{\Upsilon\}_b$ and $\delta\{J\}_b$ are vectors of the known and unknown increments of the nodal boundary values, respectively, and $[A^*]$ and $[B^*]$ are boundary element integral matrices. For the interior of the domain S the matrix form of the integral equations (14) after discretization and using the boundary conditions reads

$$\delta\begin{Bmatrix}U_s\\U_i\end{Bmatrix} = [\underline{A}^*]\delta\{Y\}_b + [\underline{B}^*]\delta\{J\}_b + [\underline{M}^*]\delta\{\ddot{U}\}_i$$
$$+[\underline{S}^*]\delta\{Q\}_i + [\underline{E}^*]\delta\{M^p\}_i + [\underline{F}^*]\delta\{f(\delta(U_3))\}_i$$
$$(17)$$

where the subscripts s and i indicate supports and inertia nodes, $\delta\{\underline{U}\}$ is the vector of unknown increments of nodal generalized interior displacements and the matrices $[\underline{A}^*]$, $[\underline{B}^*]$, $[\underline{M}^*]$, $[\underline{S}^*]$, $[\underline{E}^*]$ and $[\underline{F}^*]$ are the same as in equation (17) but they are evaluated at a point $\underset{\sim}{\xi}$ inside the domain S of the plate.

In case of rigid supports equation (17) yields to

$$\delta\begin{Bmatrix}0\\U_i\end{Bmatrix} = [\underline{A}^*]\delta\{Y\}_b + [\underline{B}^*]\delta\{J\}_b + [\underline{M}^*]\delta\{\ddot{U}\}_i$$
$$+[\underline{S}^*]\delta\{Q\}_i + [\underline{E}^*]\delta\{M^p\}_i + [\underline{F}^*]\delta\{F_s\} \quad (18)$$

where $\{F_s\}$ is the vector of the unknown reaction forces at the support nodes.

After elimination of the vector $\delta\{F_s\}$ between equations (18) one can obtain the matrix equation

$$\delta\{U_i\} = [\tilde{A}^*]\delta\{Y\}_b + [\tilde{B}^*]\delta\{J\}_b + [\tilde{S}^*]\delta\{Q\}_i$$
$$+[\tilde{M}^*]\delta\{\ddot{U}\}_i + [\tilde{E}^*]\delta\{M^p\}_i \quad (19)$$

where

$$[\tilde{A}^*] = [\underline{A}^*] + [\underline{E}^*][F^*]^{-1}[A^*]$$
$$[\tilde{M}^*] = [\underline{M}^*] + [\underline{E}^*][F^*]^{-1}[M^*]$$
$$[\tilde{S}^*] = [\underline{S}^*] + [\underline{E}^*][F^*]^{-1}[S^*] \quad (20)$$
$$[\tilde{E}^*] = [\underline{E}^*] + [\underline{E}^*][F^*]^{-1}[E^*]$$

The elimination of the matrix {J} between (19) and (20) leads to the final matrix equation

$$[A^*]'\delta\{Y\} + [M^*]'\delta\{\ddot{U}\}$$
$$= [S^*]'\delta\{Q\} + [E^*]'\delta\{M^p\} \quad (21)$$

where

$$[\tilde{A}^*]' = [\tilde{A}^*] + [\tilde{B}^*][\tilde{B}^*]^{-1}[A^*]$$
$$[\tilde{M}^*]' = [M^*] + [\tilde{B}^*][\tilde{B}^*]^{-1}[M^*] \quad (22)$$
$$[\tilde{S}^*]' = [S^*] + [\tilde{B}^*][\tilde{B}^*]^{-1}[S^*]$$

The evaluation of the coefficients of the matrices in equations (14)-(21) needs a number of complicated integration procedures. Since analytical integration of the integrals in these equations is not possible, in general, the Gaussian quadrature technique was used. For the singular cases, which occur when the field and the source point are situated over the same element, special approaches were employed.

In the case of the displacement kernel matrix $[U^*]$ the singularity is of $O(lnr)$ which is a weak singularity. This logarithmic singularity is removed by using a quadratic coordinate transformation which produces a Jacobian that eliminates this kind of singularity at the considered point. The traction kernel matrix $[P^*]$ presents $O(r^{-1})$ and $O(lnr)$ singularities in its components. The strong $O(r^{-1})$ singularity, together with the corresponding C_{ij} coefficient in equation (14) can be computed indirectly by considering three rigid body movements of the type (Van der Weeen (1982)).

(i) $u_1 = 1;$ $u_2 = 0;$ $u_3 = x_1(\xi) - x_1(x)$

(ii) $u_1 = 0;$ $u_2 = 1;$ $u_3 = x_2(\xi) - x_2(x)$

(iii) $u_1 = 0;$ $u_2 = 0;$ $u_3 = 1$

$$\quad (23)$$

This allows the diagonal (singular) block to be written in terms of the off-diagonal (non-singular) blocks.

For the case of influence matrices associated with interior elements, the integrals are also evaluated numerically. The singular surface integrals for matrices $[S^*]$' and $[E^*]$' exhibit $O(lnr)$ and $O(r^{-1})$ singularities, while those for matrices $[M^*]$' exhibit O(lnr) singularity. These singularities are removed by the use of 8 noded quadrilateral interior elements which have a Jacobian which smooths out the singular behaviour of the kernel-shape function products. For improved accuracy the singular elements are divided into triangular subelements. The common apex of all subelements is the field point $\underset{\sim}{\xi}$. Each triangular subelement is then mapped on to a flat right triangle and numerical integration is performed using a polar coordinate system (r, θ) centered at the singular apex as presented in Providakis and Beskos (1994) and Providakis (1996).

Finally, to derive the quasi-singular boundary integrals which exist in the evaluation of the moments and shear forces at internal points, in cases when these points are located very near the boundary, the same quadratic coordinate transformation mentioned above is used.

4   Elastic-plastic stress-strain relations

The Prandtl-Reuss stress-strain relations based on the Von Mises's yield condition are used to model elastoplastic material behavior. The generalized Hooke's law can be written in matrix form as

$$\delta\{\varepsilon\}^e = [D]^{e^{-1}}\delta\{\sigma\} = \delta\{\varepsilon\} - \delta\{\varepsilon\}^p, \quad (24)$$

where $[D]^e$ is the elasticity matrix and $\{\varepsilon\}$ and $\{\sigma\}$ are strain and stress vectors with the superscripts e and p denoting elastic and plastic parts, respectively. The plastic strain increment $\delta\{\varepsilon\}^p$ can be given in matrix form by the relation

$$\delta\{\varepsilon\}^p = [D]^*\delta\{\varepsilon\} \quad (25)$$

where $[D]^* = [I] - [D]^{e^{-1}}[D]^{ep}$ with $[I]$ being the identity matrix and $[D]^{ep}$ the elastoplastic matrix

having the form

$$[D]^{ep} = [D]^e - [D]^e \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D]^e$$

$$\cdot \left(H + \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D]^e \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \right)^{-1} \quad (26)$$

In the above, F is the yield surface and H is the plastic modulus, which is zero or nonzero for ideal or hardening plasticity, respectively. Matrix $[D]^{ep}$ connects the plastic stress increments with the total strain increments through the relation

$$\delta\{\sigma\}^p = [D]^{ep} \delta\{\varepsilon\} \quad (27)$$

The Von Mises' yield surface for the present case of a Reissner-Mindlin plate is given by $F = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} - \overline{\sigma}$, where $\overline{\sigma}$ is the uniaxial effective stress.

5   Solution strategy

The values of the nodal generalized displacements $\{U\}$ at every time station are obtained by integrating forward in time equation (21) through an explicit central difference predictor algorithm. The initial distribution of generalized displacements, velocities and accelerations are prescribed and set to zero. The generalized displacements can then be determined at the end of the first time step. These are now used to evaluate the partial derivatives of the displacements increments, following the FEM-type procedure described in Providakis and Beskos (1994). From these computed partial derivatives the incremental strain can be obtained along the lateral axis through the well known strain-displacement relation of the plate deformation. The increments of the stresses are obtained from the strain increments and the incremental plastic moments calculations follow in terms of the plastic strain increments after an appropriate checking at yielding. Thus the total and incremental generalized displacements are then found at time Δt and so on, and the time histories of all the variables are obtained.

6   Numerical examples

To illustrate the accuracy of the proposed method a computer program based on the analysis pre-

sented in the previous sections has been written. Three numerical examples of elastoplastic plates with different boundary and interior conditions subjected to impulsive load have been studied (Fig. 2-4).
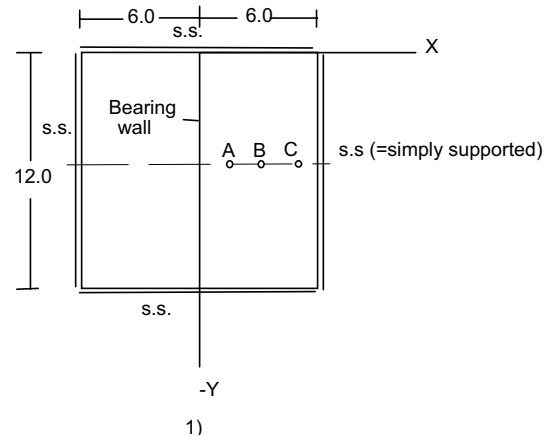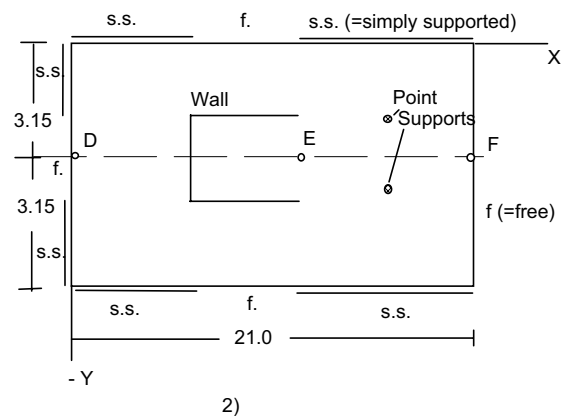

Figure 2: Geometry of example 1

Figure 3: Geometry of example 2

Example 1

Consider a square simply supported plate resting on a line support along the mid-span and subjected to a uniformly distributed suddenly applied load (Fig. 2). Figure 5 shows the dynamic elastoplastic response of the points A, B and C of the plate. Figure 6 depicts the bending moment contours as computed by the present computer program for two different time steps.
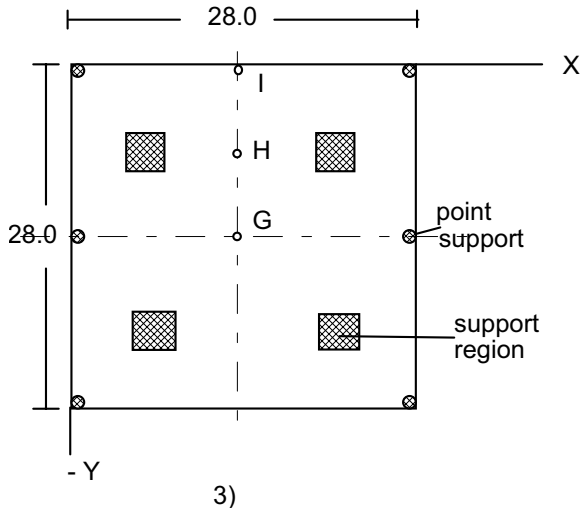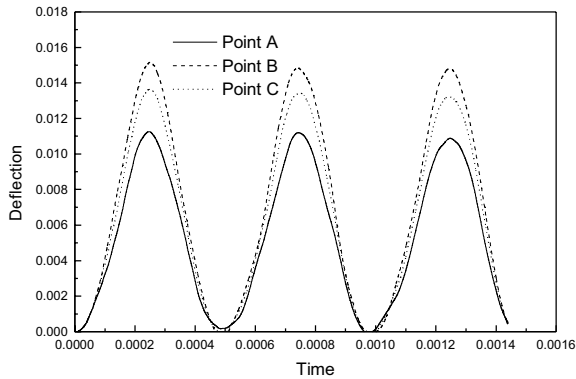
Figure 4: Geometry of example 3

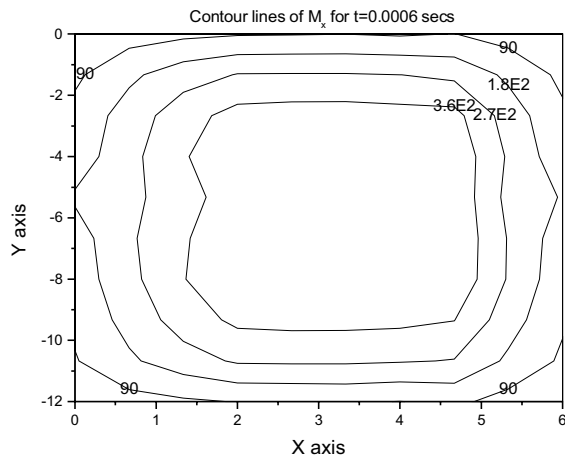Figure 5: Dynamic elastoplastic response of the plate

Figure 6: M_x contours for time step t=0.0006 secs

Example 2

As for the second example, a rectangular plate with mixed boundary conditions and complicated internal supports is considered which is subjected to a suddenly applied uniform load(Fig. 3). In Figure 7 elastoplastic time variation of the deflection at the points D, E and F of the plate is shown. Figure 8 show the bending moment contours, respectively, of the half of the plate.
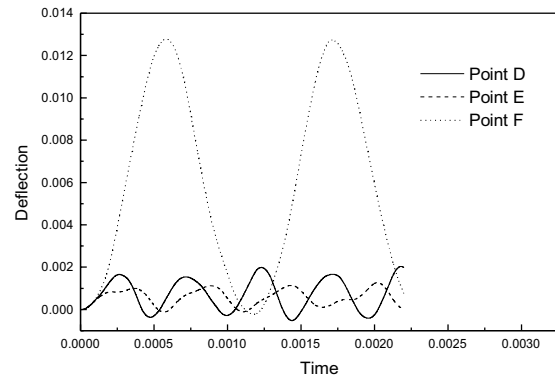

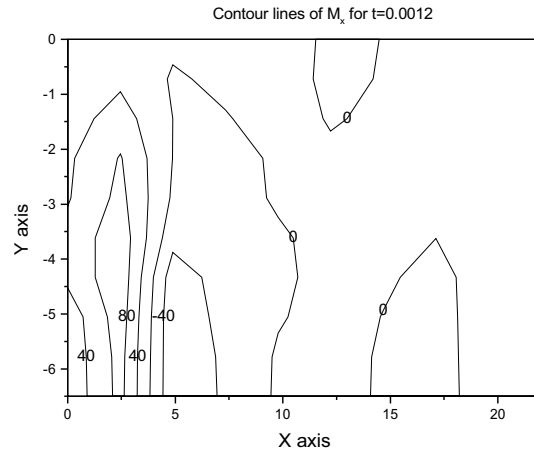Figure 7: Dynamic elastoplastic response of the plate

Figure 8: M_x contours for time step t=0.0012 secs

Example 3

In this example, a square plate with mixed boundary conditions and supported on four symmetrically located interior square regions (patches) has

been considered. The patches deform elastically with a rigidity equals to 0.1. The sides of the interior regions has been taken equal to the 7th part of the whole plate side (Fig. 4). The computed responses of the points G, H and I are depicted in Figure 9. The time dependent bending moment contours for the quarter of the plate has also been given in Figure 10.
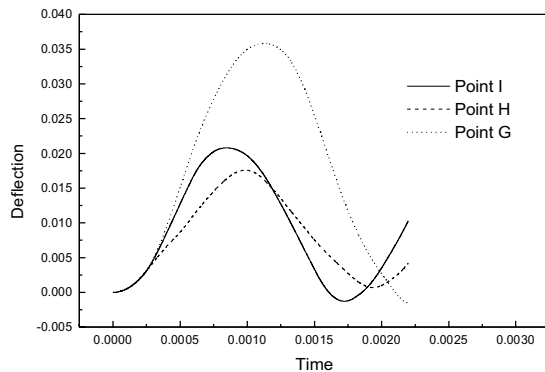

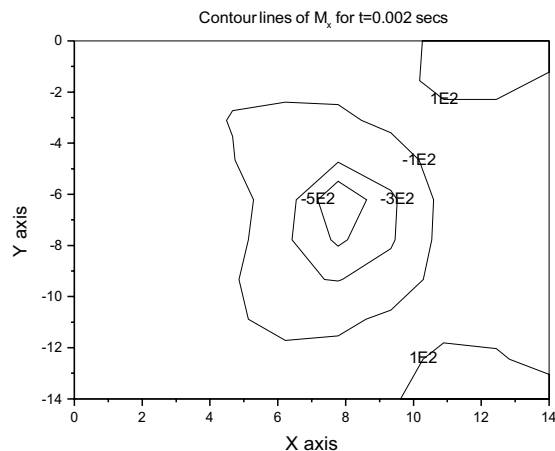Figure 9: Dynamic elastoplastic response of the plate

Figure 10: Mx contours for time step t=0.002 secs

7   Conclusions

In this paper a domain/boundary element method has been presented for solving dynamic elasto-plastic Reissner-Mindlin plate problems which, in addition to, the boundary supports are also supported, inside the domain on points (columns),

lines (walls) and regions (patches). On the basis of the preceding developments the following conclusions can be deduced

a) The BEM solution is very well suited for the solution of the dynamic elastoplastic problem of Reissner-Mindlin plates resting on internal supports.

b) Plates having an arbitrary shape, supported on all kinds of boundary conditions and subjected to any loading can be effectivelly analyzed.

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