# Comment on "Eigenmode analysis of arbitrarily shaped two-dimensional cavities by the method of point matching" [J. Acoust. Soc. Am. 107, 1153 (2000)]

J. T. Chen,[a] M. H. Chang, I. L. Chung, and Y. C. Cheng
*Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan*

(Received 6 July 2000; revised 20 August 2001; accepted 21 August 2001)

The method of point matching proposed by Kang and Lee [J. Acoust. Soc. Am. **107**, 1153–1160 (2000)] is revisited. This method can be seen as a single-layer potential approach from the viewpoint of imaginary-part dual BEM developed by Chen *et al.* [J. Chin. Inst. Eng. **12**, 729–739 (1999)]. Based on the concept of double-layer potential, an innovative method is proposed to deal with the problem of spurious eigensolution for the Neumann problem. Also, the acoustic mode is analytically derived for the circular cavity. Both the analytical study for a circular case and numerical result for a square cavity show the validity of the proposed formulation. © *2002 Acoustical Society of America.* [DOI: 10.1121/1.1410966]

PACS numbers: 43.20.Ks, 43.20.Rz [ANN]

## I. INTRODUCTION

Kang and Lee[1,2] presented a so-called method of point matching for the eigenproblems. Also, they termed the nondimensional dynamic influence function (NDIF) method in another paper.[3] Mathematically speaking, they are equivalent in essence. Based on the concept of radial basis function (RBF) expansion,[4] the method can be seen as one kind of the radial basis expansion since RBF is a function of the radial distance between the observation point and the boundary point. Since only boundary nodes are required, this is a meshless method and can be called the boundary node method. This method also belongs to the Trefftz method[5] since the approximation bases satisfy the governing equation. The main advantage of this method is simple in data preparation and no integration is required in comparison with the boundary element method (BEM). However, two disadvantages of the NDIF method,[3] spurious eigenvalues and ill-conditioned behavior, were pointed out by Chen *et al.*[6] Although many examples of the Dirichlet types were successfully worked out,[3] spurious eigensolutions occurred[1,2,6] when this method was extended to solve for the Neumann problems. Kang and Lee[1,2] filtered out the spurious eigenvalues by using the net approach, which can cancel out the embedded spurious eigenvalues. However, two influence matrices must be calculated. Based on the dual formulation developed by Chen and Hong,[7] the influence function[1–3] of the NDIF method is nothing but the imaginary part of the fundamental solution $[U(s,x)=iH_0^{(1)}(kr)]$.[8,9] The NDIF method[3] can be seen as a single-layer potential approach from the viewpoint of the imaginary-part dual BEM.[6,9] Also, the main difference between the NDIF method and the imaginary-part BEM is the distribution of density function, as shown in Table I. The former one lumps the density on the boundary point; the latter one distributes density along the boundary.[6,9] The spurious eigensolutions originate from the improper approximation of null operator since insufficient number of constraints is obtained. Many methods including the real-part, imaginary-part, and multiple reciprocity methods (MRM), suffer the problems of spurious eigenvalues since information is lost. The real-part dual BEM was developed by Chen's group and many references can be found.[8,10] Particularly, the imaginary-part formulation also results in an ill-conditioned matrix since the condition number for the influence matrix is always very large.[6] Many approaches have been employed to filter out the spurious eigensolution and to extract the true eigensolution, for example, a dual method using the residue technique, SVD (singular value decomposition) updating terms and updating documents, and the generalized SVD technique. For the circular case, it was proved by using circulants and degenerate kernels that the Kang and Lee method causes the problems of spurious eigensolutions and ill-conditioned behavior since these problems are inherent in the imaginary-part formulation.[6,9]

In this Letter, the Kang and Lee method is found to be the single-layer potential approach from the viewpoint of the imaginary-part dual formulation.[8,9] We will propose a double-layer potential approach to avoid the occurrence of a spurious eigensolution that has been filtered out using the net approach for the Neumann problems by Kang and Lee.[1,2] The acoustic modes will be derived analytically in the discrete system of a circle case using circulants and degenerate kernels. Both an analytical study and a numerical experiment will be considered to examine the solution.

## II. A UNIFIED THEORY USING DUAL FORMULATION

As mentioned earlier, spurious eigenvalues occur in the real-part BEM or MRM formulation. Also, the imaginary-part dual BEM results in spurious eigensolutions.[9] Here, we will analytically derive the true and spurious eigensolutions in the discrete system for a circular domain by using the imaginary-part dual BEM.[9] The degenerate kernels and circulants are employed to study the discrete system in an exact form. The unified theory is summarized in Table I and the comparison between the Kang and Lee method and the present method is made. The symbols in Table I follow the

[a]Author to whom correspondence should be addressed. P.O. Box 7-59, Keelung, Taiwan. Electronic mail: jtchen@mail.ntou.edu.tw

TABLE I. Comparisons of the nondimensional influence function method and the imaginary-part dual BEM.

| Method | Imaginary-part dual BEM by Chen *et al.* (Ref. 6) | Nondimensional dynamic influence function by Kang *et al.* (Refs. 1–3) |
|---|---|---|
| Auxiliary system | $J_0(k\|\mathbf{x}-\mathbf{s}\|)$ | $J_0(k\|\mathbf{x}-\mathbf{s}\|)$ |
| Density | Distributed on boundary using constant element | Concentrated on discrete points |
| Solution representation for field or boundary data | $0=\Sigma_j\int_{B_j}\{(T(s_j,x_i)u(s_j)-U(s_j,x_i)t(s_j)\}dB(s_j)$ | $u(x_i)=\Sigma J_0(k\|x_i-s_j\|)A_j$ |
| | $0=\Sigma_j\int_{B_j}\{(M(s_j,x_i)u(s_j)-L(s_j,x_i)t(s_j)\}dB(s_j)$ | $u(x_i)=\sum \frac{\partial J_0(k\|x_i-s_j\|)}{\partial n_{s_j}}B_j$ |
| Eigenequation for the Dirichlet problem | UT method: $U_{ij}t_j=0$ | UL method: $(\mathrm{SM})_{ij}A_j=0$ |
| | LM method: $L_{ij}t_j=0$ | TM method: $(\mathrm{SM}_s)_{ij}B_j=0$ |
| Eigenequation for the Neumann problem | UT method: $T_{ij}u_j=0$ | UL method: $(\mathrm{SM}_x)_{ij}A_j=0$ |
| | LM method: $M_{ij}u_j=0$ | TM method: $(\mathrm{SM}_{xs})_{ij}B_j=0$ |
| Influence matrix | $U_{ij}=\int_{B_j}U(s_j,x_i)dB(s_j)$ | $(\mathrm{SM})_{ij}=J_0(k\|x_i-s_j\|)$ |
| | $T_{ij}=\int_{B_j}T(s_j,x_i)dB(s_j)$ | $(\mathrm{SM}_s)_{ij}=\frac{\partial J_0(k\|x_i-s_j\|)}{\partial n_{s_j}}$ |
| | $L_{ij}=\int_{B_j}L(s_j,x_i)dB(s_j)$ | $(\mathrm{SM}_x)_{ij}=\frac{\partial J_0(k\|x_i-s_j\|)}{\partial n_{x_i}}$ |
| | $M_{ij}=\int_{B_j}M(s_j,x_i)dB(s_j)$ | $(\mathrm{SM}_{sx})_{ij}=\frac{\partial^2 J_0(k\|x_i-s_j\|)}{\partial n_{x_i}\partial n_{s_j}}$ |
| Spurious eigenequation for the Dirichlet problem | UT method: $J_n(k\rho)=0$ | UL method: $J_n(k\rho)=0$[a] |
| | LM method: $J_n'(k\rho)=0$ | TM method: $J_n'(k\rho)=0$ |
| True eigenequation for the Dirichlet problem | UT method: $J_n(k\rho)=0$ | UL method: $J_n(k\rho)=0$[a] |
| | LM method: $J_n(k\rho)=0$ | TM method: $J_n(k\rho)=0$ |
| Spurious eigenequation for the Neumann problem | UT method: $J_n(k\rho)=0$ | UL method: $J_n(k\rho)=0$[b] |
| | LM method: $J_n'(k\rho)=0$ | TM method: $J_n'(k\rho)=0$ |
| True eigenequation for the Neumann problem | UT method: $J_n'(k\rho)=0$ | UL method: $J_n'(k\rho)=0$[b] |
| | LM method: $J_n'(k\rho)=0$ | TM method: $J_n'(k\rho)=0$ |
| Condition number | $\frac{\mathrm{Max}(h_n)}{\mathrm{Min}(h_n)}$,[c] $n=0,1,2,\ldots$ | $\frac{\mathrm{Max}(h_n)}{\mathrm{Min}(h_n)}$, $n=0,1,2,\ldots$ |

[a]Example is available in Ref. 1.
[b]Example is not available in Ref. 1.
[c]$h_n$ can be $\lambda_l$, $\mu_l$, $\nu_l$, or $\delta_l$.

dual formulation of Chen and Hong.[7] Based on the imaginary-part dual BEM, the solution can be represented by

$$u(x_i)=\sum_{j=1}^{2N} U(s_j,x_i)A(s_j), \tag{1}$$

$$t(x_i)=\sum_{j=1}^{2N} L(s_j,x_i)A(s_j), \tag{2}$$

$$u(x_i)=\sum_{j=1}^{2N} T(s_j,x_i)B(s_j), \tag{3}$$

$$t(x_i)=\sum_{j=1}^{2N} M(s_j,x_i)B(s_j), \tag{4}$$

where $x_i$ is the $i$th observation point, $s_j$ is the $j$th boundary point, $u$ and $t$ are the potential and its normal derivative, $A(s_j)$ and $B(s_j)$ are the unknown concentrated densities at $s_j$, $2N$ is the number of boundary points, and the four imaginary-part kernels in the dual formulation can be expressed in terms of degenerate kernels[9] as shown below:

$$U(s,x)=\frac{-\pi}{2}J_0(kr) = -\sum_{m=-\infty}^{\infty}\frac{\pi}{2}J_m(kR)J_m(k\rho)\cos[m(\theta-\phi)], \tag{5}$$

$$T(s,x)=-\sum_{m=-\infty}^{\infty}\frac{\pi k}{2}J_m'(kR)J_m(k\rho)\cos[m(\theta-\phi)], \tag{6}$$

$$L(s,x)=-\sum_{m=-\infty}^{\infty}\frac{\pi k}{2}J_m(kR)J_m'(k\rho)\cos[m(\theta-\phi)], \tag{7}$$

$$M(s,x)=-\sum_{m=-\infty}^{\infty}\frac{\pi k^2}{2}J_m'(kR)J_m'(k\rho)\cos[m(\theta-\phi)], \tag{8}$$

in which $x=(\rho,\phi)$, $s=(R,\theta)$ in the polar coordinate, $J$ and $J'$ are the Bessel functions of the first kind and its derivative, respectively. For simplicity, we consider the same problem of a circular domain.[1–3] Since the rotation symmetry is preserved for a circular boundary, the four influence matrices in Eqs. (1)–(4) are denoted by $[U]$, $[T]$, $[L]$, and $[M]$ of the circulants with the elements

$$K_{ij}=K(R,\theta_j;\rho,\phi_i), \tag{9}$$

where $K$ can be $U$, $T$, $L$, or $M$, $\phi_i$ and $\theta_j$ are the angles of observation and boundary points, respectively. Based on the theory of circulants and the relation between the Riemann sum and integral,[6,9] we have

$$\lambda_l=-N\pi J_l(k\rho)J_l(k\rho), \tag{10}$$

$$\mu_l=-N\pi k\rho J_l'(k\rho)J_l(k\rho), \tag{11}$$

TABLE II. The true and spurious eigenvalues for circular and square cavities using the single- and double-layer potential approaches.

| | | Circular cavity | | Square cavity | |
|---|---|---|---|---|---|
| Boundary value problem | Eigensolution | Single-layer potential approach | Double-layer potential approach | Single-layer potential approach | Double-layer potential approach |
| Dirichlet problem | True eigensolution | $J_m(k\rho)=0$ | $J_m(k\rho)=0$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=1,2,3,...)$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=1,2,3,...)$ |
| | Spurious eigensolution | $J_m(k\rho)=0$ | $J'_m(k\rho)=0$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=1,2,3,...)$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=0,1,2,3,...)$ |
| | True eigenmode | $J_m(k\rho)e^{in\theta}$ $(m,n=0,1,2,3,...)$ | | $\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$ $(m,n=1,2,3,...)$ | |
| Neumann problem | True eigensolution | $J'_m(k\rho)=0$ | $J'_m(k\rho)=0$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=0,1,2,3,...)$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=0,1,2,3,...)$ |
| | Spurious eigensolution | $J_m(k\rho)=0$ | $J'_m(k\rho)=0$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=1,2,3,...)$ | $k_{mn}=\sqrt{\left(\frac{m}{L}\right)^2+\left(\frac{n}{L}\right)^2}\pi$ $(m,n=0,1,2,3,...)$ |
| | True eigenmode | $J_m(k\rho)e^{in\theta}$ $(m,n=0,1,2,3,...)$ | | $\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$ $(m,n=0,1,2,3,...)$ | |

$$\nu_l=-N\pi k\rho J_l(k\rho)J'_l(k\rho), \tag{12}$$

$$\delta_l=-N\pi k^2\rho J'_l(k\rho)J'_l(k\rho), \tag{13}$$

where $R$ is set to be $\rho$, $l=0,\pm1,\pm2,...,\pm(N-1)$, $N$, and $\lambda_l$, $\mu_l$, $\nu_l$, and $\delta_l$ are the eigenvalues of $[U]$, $[T]$, $[L]$, and $[M]$ matrices, respectively. The determinants for the four matrices can be obtained by multiplying all the eigenvalues. We summarize the true and spurious eigenvalues in Table II for the circular cavity using the single- and double-layer potential approaches. Also, the square case is included. Figure 1 shows the minimum singular value versus $k$ using the single-layer potential approach for the Neumann problem. It is found that both the analytical and numerical results match well and indicate that spurious eigenvalues occur. Figure 2 shows the minimum singular value versus $k$ using the double-layer potential approach for the Neumann problem. No spurious eigenvalues occur, as predicted theoretically. For a square cavity, Fig. 3 shows the minimum singular value versus $k$ using the single-layer potential approach for the Dirichlet problem. No spurious eigenvalues are found. By using the double-layer potential approach, spurious eigenvalues appear as shown in Fig. 4 for the Dirichlet problem. For the Neumann problem of a square cavity, the single-layer potential approach results in spurious eigenvalues while these values disappear in a similar way to the circular case when the double-layer potential approach is employed.

By substituting the $n$th true eigenvalue for $k$ in Eq. (10) and the $n$th true boundary mode into Eq. (1), we have

$$u_n(a,\phi)=\frac{N}{\pi}\sum_{l=0}^{2N-1}U(\rho,l\Delta\theta;a,\phi)\cos\left(\frac{\pi n}{N}l\right)\Delta\theta,$$
$$0<a<\rho,\quad 0<\phi<2\pi, \tag{14}$$

after considering the real part of the eigenvector, where $\Delta\theta=\pi/N$ is the increment of angle. By substituting the degenerate kernel of $U$ using Eq. (5), Eq. (14) reduces to

$$u_n(a,\phi)=\frac{N}{\pi}\sum_{l=0}^{2N-1}\sum_{m=-\infty}^{\infty}\frac{-\pi}{2}J_m(ka)J_m(k\rho)$$
$$\times\cos(m(l\Delta\theta-\phi))\cos\left(\frac{\pi n}{N}l\right)\Delta\theta. \tag{15}$$

When $N$ approaches infinity, the Riemann sum in Eq. (15) reduces to

$$u_n(a,\phi)=-N\pi J_n(ka)J_n(k\rho)\cos(n\phi),$$
$$0<a<\rho,\quad 0<\phi<2\pi. \tag{16}$$

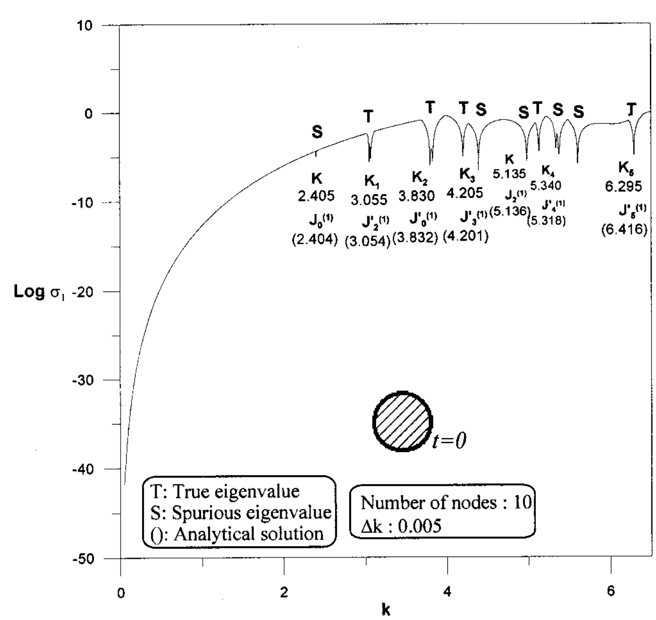


FIG. 1. The minimum singular value versus $k$ using the single-layer potential approach for the Neumann problem of a circular cavity.

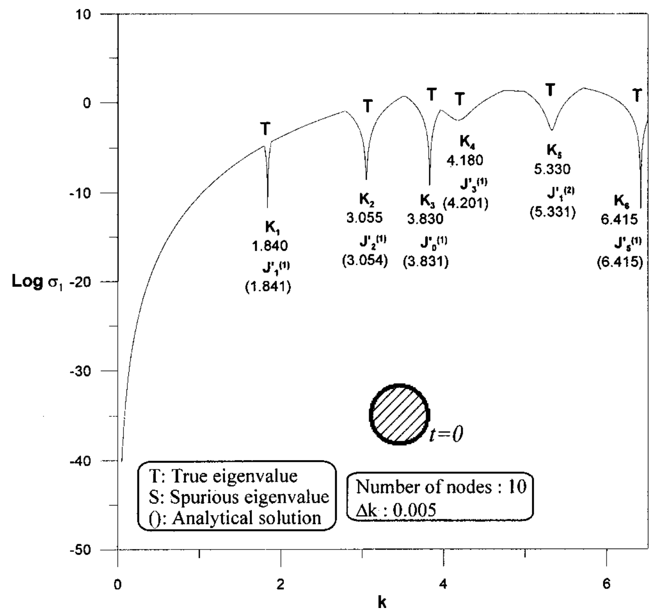


FIG. 2. The minimum singular value versus $k$ using the double-layer potential approach for the Neumann problem of a circular cavity.

It can be analytically proved that the acoustic mode is found to be trivial since $k$ satisfies the zero of $J_n(k\rho)$, as shown in Eq. (16). In the numerical implementation, the value of $J_n(k\rho)$ is not exactly zero. This is the reason why the contour plots for acoustic modes can be displayed in the papers of Kang and Lee, since a normalized value $J_n(k\rho)$ is divided.

## III. CONCLUDING REMARKS

The NDIF method or the method of point matching was classified to be the single-layer potential approach from the viewpoint of imaginary-part dual formulation. The difference between the Kang and Lee method and imaginary-part BEM is the singularity distribution of the density function, where the former one lumps the density on the boundary point and the latter one distributes the density along the boundary. This method was extended to the double-layer potential approach for avoiding the occurrence of spurious eigensolutions encountered in the Kang and Lee method. By using the degenerate kernels and the analytical properties of circulants for a circular cavity, the spurious eigensolutions were studied analytically and the spurious eigenvalues disappeared. Also, the acoustic modes were analytically proved to be trivial. An additional example of a square cavity was also considered.

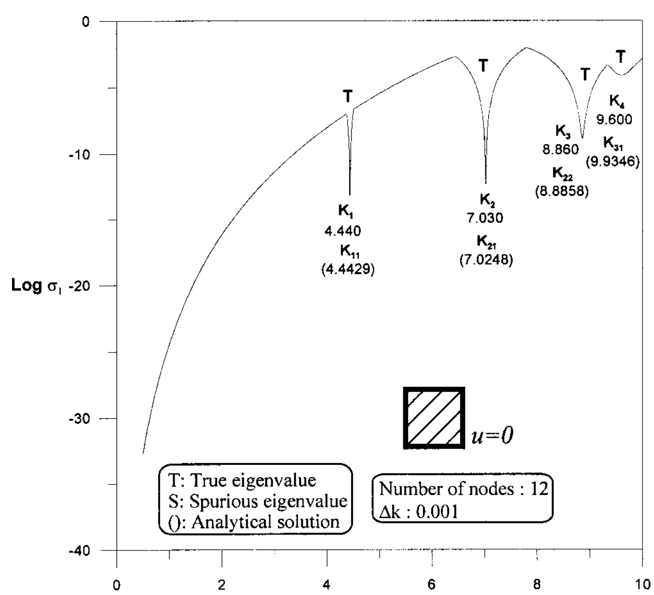


FIG. 3. The minimum singular value versus $k$ using the single-layer potential approach for the Dirichlet problem of a square cavity.

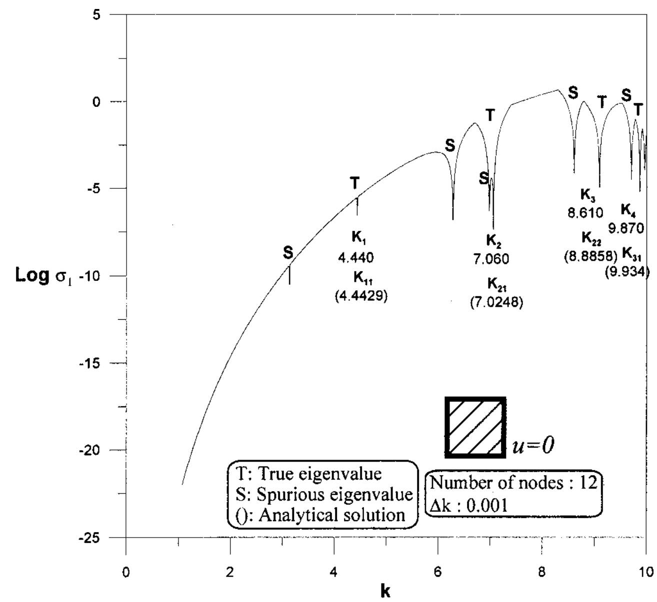


FIG. 4. The minimum singular value versus $k$ using the double-layer potential approach for the Dirichlet problem of a square cavity.


[1] S. W. Kang and J. M. Lee, "Eigenmode analysis of arbitrarily shaped two-dimensional cavities by the method of point matching," J. Acoust. Soc. Am. **107**, 1153–1160 (2000).

[2] S. W. Kang and J. M. Lee, "Authors reply to the Comments on 'Vibration analysis of arbitrary shaped membranes using nondimensional dynamic influence function,' " J. Sound Vib. **235**, 171 (2000).

[3] S. W. Kang, J. M. Lee, and Y. J. Kang, "Vibration analysis of arbitrary shaped membranes using non-dimensional dynamic influence function," J. Sound Vib. **221**, 117–132 (1999).

[4] M. A. Golberg, C. S. Chen, and H. Bowman, "Some recent results and proposals for the use of radial basis functions in the BEM," Eng. Anal. Boundary Elem. **23**, 285–296 (1999).

[5] E. Trefftz, "Ein gegenstück zum ritzschen verahren," *Proceedings of the 2nd International Congress on Applied Mechanics, 1926*, pp. 131–137.

[6] J. T. Chen, S. R. Kuo, K. H. Chen, and Y. C. Cheng, "Comments on 'Vibration analysis of arbitrary shaped membranes using nondimensional dynamic influence function,' " J. Sound Vib. **235**, 156–171 (2000).

[7] J. T. Chen and H.-K. Hong, "Review of dual integral representations with emphasis on hypersingular integrals and divergent series," Trans. ASME, J. Appl. Mech. **52**, 17–33 (1999).

[8] J. T. Chen, "Recent development of dual BEM in acoustic problems," Comput. Methods Appl. Mech. Eng. **188**, 833–845 (2000).

[9] J. T. Chen, S. R. Kuo, and K. H. Chen, "A nonsingular integral formulation for the Helmholtz eigenprobles of a circular domain," J. Chin. Univ. Sci. Technol. **12**, 729–739 (1999).

[10] S. R. Kuo, J. T. Chen, and C. X. Huang, "Analytical study and numerical experiments for true and spurious eigensolutions of a circular cavity using the real-part dual BEM," Int. J. Numer. Methods Eng. **48**, 1401–1422 (2000).