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A higher order asymptotic approximation for the fundamental frequency of a multiply connected membrane

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Abstract

The fundamental frequency of a fixed membrane is the square root of the lowest eigenvalue of negative Laplace operator with Dirichlet boundary conditions. A multiply connected membrane with inner cores of vanishing maximal dimensions $2c_j$ is considered in the present article. The modified perturbation method developed for a doubly connected membrane is extended to provide a general formula for the fundamental frequency of the multiply connected membrane. A higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency of a membrane with inner circular cores of radius c_j is specified. It is an excellent extension of the results in the literature. Moreover, a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency of a circular membrane of radius 1 with finitely many inner circular cores of small radius c is found and computed explicitly. The effects of the positions of the inner cores on the second-order asymptotic approximation are investigated. The accuracy of the second-order asymptotic approximation is also shown by the comparisons among the asymptotic approximations and the numerical values computed by other investigators.

1. Introduction

The square root of the lowest eigenvalue of negative Laplace operator with Dirichlet boundary conditions in two dimensions represents the fundamental frequency of a fixed membrane. The determination of the fundamental frequency is important in the studies of acoustics and electromagnetism [1]. It is also important in the study of the human eardrum and in the design of engineering devices such as microphones, loudspeakers, pumps, compressors, pressure regulators, antennae for space communications [1,2]. The human eardrum is a membrane with a stirrup which acts as a rigid core. However, related articles [3–7] on the fundamental frequency of a multiply connected membrane are few. None of these articles except Wang's [7] was concerned about the asymptotic case in which the maximal dimensions of the inner cores of a multiply connected membrane are vanishing.

In the present article, a multiply connected membrane with inner cores of vanishing maximal dimensions $2c_j$ is concerned. The modified perturbation method [8] developed by the author for the fundamental frequency of a doubly connected membrane is extended to provide a general formula for the fundamental frequency of the

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Nomenclature

B_j	the inner core enclosed by S_j
B_{0j}, B_{mj}, A_{mj}	the constant coefficients in the expression of W_0 as in Eq. (27)
$B_{0j}(N,p), B_{mj}(N,p), A_{mj}(N,p)$	the constant coefficients of $U_{N,p}(r_j, \theta_j)$ as in Eq. (34)
c	the radius of inner circular cores
$2c_j$	the maximal dimension of B_j
D_{0j}, C_{mj}, D_{mj}	the constant coefficients of $V_{2j}^h(r_j, \theta_j)$ as in Eqs. (41)–(44)
J_n	the nth order *Bessel* function
K	the fundamental frequency of R $(K = K_0 + F_1 + \cdots + F_m + \cdots)$
K_N	eigenvalue on R_0
$K_{s,t}$	the tth zero of J_s
K_0	the fundamental frequency of R_0
$l(N)$	the number of the eigenfunctions to K_N
M	the number of inner cores
P_j	the center of jth inner circular core
(r, θ)	the polar coordinates with the origin at the center of a circular membrane of radius 1
(r_j, θ_j)	the polar coordinates with the origin at P_j
(r_{0j}, θ_{0j})	the polar coordinates of P_j in (r, θ)
R	multiply connected membrane
R_0	simply connected membrane
(R_{ji}, ϕ_{ji})	the polar coordinates of P_i in (r_j, θ_j)
S_j	the inner boundary of R
S_0	the boundary of R_0
$U_{N,p}$	corresponding eigenfunction to K_N
$V_{2j} = V_{2j}^i + V_{2j}^h$	
W	corresponding eigenfunction to K $(W = (\sum_{j=1}^M V_{0j} + W_0) + (\sum_{j=1}^M V_{1j} + W_1) + \cdots + (\sum_{j=1}^M V_{mj} + W_m) + \cdots)$
W_0	corresponding eigenfunction to K_0
Y_n	the nth order *Neumann* function
γ	≈ 0.5772

multiply connected membrane. A higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency of a membrane with M inner circular cores of radius c_j is specified by correcting the boundary conditions, by using the results [8] for a membrane with an inner circular core, by using the translational addition theorems for circular cylindrical wave functions [4,9], and by applying the generalized Green's function [10]. It is found to be an excellent extension of the results [7,8] in the literature. Moreover, by using the generalized Green's function [8] for a circular membrane of radius 1, by applying the results [8] for a circular membrane of radius 1 with an inner circular core of small radius c, and by employing the translational addition theorems for circular cylindrical wave functions [4,9], a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency of a circular membrane of radius 1 with M inner circular cores of small radius c is found and computed explicitly. The effects of the positions of the inner cores on the second-order asymptotic approximation are investigated and the comparisons among the asymptotic approximations and the numerical values [4,5,11] computed by other investigators are made.

2. Perturbation formulation

Let R_0 be a simply connected membrane with boundary S_0 and R be a multiply connected membrane with the outer boundary S_0 and M inner boundaries S_j enclosing M inner cores B_j of vanishing maximal dimensions $2c_j, j = 1, 2, \ldots, M$. All lengths have been normalized by a characteristic length L. The governing Helmholtz equation is

$$\Delta W + K^2 W = 0, \tag{1}$$

where W is the normalized vertical displacement and K is the normalized vibrational frequency, $K = \text{frequency}\, L \sqrt{\text{density/tension per length}}$.

Consider the eigenvalue problems on R and R_0, respectively,

$$\Delta W + K^2 W = 0 \text{ in } R, \tag{2}$$

$$W = 0 \text{ on } S_0 \cup S_1 \cup \cdots \cup S_M \tag{3}$$

and

$$\Delta \breve{W} + \breve{K}^2 \breve{W} = 0 \text{ in } R_0, \tag{4}$$

$$\breve{W} = 0 \text{ on } S_0. \tag{5}$$

Let K_0^2 be the smallest eigenvalue of the problem on the membrane R_0 and W_0 be the corresponding eigenfunction. By extending the modified perturbation method [8] developed by the author for a doubly connected membrane, a general formula for the fundamental frequency K of the multiply connected membrane R is derived below.

The fundamental frequency K of the multiply connected membrane R and its corresponding eigenfunction W can be expressed as

$$K = K_0 + F_1 + \cdots + F_m + \cdots \tag{6}$$

and

$$W = \left(\sum_{j=1}^{M} V_{0j} + W_0\right) + \left(\sum_{j=1}^{M} V_{1j} + W_1\right) + \cdots + \left(\sum_{j=1}^{M} V_{mj} + W_m\right) + \cdots, \tag{7}$$

where $V_{0j} = 0$ on $R_0 \backslash B_j$,

$$\Delta W_0 + K_0^2 W_0 = 0 \text{ in } R_0, \tag{8}$$

$$W_0 = 0 \text{ on } S_0, \tag{9}$$

$$\Delta V_{1j} + K_0^2 V_{1j} = 0 \text{ in } R_0 \backslash B_j, \tag{10}$$

$$V_{1j} = -W_0 \text{ on } S_j, \tag{11}$$

$$\Delta W_1 + K_0^2 W_1 = -2K_0 F_1 W_0 \text{ in } R_0, \tag{12}$$

$$W_1 = -\sum_{j=1}^{M} V_{1j} \text{ on } S_0, \tag{13}$$

$$\Delta V_{2j} + K_0^2 V_{2j} = -2K_0 F_1 V_{1j} \text{ in } R_0 \backslash B_j, \tag{14}$$

$$V_{2j} = -W_1 - \sum_{l=1,\ l \neq j}^{M} V_{1l} \text{ on } S_j, \tag{15}$$

$$\Delta W_2 + K_0^2 W_2 = -2K_0 F_1 W_1 - 2K_0 F_2 W_0 - F_1^2 W_0 \text{ in } R_0, \tag{16}$$

$$W_2 = -\sum_{j=1}^{M} V_{2j} \text{ on } S_0, \tag{17}$$

$$\Delta V_{mj} + K_0^2 V_{mj} = -2K_0 \sum_{p=1}^{m} F_p V_{(m-p)j} - \sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t V_{(m-s-t)j} \text{ in } R_0 \backslash B_j, \tag{18}$$

$$V_{mj} = -W_{m-1} - \sum_{l=1,\ l \neq j}^{M} V_{(m-1)l} \text{ on } S_j,\ m = 3, 4, 5, \ldots, \tag{19}$$

$$\Delta W_m + K_0^2 W_m = -2K_0 \sum_{p=1}^{m} F_p W_{m-p} - \sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t W_{m-s-t} \text{ in } R_0, \tag{20}$$

$$W_m = -\sum_{j=1}^{M} V_{mj} \text{ on } S_0,\ m = 3, 4, 5, \ldots\ . \tag{21}$$

The conditions for the existence of W_n, $n = 1, 2, 3, \ldots$, are provided by Fredholm Alternative theorem [10]. These conditions give formulas to the correction terms of the fundamental frequency K. The formulas are as follows:

$$F_1 = \frac{\sum_{j=1}^{M} \oint_{S_0} \frac{\partial W_0}{\partial n} V_{1j}\, \mathrm{d}s}{-2K_0 \int_{R_0} W_0^2\, \mathrm{d}A}, \tag{22}$$

$$F_2 = \frac{\int_{R_0} (2K_0 F_1 W_1 + F_1^2 W_0) W_0\, \mathrm{d}A + \sum_{j=1}^{M} \oint_{S_0} \frac{\partial W_0}{\partial n} V_{2j}\, \mathrm{d}s}{-2K_0 \int_{R_0} W_0^2\, \mathrm{d}A}, \tag{23}$$

$$F_m = \frac{\int_{R_0} (2K_0 \sum_{p=1}^{m-1} F_p W_{m-p}) W_0\, \mathrm{d}A}{-2K_0 \int_{R_0} W_0^2\, \mathrm{d}A} - \frac{\sum_{j=1}^{M} \oint_{S_0} \frac{\partial W_0}{\partial n} V_{mj}\, \mathrm{d}s}{2K_0 \int_{R_0} W_0^2\, \mathrm{d}A} - \frac{\int_{R_0} (\sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t W_{m-s-t}) W_0\, \mathrm{d}A}{2K_0 \int_{R_0} W_0^2\, \mathrm{d}A}, \quad m = 3, 4, 5, \ldots, \tag{24}$$

where the derivative on S_0 is the outward normal derivative.

Hence, a general formula for the fundamental frequency K of the multiply connected membrane R is found to be the formula Eq. (6) with Eqs. (22)–(24). However, Eqs. (22)–(24) will not be specified without the specifications of W_n's and V_{nj}'s. The specification for W_n's will be done by correcting the boundary conditions on the inner boundaries S_j, $j = 1, 2, \ldots, M$, and the specification for V_{nj}'s will be done by correcting the boundary conditions on the outer boundary S_0. This will be shown in the next section.

3. A membrane with M inner circular cores of small radius c_j

A higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency K of a membrane with M inner circular cores of small radius c_j centered at P_j is specified in this section.

By correcting the boundary condition for $W_0 + V_{1j}$ on the outer boundary S_0 to $O(1/|\ln c_j|)$, by using the results [8, Eqs. (54) and (55)] for a membrane with an inner circular core, and by using Eq. (22), V_{1j} and the first correction term F_1 are specified as

$$V_{1j}(r_j, \theta_j) = \frac{-B_{0j} J_0(K_0 c_j)}{Y_0(K_0 c_j)} Y_0(K_0 r_j) - \sum_{m=1}^{\infty} \frac{J_m(K_0 c_j)}{Y_m(K_0 c_j)} Y_m(K_0 r_j)(A_{mj} \sin(m\theta_j) + B_{mj} \cos(m\theta_j)) \tag{25}$$

and

$$F_1 = \sum_{j=1}^{M} \left(\frac{\pi B_{0j}^2}{K_0 \int_{R_0} W_0^2\, \mathrm{d}A} \right) \frac{1}{|\ln c_j|} + \sum_{j=1}^{M} \left(\frac{\pi(\ln K_0 + \gamma - \ln 2) B_{0j}^2}{K_0 \int_{R_0} W_0^2\, \mathrm{d}A} \right) \frac{1}{|\ln c_j|^2} + \cdots, \tag{26}$$

where B_{0j}, B_{mj}, A_{mj} are the constant coefficients in the expression of W_0:

$$W_0(r_j,\theta_j) = B_{0j}J_0(K_0r_j) + \sum_{m=1}^{\infty} J_m(K_0r_j)(A_{mj}\sin(m\theta_j) + B_{mj}\cos(m\theta_j)), \tag{27}$$

(r_j,θ_j) is the polar coordinates with the origin at P_j, $\gamma \approx 0.5772$, J_n is the nth order *Bessel* function, Y_n is the nth order *Neumann* function, and [12]

$$J_n(z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!}\left(\frac{z}{2}\right)^{(n+2l)}, \quad n = 0,1,2,\ldots, \tag{28}$$

$$Y_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2}+\gamma\right)J_0(z) - \frac{2}{\pi}\sum_{l=1}^{\infty}\frac{(-1)^l}{(l!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{l}\right)\left(\frac{z}{2}\right)^{2l}, \tag{29}$$

$$\begin{aligned} Y_m(z) = {} & \frac{2}{\pi}\left(\ln\frac{z}{2}\right)J_m(z) - \frac{1}{\pi}\sum_{j=0}^{m-1}\frac{(m-j-1)!}{j!}\left(\frac{2}{z}\right)^{(m-2j)} \\ & -\frac{1}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!(m+l)!}(\psi(m+l+1)+\psi(l+1))\left(\frac{z}{2}\right)^{(m+2l)}, \end{aligned}$$

$$\psi(1) = -\gamma, \ \psi(m+1) = \left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) - \gamma, \ m = 1,2,3,\ldots \ . \tag{30}$$

Notice that V_{1j} has $O(1/|\ln c_j|)$ boundary value on the inner boundaries S_i for all $i \neq j$ because the translational addition theorems for circular cylindrical wave functions [4,9] give

$$Y_0(K_0r_i) = \sum_{l=-\infty}^{\infty} Y_l(K_0R_{ji})J_l(K_0r_j)\cos\left(l\phi_{ji} - l\theta_j\right), \tag{31}$$

$$\begin{aligned} Y_p(K_0r_i)\sin(p\theta_i) = {} & \sum_{l=-\infty}^{\infty} Y_{l+p}(K_0R_{ji})J_l(K_0r_j)\sin((l+p)\phi_{ji} - l\theta_j) \\ & \times(-1)^p, \end{aligned} \tag{32}$$

$$\begin{aligned} Y_p(K_0r_i)\cos(p\theta_i) = {} & \sum_{l=-\infty}^{\infty} Y_{l+p}(K_0R_{ji})J_l(K_0r_j)\cos((l+p)\phi_{ji} - l\theta_j) \\ & \times(-1)^p, \end{aligned} \tag{33}$$

where (R_{ji}, ϕ_{ji}) is the polar coordinates representation of the center point P_i in the polar coordinates (r_j,θ_j) and $p = 1,2,3,\ldots$.

To specify the second correction term F_2, the specifications of W_1 and V_{2j}'s are required. Let $U_{N,p}$, $p = 1,2,3,\ldots,l(N)$, be the corresponding eigenfunctions to the eigenvalues $K_N(\neq K_0)$, $N = 1,2,3,\ldots$, of the eigenvalue problem on the simply connected membrane R_0, then $U_{N,p}$ can be expressed as

$$\begin{aligned} U_{N,p}(r_j,\theta_j) = {} & B_{0j}(N,p)J_0(K_Nr_j) + \sum_{m=1}^{\infty} J_m(K_Nr_j)(A_{mj}(N,p)\sin(m\theta_j) \\ & + B_{mj}(N,p)\cos(m\theta_j)), \end{aligned} \tag{34}$$

with appropriate constant coefficients $B_{0j}(N,p)$, $B_{mj}(N,p)$, and $A_{mj}(N,p)$ determined by the boundary condition on S_0, Eq. (5). Thus, by using Green's second identity [10], by using the generalized Green's function G [10] for the boundary value problem consisting of Eqs. (8) and (9), and by correcting the boundary condition for $W_0 + \sum_{i=1}^{M} V_{1i} + W_1$ on the inner boundary S_j to $\sum_{i=1}^{M} O(1/|\ln c_i|)$, $j = 1,2,\ldots,M$, W_1 is specified as

$$W_1 = \sum_{i=1}^{M} \oint_{S_0} \frac{\partial G}{\partial n} V_{1i}\,\mathrm{d}s \tag{35}$$

$$= \sum_{N=1}^{\infty} \sum_{p=1}^{l(N)} \frac{U_{N,p}}{(K_N^2 - K_0^2)\|U_{N,p}\|^2} \left(\sum_{i=1}^{M} \oint_{S_0} \frac{\partial U_{N,p}}{\partial n} V_{1i}\,\mathrm{d}s \right). \tag{36}$$

To specify V_{2j}'s, let $V_{2j} = V_{2j}^i + V_{2j}^h$ with

$$\Delta V_{2j}^i + K_0^2 V_{2j}^i = -2K_0 F_1 V_{1j} \text{ in } R_0 \backslash B_j, \tag{37}$$

$$\Delta V_{2j}^h + K_0^2 V_{2j}^h = 0 \text{ in } R_0 \backslash B_j, \tag{38}$$

$$V_{2j}^h = -W_1 - \sum_{l=1,\ l \neq j}^{M} V_{1l} - V_{2j}^i \text{ on } S_j, \tag{39}$$

then

$$V_{2j}^i(r_j, \theta_j) = -F_1 \left[\frac{B_{0j} J_0(K_0 c_j)}{Y_0(K_0 c_j)} r_j Y_0'(K_0 r_j) + \sum_{m=1}^{\infty} \frac{J_m(K_0 c_j)}{Y_m(K_0 c_j)} r_j Y_m'(K_0 r_j) \times (A_{mj} \sin(m\theta_j) + B_{mj} \cos(m\theta_j)) \right] \tag{40}$$

and

$$V_{2j}^h(r_j, \theta_j) = \tilde{D}_{0j}(c_j) \left(J_0(K_0 r_j) - \frac{J_0(K_0 c_j)}{Y_0(K_0 c_j)} Y_0(K_0 r_j) \right) + \sum_{m=1}^{\infty} (\tilde{C}_{mj}(c_j) \times \sin(m\theta_j) + \tilde{D}_{mj}(c_j) \cos(m\theta_j)) \left(J_m(K_0 r_j) - \frac{J_m(K_0 c_j)}{Y_m(K_0 c_j)} Y_m(K_0 r_j) \right) + D_{0j} Y_0(K_0 r_j) + \sum_{m=1}^{\infty} Y_m(K_0 r_j)(C_{mj} \sin(m\theta_j) + D_{mj} \cos(m\theta_j)), \tag{41}$$

where

$$D_{0j} = \frac{-1}{2\pi Y_0(K_0 c_j)} \int_0^{2\pi} \left(W_1 + \sum_{l=1,\ l \neq j}^{M} V_{1l} \right)(c_j, \theta_j)\,\mathrm{d}\theta_j + \frac{B_{0j} F_1 c_j J_0(K_0 c_j) Y_0'(K_0 c_j)}{Y_0^2(K_0 c_j)}, \tag{42}$$

$$C_{mj} = \frac{-1}{\pi Y_m(K_0 c_j)} \int_0^{2\pi} \sin(m\theta_j) \left(W_1 + \sum_{l=1,\ l \neq j}^{M} V_{1l} \right)(c_j, \theta_j)\,\mathrm{d}\theta_j + \frac{A_{mj} F_1 c_j J_m(k_0 c_j) Y_m'(K_0 c_j)}{Y_m^2(K_0 c_j)}, \tag{43}$$

$$D_{mj} = \frac{-1}{\pi Y_m(K_0 c_j)} \int_0^{2\pi} \cos(m\theta_j) \left(W_1 + \sum_{l=1,\ l \neq j}^{M} V_{1l} \right)(c_j, \theta_j)\,\mathrm{d}\theta_j + \frac{B_{mj} F_1 c_j J_m(k_0 c_j) Y_m'(K_0 c_j)}{Y_m^2(K_0 c_j)}. \tag{44}$$

Moreover, the specification for V_{2j}^h is made to the following. The asymptotic expansion of $J_0(K_0c_j)/Y_0(K_0c_j)$ as $c_j \to 0$ is

$$\frac{J_0(K_0c_j)}{Y_0(K_0c_j)} = \frac{-\pi}{2}\frac{1}{(-\ln c_j)} + \frac{\pi}{2}(\ln 2 - \gamma - \ln K_0)\frac{1}{(\ln c_j)^2} + \cdots, \tag{45}$$

where $\gamma \approx 0.5772$. Thus,

$$\int_0^{2\pi} W_1(c_j,\theta_j)\,\mathrm{d}\theta_j = \sum_{i=1}^{M}\left[\sum_{N=1}^{\infty}\sum_{p=1}^{l(N)}\frac{\pi^2 B_{0i}B_{0j}(N,p)J_0(K_Nc_j)}{(K_N^2-K_0^2)\|U_{N,p}\|^2} \times \oint_{S_0}\frac{\partial U_{N,p}(r_i,\theta_i)}{\partial n}Y_0(K_0r_i)\,\mathrm{d}s\right]\frac{1}{|\ln c_i|} + \cdots, \tag{46}$$

$$\int_0^{2\pi} \sin(m\theta_j)W_1(c_j,\theta_j)\,\mathrm{d}\theta_j = \sum_{i=1}^{M}\left[\sum_{N=1}^{\infty}\sum_{p=1}^{l(N)}\frac{\pi^2 B_{0i}A_{mj}(N,p)J_m(K_Nc_j)}{2(K_N^2-K_0^2)\|U_{N,p}\|^2} \times \oint_{S_0}\frac{\partial U_{N,p}(r_i,\theta_i)}{\partial n}Y_0(K_0r_i)\,\mathrm{d}s\right]\frac{1}{|\ln c_i|} + \cdots, \tag{47}$$

$$\int_0^{2\pi} \cos(m\theta_j)W_1(c_j,\theta_j)\,\mathrm{d}\theta_j = \sum_{i=1}^{M}\left[\sum_{N=1}^{\infty}\sum_{p=1}^{l(N)}\frac{\pi^2 B_{0i}B_{mj}(N,p)J_m(K_Nc_j)}{2(K_N^2-K_0^2)\|U_{N,p}\|^2} \times \oint_{S_0}\frac{\partial U_{N,p}(r_i,\theta_i)}{\partial n}Y_0(K_0r_i)\,\mathrm{d}s\right]\frac{1}{|\ln c_i|} + \cdots, \tag{48}$$

and by Eqs. (31)–(33),

$$\sum_{l=1,\ l\neq j}^{M}\int_0^{2\pi} V_{1l}(c_j,\theta_j)\,\mathrm{d}\theta_j = \sum_{l=1,\ l\neq j}^{M}\pi^2 B_{0l}J_0(K_0c_j)Y_0(K_0R_{jl})\frac{1}{|\ln c_l|} + \cdots, \tag{49}$$

$$\sum_{l=1,\ l\neq j}^{M}\int_0^{2\pi} \sin(m\theta_j)V_{1l}(c_j,\theta_j)\,\mathrm{d}\theta_j = \sum_{l=1,\ l\neq j}^{M} 0\frac{1}{|\ln c_l|} + \cdots, \tag{50}$$

$$\sum_{l=1,\ l\neq j}^{M}\int_0^{2\pi} \cos(m\theta_j)V_{1l}(c_j,\theta_j)\,\mathrm{d}\theta_j = \sum_{l=1,\ l\neq j}^{M} 0\frac{1}{|\ln c_l|} + \cdots. \tag{51}$$

Green's second identity [10] gives

$$\oint_{S_0}\frac{\partial W_0}{\partial n}V_{2j}\,\mathrm{d}s = \oint_{S_0}\frac{\partial W_0}{\partial n}V_{2j}^i\,\mathrm{d}s - \oint_{S_j}\left(\frac{\partial W_0}{\partial n}V_{2j}^h - \frac{\partial V_{2j}^h}{\partial n}W_0\right)\mathrm{d}s. \tag{52}$$

Also,

$$\begin{aligned}\oint_{S_0}\frac{\partial W_0}{\partial n}V_{2j}^i\,\mathrm{d}s = & -F_1\left\{\frac{B_{0j}J_0(K_0c_j)}{Y_0(K_0c_j)}\oint_{S_0}\frac{\partial W_0}{\partial n}r_jY_0'(K_0r_j)\,\mathrm{d}s\right.\\ &\left. + \sum_{m=1}^{\infty}\frac{J_m(K_0c_j)}{Y_m(K_0c_j)}\oint_{S_0}\frac{\partial W_0}{\partial n}\big(A_{mj}\sin(m\theta_j) + B_{mj}\cos(m\theta_j)\big)r_jY_m'(K_0r_j)\,\mathrm{d}s\right\}\\ = & \sum_{i=1}^{M}\left(\frac{\pi^2 B_{0i}^2B_{0j}}{2K_0\int_{R_0}W_0^2\,\mathrm{d}A}\oint_{S_0}\frac{\partial W_0}{\partial n}r_jY_0'(K_0r_j)\,\mathrm{d}s\right)\frac{1}{|\ln c_j||\ln c_i|} + \cdots\end{aligned} \tag{53}$$

and Wronskian $\mathscr{W}$ of $J_m(z)$ and $Y_m(z)$ [12], $\mathscr{W}(J_m(z), Y_m(z)) = 2/\pi z$, gives

$$\begin{aligned}
&\oint_{S_j}\left(\frac{\partial W_0}{\partial n}V_{2j}^h - \frac{\partial V_{2j}^h}{\partial n}W_0\right)\mathrm{d}s \\
&= \int_0^{2\pi}\left(\left.\frac{-\partial W_0(r_j,\theta_j)}{\partial r_j}\right|_{r_j=c_j}V_{2j}^h(c_j,\theta_j) + \left.\frac{\partial V_{2j}^h(r_j,\theta_j)}{\partial r_j}\right|_{r_j=c_j}W_0(c_j,\theta_j)\right)c_j\,\mathrm{d}\theta_j \\
&= -4B_{0j}\tilde{D}_{0j}(c_j)\frac{J_0(K_0c_j)}{Y_0(K_0c_j)} - 2\sum_{m=1}^{\infty}(A_{mj}\tilde{C}_{mj}(c_j) + B_{mj}\tilde{D}_{mj}(c_j))\frac{J_m(K_0c_j)}{Y_m(K_0c_j)} \\
&\quad + 4B_{0j}D_{0j} + 2\sum_{m=1}^{\infty}(A_{mj}C_{mj} + B_{mj}D_{mj}).
\end{aligned} \tag{54}$$

Thus, by correcting the boundary condition for $W_0 + \sum_{i=1}^{M}V_{1i} + W_1 + V_{2j}$ on the outer boundary S_0 to $\sum_{i=1}^{M}O(1/(|\ln c_i||\ln c_j|))$ and by Eq. (23), V_{2j}^h and F_2 are specified as

$$V_{2j}^h(r_j,\theta_j) = D_{0j}Y_0(K_0r_j) + \sum_{m=1}^{\infty}Y_m(K_0r_j)(C_{mj}\sin(m\theta_j) + D_{mj}\cos(m\theta_j)) \tag{55}$$

and

$$\begin{aligned}
F_2 = &\sum_{j=1}^{M}\sum_{i=1}^{M}\left[\frac{\pi^2B_{0j}^2B_{0i}^2}{-2K_0^3\left(\int_{R_0}W_0^2\,\mathrm{d}A\right)^2} + \frac{\pi^2B_{0i}^2B_{0j}}{-4K_0^2\left(\int_{R_0}W_0^2\,\mathrm{d}A\right)^2}\oint_{S_0}\frac{\partial W_0}{\partial n}r_jY_0'(K_0r_j)\,\mathrm{d}s\right. \\
&\left.+\sum_{N=1}^{\infty}\sum_{p=1}^{l(N)}\left(\frac{\pi^2B_{0i}B_{0j}B_{0j}(N,p)}{2K_0(K_N^2-K_0^2)\|U_{N,p}\|^2\int_{R_0}W_0^2\,\mathrm{d}A}\right)\oint_{S_0}\frac{\partial U_{N,p}}{\partial n}Y_0(K_0r_i)\,\mathrm{d}s\right] \\
&\times\frac{1}{|\ln c_i||\ln c_j|} + \sum_{j=1}^{M}\sum_{l=1,\ l\neq j}^{M}\left(\frac{\pi^2B_{0l}B_{0j}Y_0(K_0R_{jl})}{2K_0\int_{R_0}W_0^2\,\mathrm{d}A}\right)\frac{1}{|\ln c_l||\ln c_j|} + \cdots.
\end{aligned} \tag{56}$$

Therefore, a higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency K of a membrane with M inner circular cores of small radius c_j centered at P_j is specified as

$$\begin{aligned}
K = &K_0 + \sum_{j=1}^{M}\left(\frac{\pi B_{0j}^2}{K_0\int_{R_0}W_0^2\,\mathrm{d}A}\right)\frac{1}{|\ln c_j|} + \sum_{j=1}^{M}\left(\frac{\pi(\ln K_0 + \gamma - \ln 2)B_{0j}^2}{K_0\int_{R_0}W_0^2\,\mathrm{d}A}\right)\frac{1}{|\ln c_j|^2} \\
&+\sum_{j=1}^{M}\sum_{i=1}^{M}\left[\frac{\pi^2B_{0j}^2B_{0i}^2}{-2K_0^3\left(\int_{R_0}W_0^2\,\mathrm{d}A\right)^2} + \frac{\pi^2B_{0i}^2B_{0j}}{-4K_0^2\left(\int_{R_0}W_0^2\,\mathrm{d}A\right)^2}\oint_{S_0}\frac{\partial W_0}{\partial n}r_jY_0'(K_0r_j)\,\mathrm{d}s\right. \\
&\left.+\sum_{N=1}^{\infty}\sum_{p=1}^{l(N)}\left(\frac{\pi^2B_{0i}B_{0j}B_{0j}(N,p)}{2K_0(K_N^2-K_0^2)\|U_{N,p}\|^2\int_{R_0}W_0^2\,\mathrm{d}A}\right)\oint_{S_0}\frac{\partial U_{N,p}}{\partial n}Y_0(K_0r_i)\,\mathrm{d}s\right] \\
&\times\frac{1}{|\ln c_i||\ln c_j|} + \sum_{j=1}^{M}\sum_{l=1,\ l\neq j}^{M}\left(\frac{\pi^2B_{0l}B_{0j}Y_0(K_0R_{jl})}{2K_0\int_{R_0}W_0^2\,\mathrm{d}A}\right)\frac{1}{|\ln c_l||\ln c_j|} + \cdots\ .
\end{aligned} \tag{57}$$

This higher order asymptotic approximation agrees with the result [8, Eq. (85)] derived by the author for a membrane with a small inner circular core, and extends the lower order asymptotic approximation [7, Eq. (8)] derived by Wang for a membrane with M small inner circular cores.

4. A circular membrane of radius 1 with M inner circular cores of small radius c

Now, consider a circular membrane of radius 1 with M inner circular cores of small radius c centered at P_j. Let (r, θ) be the polar coordinates with the origin at the center of the circular membrane without the inner cores and $P_j = (r_{0j}, \theta_{0j})$. By the results [8, Eqs. (96)–(98), (101)–(104), (106)–(108)] derived by the author for a circular membrane of radius 1 with an inner circular core of small radius c, a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency K of a circular membrane of radius 1 with M inner circular cores of small radius c centered at (r_{0j}, θ_{0j}) is found explicitly to be

$$\begin{aligned}K = {} & K_0 + \sum_{j=1}^{M}\left(\frac{J_0^2(K_0 r_{0j})}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|} \\ & + \left\{\sum_{j=1}^{M}\left(\frac{(\ln K_0 + \gamma - \ln 2)J_0^2(K_0 r_{0j})}{K_0 J_1^2(K_0)}\right) + \sum_{j=1}^{M}\sum_{i=1}^{M}\left[-\frac{J_0^2(K_0 r_{0j})J_0^2(K_0 r_{0i})}{2K_0^3 J_1^4(K_0)}\right.\right. \\ & - \frac{J_0(K_0 r_{0j})J_0^2(K_0 r_{0i})\int_0^{2\pi}\sqrt{1 + r_{0j}^2 - 2r_{0j}\cos\theta}\, Y_1(K_0\sqrt{1 + r_{0j}^2 - 2r_{0j}\cos\theta})\,\mathrm{d}\theta}{4K_0 J_1^3(K_0)} \\ & + \frac{2\pi J_0(K_0 r_{0i})J_0(K_0 r_{0j})}{K_0 J_1^2(K_0)}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{K_{p,m}J_p(K_{p,m}r_{0j})J_p(K_0 r_{0i})Y_p(K_0)(\cos p(\theta_{0i} - \theta_{0j}))}{(K_{p,m}^2 - K_0^2)J_p'(K_{p,m})}\right) \\ & \left. - \frac{\pi J_0^2(K_0 r_{0i})J_0(K_0 r_{0j})Y_0(K_0)}{K_0 J_1^2(K_0)}\sum_{n=2}^{\infty}\left(\frac{K_{0,n}J_0(K_{0,n}r_{0j})}{(K_{0,n}^2 - K_0^2)J_1(K_{0,n})}\right)\right] \\ & \left. + \sum_{j=1}^{M}\sum_{l=1,\ l\neq j}^{M}\left(\frac{\pi J_0(K_0 r_{0l})J_0(K_0 r_{0j})Y_0(K_0 R_{jl})}{2K_0 J_1^2(K_0)}\right)\right\}\frac{1}{|\ln c|^2} + \cdots,\end{aligned} \tag{58}$$

where $\gamma \approx 0.5772$, R_{jl} is the distance between the center points P_j and P_l, J_s is the sth order *Bessel* function, Y_s is the sth order *Neumann* function, $K_{s,t}$ is the tth zero of J_s, $s = 0, 1, 2, 3, \ldots, t = 1, 2, 3, \ldots$, and $K_0 = K_{0,1} \approx 2.4048$.

Moreover, the generalized Green's function G has a compact expression for a circular membrane of radius 1, a more compact expression for the second-order asymptotic approximation Eq. (58), is found below.

The compact expression of the generalized Green's function G for a circular membrane of radius 1 [8] is

$$G(r,\theta;\tilde{r},\tilde{\theta}) = \begin{cases} -\dfrac{1}{4}\left[J_0(K_0 r)Y_0(K_0\tilde{r}) + \dfrac{Y_0(K_0)J_0(K_0 r)J_0(K_0\tilde{r})}{K_0 J_1(K_0)} - \dfrac{rY_0(K_0)J_0(K_0\tilde{r})J_1(K_0 r)}{J_1(K_0)}\right. \\ \left. -\dfrac{(J_0(K_0 r)J_0(K_0\tilde{r})Y_1(K_0) + \tilde{r}J_0(K_0 r)Y_0(K_0)J_1(K_0\tilde{r}))}{J_1(K_0)}\right] \\ -\dfrac{1}{2}\sum\limits_{m=1}^{\infty}\cos(m(\theta - \tilde{\theta}))\left[J_m(K_0 r)Y_m(K_0\tilde{r})\right. \\ \left. -\dfrac{J_m(K_0 r)Y_m(K_0)J_m(K_0\tilde{r})}{J_m(K_0)}\right], \quad r \leqslant \tilde{r} \\ \text{interchange } r \text{ and } \tilde{r} \text{ in the above result of } r\leqslant\tilde{r}, \quad r \geqslant \tilde{r}, \end{cases} \tag{59}$$

the translational addition theorems for circular cylindrical wave functions [4,9] give

$$Y_0(K_0 r_j) = \sum_{l=-\infty}^{\infty} J_l(K_0 r_{0j})Y_l(K_0 r)\cos l(\theta - \theta_{0j}), \tag{60}$$

$$Y_m(K_0 r_j)\sin(m\theta_j) = \sum_{l=-\infty}^{\infty} J_{l-m}(K_0 r_{0j})Y_l(K_0 r)\sin(l\theta - (l-m)\theta_{0j}), \tag{61}$$

$$Y_m(K_0 r_j)\cos(m\theta_j) = \sum_{l=-\infty}^{\infty} J_{l-m}(K_0 r_{0j}) Y_l(K_0 r)\cos(l\theta - (l-m)\theta_{0j}), \tag{62}$$

$m = 1, 2, 3, \ldots$, and $J_{-i}(z) = (-1)^i J_i(z),\ Y_{-i}(z) = (-1)^i Y_i(z)$ [12], $i = 1, 2, 3, \ldots$. Hence,

$$\begin{aligned}
W_1(\tilde{r}, \tilde{\theta}) = \sum_{i=1}^{M} \int_0^{2\pi} \left.\frac{\partial G(r,\theta;\tilde{r},\tilde{\theta})}{\partial r}\right|_{r=1} V_{1i}(1,\theta)\,\mathrm{d}\theta = \sum_{i=1}^{M} \Bigg\{ -\frac{1}{4}\Bigg[& J_0(K_0\tilde{r}) Y_0'(K_0) K_0 \\
& + \frac{Y_0(K_0) J_0(K_0\tilde{r}) J_0'(K_0)}{J_1(K_0)} - \frac{\tilde{r} Y_0(K_0) J_1(K_0\tilde{r}) J_0'(K_0) K_0}{J_1(K_0)} \\
& - \frac{J_0(K_0\tilde{r}) J_0'(K_0) K_0 Y_1(K_0) + J_0(K_0\tilde{r}) Y_0(K_0)(J_1(K_0) + J_1'(K_0) K_0)}{J_1(K_0)}\Bigg] \\
& \times \Bigg[\frac{-B_{0i} J_0(K_0 c)}{Y_0(K_0 c)} 2\pi J_0(K_0 r_{0i}) Y_0(K_0) \\
& - \sum_{m=1}^{\infty} \frac{J_m(K_0 c)}{Y_m(K_0 c)} 2\pi A_{mi} J_{-m}(K_0 r_{0i}) Y_0(K_0)\sin(m\theta_{0i}) \\
& - \sum_{m=1}^{\infty} \frac{J_m(K_0 c)}{Y_m(K_0 c)} 2\pi B_{mi} J_{-m}(K_0 r_{0i}) Y_0(K_0)\cos(m\theta_{0i}) \Bigg]\Bigg\} \\
& - \Bigg\{ \frac{\pi}{2} \sum_{m=1}^{\infty} \Bigg[J_m(K_0\tilde{r}) Y_m'(K_0) K_0 - \frac{J_m(K_0\tilde{r}) Y_m(K_0) J_m'(K_0) K_0}{J_m(K_0)} \Bigg] \\
& \times \Bigg[\frac{-B_{0i} J_0(K_0 c)}{Y_0(K_0 c)} 2 J_m(K_0 r_{0i}) Y_m(K_0)\cos(m\tilde{\theta} - m\theta_{0i}) \\
& + \sum_{p=1}^{\infty} A_{pi} \frac{J_p(K_0 c)}{Y_p(K_0 c)} (J_{m-p}(K_0 r_{0i}) Y_m(K_0)\sin(m\tilde{\theta} - (m-p)\theta_{0i}) \\
& + J_{-m-p}(K_0 r_{0i}) Y_{-m}(K_0)\sin(m\tilde{\theta} - (m+p)\theta_{0i})) \\
& - \sum_{p=1}^{\infty} B_{pi} \frac{J_p(K_0 c)}{Y_p(K_0 c)} (J_{m-p}(K_0 r_{0i}) Y_m(K_0)\cos(m\tilde{\theta} - (m-p)\theta_{0i}) \\
& + J_{-m-p}(K_0 r_{0i}) Y_{-m}(K_0)\cos(m\tilde{\theta} - (m+p)\theta_{0i})) \Bigg]\Bigg\}.
\end{aligned} \tag{63}$$

The translational addition theorems for circular cylindrical wave functions [4,9] give

$$J_0(K_0\tilde{r}) = \sum_{l=-\infty}^{\infty} J_l(K_0 r_{0j}) J_l(K_0 r_j)\cos\big(l\theta_j - l(\theta_{0j} + \pi)\big), \tag{64}$$

$$J_p(K_0\tilde{r})\sin(p\tilde{\theta}) = \sum_{l=-\infty}^{\infty} J_{l-p}(K_0 r_{0j}) J_l(K_0 r_j)\sin(l\theta_j - (l-p)(\theta_{0j} + \pi)), \tag{65}$$

$$J_p(K_0\tilde{r})\cos(p\tilde{\theta}) = \sum_{l=-\infty}^{\infty} J_{l-p}(K_0 r_{0j}) J_l(K_0 r_j)\cos(l\theta_j - (l-p)(\theta_{0j} + \pi)), \tag{66}$$

$p = 1, 2, 3, \ldots$. Hence, by *Law of cosine*, $J_0'(z) = -J_1(z)$, $Y_0'(z) = -Y_1(z)$, Wronskian $\mathscr{W}$ of $J_m(z)$ and $Y_m(z)$ [12] which is $\mathscr{W}(J_m(z), Y_m(z)) = 2/\pi z$, and $B_{0i} = J_0(K_0 r_{0i})$ [8, Eq. (96)],

$$\begin{aligned}
&\frac{-1}{2\pi Y_0(K_0 c)} \int_0^{2\pi} W_1(r_j, \theta_j)\bigg|_{r_j=c} \mathrm{d}\theta_j \\
&= \sum_{i=1}^{M} \frac{J_0(K_0 r_{0i})\pi^2}{2} \left[\frac{\pi J_0(K_0 r_{0i}) J_0(K_0 r_{0j}) Y_0^2(K_0)}{2} - \frac{\pi r_{0j} K_0 J_0(K_0 r_{0i}) J_1(K_0 r_{0j}) Y_0^2(K_0)}{4} \right. \\
&\quad + \frac{\pi K_0 J_0(K_0 r_{0i}) J_0(K_0 r_{0j}) Y_0^2(K_0) J_1'(K_0)}{4 J_1(K_0)} \\
&\quad \left. - \sum_{m=1}^{\infty} \frac{J_m(K_0 r_{0i}) J_m(K_0 r_{0j}) Y_m(K_0) \cos(m\theta_{0i} - m\theta_{0j})}{J_m(K_0)} \right] \frac{1}{|\ln c|^2} + \cdots . \qquad (67)
\end{aligned}$$

Therefore, following the derivation for the second correction term F_2 in Section 3, a more compact expression for the second-order asymptotic approximation Eq. (58) is found to be

$$\begin{aligned}
K = K_0 &+ \sum_{j=1}^{M} \left(\frac{J_0^2(K_0 r_{0j})}{K_0 J_1^2(K_0)} \right) \frac{1}{|\ln c|} \\
&+ \left\{ \sum_{j=1}^{M} \left(\frac{(\ln K_0 + \gamma - \ln 2) J_0^2(K_0 r_{0j})}{K_0 J_1^2(K_0)} \right) + \sum_{j=1}^{M} \sum_{i=1}^{M} \left[-\frac{J_0^2(K_0 r_{0j}) J_0^2(K_0 r_{0i})}{2 K_0^3 J_1^4(K_0)} \right. \right. \\
&- \frac{J_0(K_0 r_{0j}) J_0^2(K_0 r_{0i}) \int_0^{2\pi} \sqrt{1 + r_{0j}^2 - 2 r_{0j} \cos\theta}\, Y_1(K_0 \sqrt{1 + r_{0j}^2 - 2 r_{0j} \cos\theta})\, \mathrm{d}\theta}{4 K_0 J_1^3(K_0)} \\
&+ \frac{\pi^2 J_0^2(K_0 r_{0i}) J_0^2(K_0 r_{0j}) Y_0^2(K_0)}{2 K_0 J_1^2(K_0)} + \frac{\pi^2 J_0^2(K_0 r_{0i}) J_0^2(K_0 r_{0j}) Y_0^2(K_0) J_1'(K_0)}{4 J_1^3(K_0)} \\
&- \frac{\pi^2 r_{0j} J_0^2(K_0 r_{0i}) J_0(K_0 r_{0j}) J_1(K_0 r_{0j}) Y_0^2(K_0)}{4 J_1^2(K_0)} \\
&\left. - \frac{\pi J_0(K_0 r_{0i}) J_0(K_0 r_{0j})}{K_0 J_1^2(K_0)} \sum_{m=1}^{\infty} \frac{J_m(K_0 r_{0i}) J_m(K_0 r_{0j}) Y_m(K_0) \cos(m\theta_{0i} - m\theta_{0j})}{J_m(K_0)} \right] \\
&\left. + \sum_{j=1}^{M} \sum_{l=1,\ l \neq j}^{M} \left(\frac{\pi J_0(K_0 r_{0l}) J_0(K_0 r_{0j}) Y_0(K_0 R_{jl})}{2 K_0 J_1^2(K_0)} \right) \right\} \frac{1}{|\ln c|^2} + \cdots, \qquad (68)
\end{aligned}$$

where $\gamma \approx 0.5772$, R_{jl} is the distance between the centers of the inner circular cores B_j and B_l, whose center points are $P_j = (r_{0j}, \theta_{0j})$ and $P_l = (r_{0l}, \theta_{0l})$, respectively. J_s is the sth order *Bessel* function, Y_s is the sth order *Neumann* function, and $K_0 = K_{0,1} \approx 2.4048$ is the *first* zero of J_0.

Table 1
The fundamental frequency K for a circular membrane of radius 1 with an eccentric circular core of radius 0.1 centered at $(r, \theta) = (r_0, 0)$

r_0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Lin	3.325	3.328	3.188	3.083	2.980	2.868	2.780	2.700	2.620	2.500
Nagaya	3.310	3.241	3.145	3.042	2.880	2.759	2.639	2.578	2.518	—
The first-order approximation	3.075	3.056	3.001	2.917	2.814	2.706	2.603	2.516	2.453	2.416
The second-order approximation	3.299	3.261	3.157	3.014	2.864	2.731	2.622	2.536	2.470	2.424
The exact value	3.314	—	—	—	—	—	—	—	—	—

Table 2
The fundamental frequency K for a circular membrane of radius 1 with two eccentric circular cores of radius 0.1 centered at $(r,\theta)=(r_0,0)$ and $(r,\theta)=(r_0,\pi)$, respectively

r_0	0	0.08	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Nagaya and Poltorak	3.313	3.543	3.802	3.842	3.629	3.295	3.002	2.783	2.625	2.516
The first-order approximation	3.075	3.720	3.597	3.429	3.224	3.006	2.801	2.628	2.501	2.428
The second-order approximation	3.299	3.571	3.909	3.792	3.511	3.190	2.907	2.692	2.540	2.443
The exact value	3.314	—	—	—	—	—	—	—	—	—

Table 3
The fundamental frequency K for a circular membrane of radius 1 with two eccentric circular cores of radius c centered at $(r,\theta)=(0.5,0)$ and $(r,\theta)=(0.5,\pi)$, respectively

c	0.1	0.2	0.3	0.4
Nagaya and Poltorak	3.295	3.882	4.548	5.194
The first-order approximation	3.006	3.265	3.555	3.916
The second-order approximation	3.190	3.641	4.226	5.074

Table 4
The fundamental frequency K for a circular membrane of radius 1 with four eccentric circular cores of radius 0.1 centered at $(r,\theta)=(r_0,[(j-1)\pi]/2)$, $j=1,2,3,4$, respectively

r_0	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Nagaya and Poltorak	4.206	4.718	5.186	4.655	3.792	3.229	2.863	2.652
The first-order approximation	4.788	4.453	4.043	3.608	3.196	2.850	2.597	2.450
The second-order approximation	4.452	4.975	4.700	4.102	3.491	3.011	2.683	2.482

Table 5
The fundamental frequency K for a circular membrane of radius 1 with four eccentric circular cores of radius c centered at $(r,\theta)=(0.5,[(j-1)\pi]/2)$, $j=1,2,3,4$, respectively

c	0.05	0.1	0.15	0.2	0.25
Nagaya and Poltorak	3.906	4.655	5.614	6.710	7.620
The first-order approximation	3.329	3.608	3.865	4.126	4.403
The second-order approximation	3.621	4.102	4.593	5.137	5.766

There are also some existing results on the fundamental frequency of a circular membrane with inner circular cores. In 1977, Nagaya [11] considered a circular membrane with an eccentric inner circular core. The author used the exact solution of the equation of motion which satisfies the inner boundary condition and adopted the Fourier expansion method on the outer boundary condition to calculate the fundamental frequency of the membrane. In 1981, Lin [4] considered an equivalent problem to the fundamental frequency of a circular membrane with inner circular cores, in which both the equation of motion and the boundary conditions were exactly satisfied and the technique of transformation of cylindrical wave functions was then used to determinate the fundamental frequency of the membrane. In 1989, Nagaya and Poltorak [5] considered a circular membrane with a number of eccentric inner circular cores. The authors used the point-matching approach to treat the inner boundary conditions and presented the expression to find the fundamental frequency of the membrane. The comparisons among the asymptotic approximations Eq. (68) and the numerical values computed by the above investigators are shown in Tables 1–5.

5. Results and discussion

The fundamental frequency of a multiply connected membrane with inner cores of vanishing maximal dimensions is concerned in the present article. A general formula for the fundamental frequency is derived by the extension of the modified perturbation method [8] and found to be the formula Eq. (6) with Eqs. (22)–(24). A higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency of a membrane with inner circular cores of radius c_j is specified as the approximation Eq. (57). It is an excellent extension for the second-order result [8, Eq. (85)] derived by the author for a doubly connected membrane with a vanishing inner circular core and an excellent extension for the lower order result [7, Eq. (8)] derived by Wang for a multiply connected membrane with vanishing inner circular cores.

Moreover, a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency of a circular membrane of radius 1 with M inner circular cores of small radius c is found explicitly as the approximation Eq. (68). Observing from the second-order asymptotic approximation Eq. (68), it is found that the positions of the inner cores related to the membrane affect the approximation starting at the first correction term, while the inter-positions of the inner cores affect the approximation starting at the second correction term. Also, the comparisons among the asymptotic approximations and the numerical values [4,5,11] computed by other investigators are made and shown in Tables 1–5. Observing from the tables, it is found that the second-order asymptotic approximation achieves more accuracy than the first-order asymptotic approximation does. Moreover, Nagaya and Poltorak [5] pointed out that the fundamental frequency decreases as the eccentricity increases for a circular membrane with an inner circular core and the fundamental frequency increases first and then decreases as the eccentricity increases for a circular membrane with more than one inner circular cores. The second-order asymptotic approximation depicts this phenomenon, while the first-order asymptotic approximation does not.

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