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Short Communication

Small amplitude, transverse vibrations of circular plates with an eccentric rectangular perforation elastically restrained against rotation and translation on both edges

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Abstract

The Rayleigh–Ritz variational method is applied to the determination of the first four frequency coefficients for small amplitude, transverse vibrations of circular plates with an eccentric, rectangular perforation that is elastically restrained against rotation and translation on both edges. Coordinate functions are used which identically satisfy the boundary conditions at the outer circular edge, while the restraining boundary conditions at the inner edge of the cutout are dealt with directly through the energetic terms in the functional expressions. The procedure seems to show very good numerical stability and convergence properties. As an added bonus, the method allows for increased flexibility in dealing with boundary conditions at the edge of the cutout.

1. Introduction

Although there is a large amount of papers on membrane vibration with holes in the open technical literature, only relatively a few can be found for vibrating plates. A rather complete literature review on the subject can be found in Ref. [1].

This paper focuses on a situation which may appear in real life vibrating systems. A degree of eccentricity in an internal boundary may be required by a practical reason like passage of a cable or any other type of conduit of smaller size than the circular plate. If, additionally, the cutout is meant to keep both sides of the plate airtight, for example, then boundary conditions involving restraints against rotation and translation on both edges of the plate will arise.

Previous work has shown that the use of coordinate functions that identically satisfy the boundary conditions at the (outer) circular boundary has the added benefit of increased accuracy in finding the frequency coefficients of vibrating circular plates with edges elastically restrained against rotation and

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translation [2]. In this work, the problem of dealing with the restrained (inner) rectangular edges is tackled by introducing the boundary restrains directly into the energetic terms of the functional expression, which additionally provides a greater degree of flexibility. Rayleigh–Ritz procedure ensures that the approximated solution will converge to the actual one for the system in hand. The numerical experiments seem to show a greater degree of accuracy and an improvement in the convergence properties of the results.

2. Approximate analytical solution

In the case of normal modes of vibration of the vibrating system shown in Fig. 1, one takes

$$w'(r',\theta,t) = W'(r',\theta)e^{i\omega t} \tag{1}$$

for the plate transverse displacement and then introduces the following approximation, convenient in the case of both axisymmetric and antisymmetric modes of vibration, see for example Refs. [2,3]:

$$W'(r',\theta) \cong W'_a(r',\theta) = \sum_{j=0}^{J} A_{j0}(\alpha_{jk}r'^4 + \beta_{jk}r'^2 + 1)r'^{2j} + \sum_{k=1}^{K}\cos(k\theta)\sum_{j=1}^{J} A_{jk}(\alpha_{jk}r'^4 + \beta_{jk}r'^2 + 1)r'^{j+k}, \tag{2}$$

where α's and β's of each coordinate function are determined substituting each functional relation in the governing boundary conditions at the external (circular) contour. For an elastically restrained edge against rotation and translation, those conditions are:

$$k_O W'(a,\theta) = V_r(a,\theta), \quad \frac{\partial W'(r',\theta)}{\partial r'}(a,\theta) = \phi_O M_r(a,\theta), \tag{3}$$

where M_r, V_r are the radial flexural-moment and shear-force, ϕ_O, k_O are the elastic constants of the rotational and translational springs at the outer circular boundary, and a is the radius of the circular plate.

Note that in Eq. (2) a term in $\sin(k\theta)$ has been not included. The reason is that, as shown in Ref. [4], antisymmetric modes are already present in the $\cos(k\theta)$ term and, furthermore, when the analytical expressions are written down for the geometry shown in Fig. 1 it is seen that any contribution to the functional coming from the sinusoidal term vanishes due to the symmetry of the hole with respect to the x-axis.

The Rayleigh–Ritz variational approach requires minimization of the functional

$$J[W'] = U_T[W'] - T[W'], \tag{4}$$

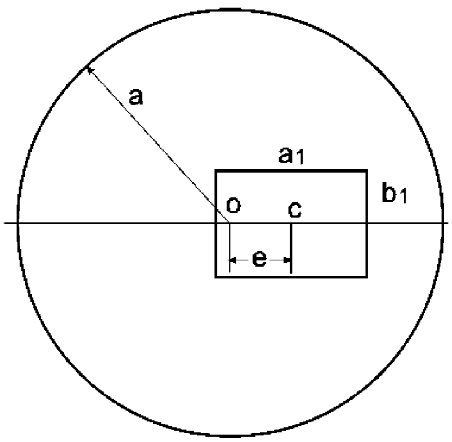

Fig. 1. Vibrating mechanical system. Case of an eccentric rectangular cutout with edges elastically restrained against rotation and translation. In the figure, o and c are the centers of the circular plate and rectangular cutout, respectively.

where $T[W']$ and $U_T[W']$ are the (maximum) kinetic and potential energies for the (true) displacement amplitude W' of the plate. The potential energy term in Eq. (4) can be written as

$$U_T[W'] = U_P[W'] + U_B[W'], \tag{5}$$

where $U_P[W']$ is the maximum strain energy of the plate and $U_B[W']$ is the sum of potential energies for the translational and rotational springs at the outer boundary of the circular plate and the inner contour of the rectangular cutout.

As has been shown elsewhere, see for example Ref. [5], in the case of a circular plate each term in Eq. (5) can be written as

$$U_P[W'] = \frac{D}{2}\iint\left\{\left[\left(\frac{\partial^2 W'}{\partial r'^2}+\frac{1}{r'}\frac{\partial W'}{\partial r'}\right)+\left(\frac{1}{r'^2}\frac{\partial^2 W'}{\partial \theta^2}\right)\right]^2 - 2(1-\mu)\left[\frac{\partial^2 W'}{\partial r'^2}\left(\frac{1}{r'}\frac{\partial W'}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2 W'}{\partial \theta^2}\right)-\left(\frac{1}{r'}\frac{\partial^2 W'}{\partial \theta^2 \partial r'}-\frac{1}{r'^2}\frac{\partial W}{\partial \theta}\right)^2\right]\right\} r'\,\mathrm{d}r'\,\mathrm{d}\theta, \tag{6}$$

where D is the flexural rigidity of the plate, μ is its Poison ratio, and

$$T[W'] = \frac{\rho\omega^2 h}{2}\iint W'^2 r'\,\mathrm{d}r'\,\mathrm{d}\theta. \tag{7}$$

The integrals in the expressions given in Eqs. (6) and (7) extend over the actual area of the double connected plate under study, i.e. over the surface of the circular plate not including the inner rectangular hole of size a_1 and b_1 centered at $x = e$, see Fig. 1.

The expressions for the potential energies of the restraining (linear) translational and rotational springs are the well-known classical expressions

$$U_B[W'] = \frac{1}{2}\int_{OB}\left[k_O W'^2 + \frac{1}{\phi_O}\left(\frac{\partial W'}{\partial r'}\right)^2\right]\mathrm{d}s + \frac{1}{2}\int_{IB}\left[k_I W'^2 + \frac{1}{\phi_I}\left(\frac{\partial W'}{\partial r'}\right)^2\right]\mathrm{d}s, \tag{8}$$

where the first integration is to be taken along the path of the outer circular edge of the plate (OB) and the second integral is to be taken along path of the inner boundary of the rectangular cutout (IB).

Introducing the non-dimensional variables:

$$W = W'/a, \quad r = r'/a, \tag{9}$$

all equations can be recast in a non-dimensional form. In this case, the elastic constants k_O and k_I of Eq. (8) are transformed into the non-dimensional counterparts:

$$k_O a^3/D \text{ and } k_I a^3/D,$$

respectively. Likewise, the rotational-spring elastic coefficients $(1/\Phi_O)$ and $(1/\Phi_I)$ are transformed into:

$$a/(\Phi_O D) \text{ and } a/(\Phi_I D),$$

respectively.

Minimizing the governing functional with respect to the A_{jk}'s expression (4) yields an $(J \times K)$ homogeneous, linear system of equations in the A_{jk}'s. A secular determinant in the natural non-dimensional frequency coefficients of the system $\Omega_i = \sqrt{\rho h/D}\,\omega_i a^2$ results from the non-triviality condition.

The present study is concerned with the determination of the first four frequency coefficients, Ω_1 to Ω_4, in the case of circular plates with an eccentric rectangular cutout with both the external boundary of the circular plate and the inner edge of the rectangular cutout elastically restrained against rotation and translation.

3. Numerical results

All calculations were performed for circular plates of uniform thickness simply supported and clamped at the outer edge $r = a$ while the inner edge of the eccentric rectangular perforation was taken to be elastically restrained against rotation and translation. In all cases, the Poisson coefficient was taken to be $\mu = 0.3$.

A total of eight tables are presented. Tables 1 and 2 show a comparison between this work and a finite element calculation [6] for the first three frequency coefficients. As can been seen from the tables the agreement looks very good. Tables 3–5 depict values for the first four frequency coefficients for a rectangular cutout of size $a_1/a = 0.15$ and $b_1/a = 0.1$ when the center of the cutout is displaced along a radial line of the circular plate. Likewise, Tables 6–8 depict values for the case in which a square cutout of size $a_1/a = 0.3$ is being

Table 1
A comparison of frequency coefficients for the first three modes with a finite element (FE) calculation for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Method	Ω_1	Ω_2	Ω_3
Simply supported	0	0	This work	4.901	13.89	25.50
			FE	4.870	13.77	25.07
		∞	This work	5.070	18.01	26.22
			FE	5.010	16.11	25.27
	∞	0	This work	14.51	15.91	25.51
			FE	14.63	16.41	25.12
		∞	This work	17.10	19.10	26.26
			FE	16.45	18.04	25.61
Clamped	0	0	This work	10.25	21.25	34.71
			FE	10.08	20.94	33.92
		∞	This work	10.56	26.09	35.84
			FE	10.38	23.83	34.22
	∞	0	This work	22.32	23.93	34.72
			FE	22.47	24.50	34.00
		∞	This work	25.93	28.49	35.91
			FE	25.04	26.75	34.77

Case of a circular plate with rectangular cutout of size $a_1/a = 0.15$, and $b_1/a = 0.1$ centered at $e/a = 0.0$.

Table 2
A comparison of frequency coefficients for the first three modes with a finite element (FE) calculation for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Method	Ω_1	Ω_2	Ω_3
Simply supported	0	0	This work	4.874	13.88	25.95
			FE	4.770	13.52	25.09
	∞	∞	This work	10.75	24.43	35.52
			FE	10.25	23.67	30.86
Clamped	0	0	This work	10.31	21.77	35.88
			FE	10.12	20.88	33.24
	∞	∞	This work	16.42	33.38	44.56
			FE	15.64	32.16	40.63

Case of a circular plate with square cutout of size $a_1/a = b_1/a = 0.3$ centered at $e/a = 0.5$.

Table 3
Values of the first four frequency coefficients in the case of a circular plate with rectangular cutout of size $a_1/a = 0.15$, and $b_1/a = 0.1$ centered at $e/a = 0.1$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I\, a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.908	13.89	25.51	29.60
		1	4.920	14.01	25.53	29.72
		∞	5.114	18.26	26.82	32.58
	1000	0	10.96	15.22	25.58	36.18
		1	10.99	15.33	25.60	36.21
		∞	11.94	19.15	27.13	37.57
	∞	0	13.02	17.63	25.83	39.94
		1	13.08	17.70	25.84	39.94
		∞	14.89	20.71	27.40	40.23
Clamped	0	0	10.20	21.25	34.73	39.63
		1	10.23	21.39	34.75	39.79
		∞	10.65	26.92	36.71	44.23
	1000	0	16.81	22.96	34.85	45.66
		1	16.84	23.08	34.87	45.71
		∞	17.84	27.96	37.21	48.04
	∞	0	20.02	26.46	35.30	51.00
		1	20.09	26.56	35.33	51.01
		∞	22.61	30.45	37.95	51.54

Table 4
Values of the first four frequency coefficients in the case of a circular plate with rectangular cutout of size $a_1/a = 0.15$, and $b_1/a = 0.1$ centered at $e/a = 0.5$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I\, a^3/D$ at the inner edge	$a/(\Phi_I\, D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.935	13.91	25.67	29.67
		1	5.006	13.92	25.69	29.79
		∞	6.751	14.52	26.25	32.98
	1000	0	7.530	18.72	29.22	30.92
		1	7.571	18.73	29.25	30.99
		∞	8.808	19.17	29.62	33.17
	∞	0	8.375	20.78	30.53	34.79
		1	8.390	20.79	30.56	34.79
		∞	9.314	21.45	32.52	35.07
Clamped	0	0	10.23	21.35	35.02	39.73
		1	10.33	21.39	35.04	39.81
		∞	13.05	23.24	35.80	42.27
	1000	0	12.12	25.51	38.15	41.46
		1	12.17	25.55	38.17	41.48
		∞	14.00	27.06	38.96	42.69
	∞	0	13.22	28.28	39.62	45.31
		1	13.24	28.30	39.64	45.31
		∞	14.51	29.61	41.50	45.55

Table 5
Values of the first four frequency coefficients in the case of a circular plate with rectangular cutout of size $a_1/a = 0.15$ and $b_1/a = 0.1$ centered at $e/a = 0.75$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I\, a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.932	13.90	25.64	29.71
		1	5.041	14.02	25.72	29.73
		∞	7.182	18.47	28.77	32.51
	1000	0	6.207	16.57	27.70	31.61
		1	6.235	16.62	27.73	31.64
		∞	7.247	18.82	28.90	33.48
	∞	0	6.731	18.01	28.38	33.65
		1	6.740	18.03	28.39	33.66
		∞	7.323	19.21	29.01	34.50
Clamped	0	0	10.19	21.25	34.92	39.81
		1	10.26	21.41	35.07	39.87
		∞	11.81	25.64	38.06	43.90
	1000	0	10.65	22.59	36.30	40.83
		1	10.68	22.66	36.35	40.87
		∞	11.82	25.68	38.07	44.00
	∞	0	11.14	24.10	37.37	42.70
		1	11.14	24.12	37.38	42.72
		∞	11.84	25.78	38.11	44.26

Table 6
Values of the first four frequency coefficients in the case of a circular plate with square cutout of size $a_1/a = 0.3$, centered at $e/a = 0.1$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.759	13.78	25.11	30.21
		1	4.857	14.08	25.21	30.66
		∞	5.828	23.02	30.50	41.14
	1000	0	13.29	18.11	25.68	39.74
		1	13.48	18.31	25.80	39.77
		∞	17.27	24.89	31.59	42.07
	∞	0	15.72	22.57	26.86	40.14
		1	15.89	22.69	27.02	40.17
		∞	20.15	27.30	33.54	42.50
Clamped	0	0	10.27	21.09	34.13	40.81
		1	10.42	21.43	34.26	41.30
		∞	12.32	33.41	42.42	54.41
	1000	0	20.55	26.96	35.27	50.66
		1	20.73	27.15	35.44	50.72
		∞	24.94	35.23	43.93	54.59
	∞	0	24.12	33.09	37.98	51.37
		1	24.35	33.18	38.25	51.41
		∞	30.23	38.98	48.48	55.32

Table 7
Values of the first four frequency coefficients in the case of a circular plate with square cutout of size $a_1/a = 0.3$, centered at $e/a = 0.5$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I\, a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.874	13.88	25.95	29.38
		1	5.077	14.05	26.06	29.67
		∞	8.491	16.58	28.31	36.17
	1000	0	8.443	20.69	30.74	34.06
		1	8.514	20.71	30.87	34.09
		∞	10.46	22.79	33.79	37.03
	∞	0	9.025	22.00	31.90	36.34
		1	9.087	22.03	32.00	36.38
		∞	10.75	24.43	35.52	40.35
Clamped	0	0	10.31	21.77	35.88	39.66
		1	10.52	22.01	36.03	39.95
		∞	15.32	27.43	39.27	45.71
	1000	0	13.36	28.34	39.74	44.24
		1	13.45	28.37	39.85	44.29
		∞	16.09	31.51	43.18	47.31
	∞	0	14.14	30.26	40.85	47.17
		1	14.22	30.32	40.96	47.20
		∞	16.42	33.38	44.56	51.60

Table 8
Values of the first four frequency coefficients in the case of a circular plate with square cutout of size $a_1/a = 0.3$, centered at $e/a = 0.75$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I\, a^3/D$ at the inner edge	$a/(\Phi_I D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.902	13.87	25.81	29.64
		1	5.163	14.20	26.06	29.80
		∞	8.103	20.97	29.74	36.51
	1000	0	6.782	18.17	28.41	33.56
		1	6.829	18.23	28.44	33.58
		∞	8.108	21.01	29.75	36.68
	∞	0	7.163	19.19	28.84	35.15
		1	7.192	19.23	28.86	35.17
		∞	8.131	21.18	29.76	37.28
Clamped	0	0	10.13	21.31	35.21	39.96
		1	10.26	21.52	35.35	40.04
		∞	12.77	28.09	38.79	47.09
	1000	0	11.19	24.29	37.39	42.74
		1	11.24	24.37	37.43	42.78
		∞	12.77	28.10	38.79	47.13
	∞	0	11.62	25.57	37.95	44.65
		1	11.66	25.63	37.97	44.69
		∞	12.78	28.15	38.79	47.39

Table 9
Values of the first four frequency coefficients in the case of a circular plate with square cutout of size $a_1/a = 0.5$, centered at $e/a = 0.5$ for different values of the inner non-dimensional elastic constants of the translational and rotational springs

Outer circular border	$k_I\,a^3/D$ at the inner edge	$a/(\Phi_I\,D)$ at the inner edge	Ω_1	Ω_2	Ω_3	Ω_4
Simply supported	0	0	4.659	14.11	26.86	28.21
		1	5.123	14.83	27.39	28.64
		∞	9.507	19.19	32.37	40.32
	1000	0	9.503	22.38	32.79	35.72
		1	9.701	22.46	33.08	35.78
		∞	12.02	25.46	36.95	41.15
	∞	0	10.10	24.43	34.16	39.47
		1	10.29	24.54	34.46	39.60
		∞	12.52	27.81	39.42	44.01
Clamped	0	0	10.22	22.58	37.11	40.78
		1	10.47	23.09	37.36	41.36
		∞	15.99	27.42	42.14	51.14
	1000	0	14.87	31.08	41.71	45.84
		1	15.10	31.15	42.00	45.94
		∞	17.99	34.18	46.14	52.18
	∞	0	15.61	33.79	43.21	51.04
		1	15.84	33.92	43.52	51.18
		∞	18.99	37.96	48.26	57.10

displaced along a radial line. Finally, Table 9 shows values for the frequency coefficients for a square hole at a fixed position $e/a = 0.5$ of its center.

For the double series, Eq. (2), up to 96 terms have been taken ($J = 16$ and $K = 6$) for all situations. Although satisfactory convergence is already achieved for $J = 8$ and $K = 4$, such high values of J and K have been used taking advantage of the speed of modern desktop computers. As usual, special care has been taken to manipulate the numerical solving of the involved determinants and 80 bits floating point variables (IEEE-standard temporary reals) have been employed to satisfy accuracy requirements. It is worth noting that computations are very stable and all frequency coefficients uniformly converge as the number of terms in the double series is increased.

As stated above, the methodology has the added benefit of providing a great degree of flexibility in dealing with different types of boundary conditions at the edge of the cutout and it could be generalized to rectangular cutouts of arbitrary orientation or even cutouts of any polygonal shape. In these cases terms in $\sin(k\theta)$ should be added to Eq. (2). As a general conclusion one may say that the mathematical model seems to be quite realistic and accurate, within the realm of the classical theory of vibrating plates.

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