Error Estimate (Accuracy) and Effective Condition Number (Stability) of Collocation Trefftz Method, Method of Fundamental Solutions, and Radial Basis Function Method

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(Point) Collocation Methods

- Trefftz method
- Method of fundamental solutions
- Radial basis function collocation

Why Collocation Method?

- High accuracy (exponential error convergence)
- Simplicity in formulation
- Meshless
- Boundary method (Trefftz, MFS)
- Solve ill-posed BVP without iteration
- Easy to adapt to *n*-dimensional problem

CONSIDERATIONS OF NUMERICAL METHODS

Considerations

- Is it suitable for engineering applications, such as arbitrary geometry?
- Is it efficient? (CPU time)
- Is it accurate? (Coupled with efficiency)
- Is the theory easy to understand?
- Is it easy to write computer program?

- Is it general enough to solve linear or nonlinear, homogeneous or inhomogeneous, constant or variable coefficients, and all kinds of governing equations?
- Are commercial software widely available?
- Is there inertia (people are comfortable with the method they use)?

TREFFTZ METHOD

Original Trefftz Method

- Solve the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{in } \Omega$$

$$u = f(\boldsymbol{x}) \quad \text{in } \Gamma$$

- Utilize the Ritz method

Ritz Method

- Approximate the solution using a set of trial functions $\phi_i(\boldsymbol{x})$

$$u(\boldsymbol{x}) \approx \sum_{i=1}^{n} \alpha_i \, \phi_i(\boldsymbol{x})$$

- Minimize the functional

$$\Pi = -\iiint_{\Omega} \frac{1}{2} u \nabla^2 u \, d\boldsymbol{x} - \iint_{\Gamma} \frac{\partial u}{\partial n} \left(\frac{1}{2} u - f \right) d\boldsymbol{x}$$

with respect to the trial functions

Walter Ritz
(1878-1909)

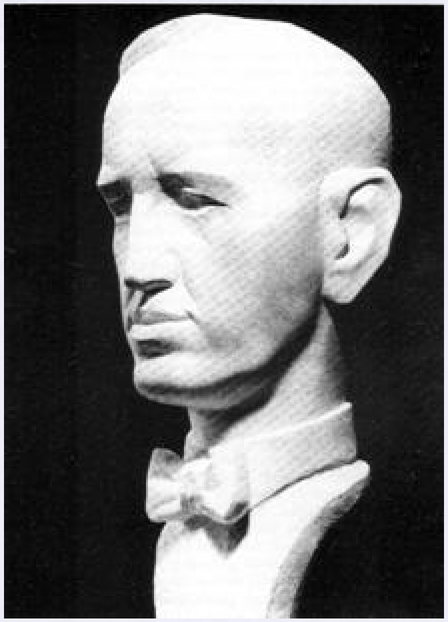
Trefftz's Contribution

- Require trial functions to satisfy the governing equation

$$\nabla^2 \phi_i = 0$$

- The functional reduces to

$$\Pi = -\iint_{\Gamma} \sum_{i=1}^{n} \alpha_i \frac{\partial \phi_i}{\partial n} \left(\frac{1}{2} \sum_{i=1}^{n} \alpha_i \phi_i - f \right) d\boldsymbol{x}$$



Erich Trefftz
(1888-1937)

Solution System

- Use harmonic polynomials

$$\phi_i = \left\{1, x, y, z, x^2 - y^2, y^2 - z^2, xy, yz, xz, \ldots\right\}$$

- Linear solution system

$$\sum_{i=1}^{n} a_{ij}\alpha_i = b_j; \quad j = 1, \ldots, n$$

$$a_{ij} = \frac{1}{2}\iint_{\Gamma} \frac{\partial \phi_i \phi_j}{\partial n}\, d\mathbf{x}; \quad b_j = \iint_{\Gamma} f\, \frac{\partial \phi_j}{\partial n}\, d\mathbf{x}$$

Other Formulations of Trefftz Method

- Trefftz-Herrera formulation
- Weighted residual formulation
- Collocation formulation
- Hybrid methods

METHOD OF FUNDAMENTAL SOLUTIONS

Kupradze Formulation

- Solution given by

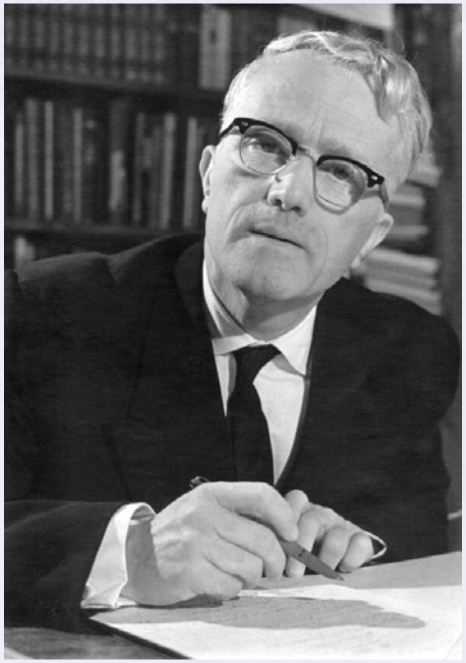
$$u(\boldsymbol{x}) = \frac{1}{2\pi}\iint_{\Gamma} f(\boldsymbol{x}')\frac{\partial G(\boldsymbol{x},\boldsymbol{x}')}{\partial n(\boldsymbol{x}')}\,d\boldsymbol{x}'$$

$$+\frac{1}{\pi}\iint_{\Gamma}\sigma(\boldsymbol{x}')G(\boldsymbol{x},\boldsymbol{x}')\,d\boldsymbol{x}'; \quad \boldsymbol{x}\in\Omega$$

where $\sigma(\boldsymbol{x}')$ is found from

$$0 = \frac{1}{2\pi}\iint_{\Gamma} f(\boldsymbol{x}')\frac{\partial G(\boldsymbol{x},\boldsymbol{x}')}{\partial n(\boldsymbol{x}')}\,d\boldsymbol{x}'$$

$$+\frac{1}{\pi}\iint_{\Gamma}\sigma(\boldsymbol{x}')G(\boldsymbol{x},\boldsymbol{x}')\,d\boldsymbol{x}'; \quad \boldsymbol{x}\in\Gamma'$$



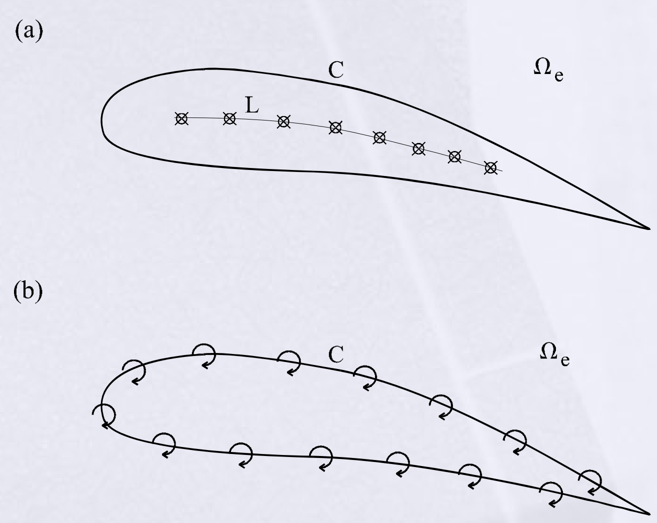
Viktor Dmitrievich Kupradze (1903-1985)

Von Karman Formulation

- Potential flow around objects

$$u(x) \approx Ux + \sum_{i=1}^{n+1} \frac{\sigma_i}{4\pi r(x, x_i)}$$

$$\sum_{i=1}^{n+1} \sigma_i = 0$$



Theodore von Karman
(1881-1963)

Other Formulations

- Point collocation (Mathon & Johnson, 1997)
- Least square, optimization formulation (Fairweather)
- Boundary integral equations
- Boundary element method.

Cheng, A.H.-D. and Cheng, D.T., "Heritage and early history of the boundary element method," ENGINEERING ANALYSIS WITH BOUNDARY ELEMENTS, Vol. 29, No. 3, pp. 268–302, 2005.

UNIFIED VIEW—METHOD OF WEIGHTED RESIDUALS

Method of Weighted Residuals

- Governing equation

$$\mathcal{L}\{u(\boldsymbol{x})\} = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$$

- Essential and natural boundary conditions

$$\mathcal{S}\{u(\boldsymbol{x})\} = g_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_S$$
$$\mathcal{N}\{u(\boldsymbol{x})\} = g_2(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_N$$

Minimizing Weighted Residual

- Approximation by trial functions

$$u(\boldsymbol{x}) \approx \sum_{i=1}^{n} \alpha_i \phi_i(\boldsymbol{x})$$

- Minimizing the residual

$$\iiint_{\Omega} \left[\mathcal{L}\{u(\boldsymbol{x})\} - f(\boldsymbol{x}) \right] w_i \, d\boldsymbol{x} + \iint_{\Gamma_S} \left[\mathcal{S}\{u(\boldsymbol{x})\} - g_1(\boldsymbol{x}) \right] w_i \, d\boldsymbol{x}$$

$$+ \iint_{\Gamma_N} \left[\mathcal{N}\{u(\boldsymbol{x})\} - g_2(\boldsymbol{x}) \right] w_i' \, d\boldsymbol{x} = 0$$

with respect to trial functions

Choosing Different Weight

- Galerkin method: Trial function as weight

$$w_i = \phi_i$$

- Subdomain method: Step function
- Collocation method: Dirac delta function

$$w_i = \delta(\boldsymbol{x}, \boldsymbol{x}_i)$$

A Simple Example

$$\frac{d^2h}{dx^2} + \frac{H_o - h}{\lambda} = 0,$$

$$h(0) = h_1 \text{ and } h(L) = h_2,$$

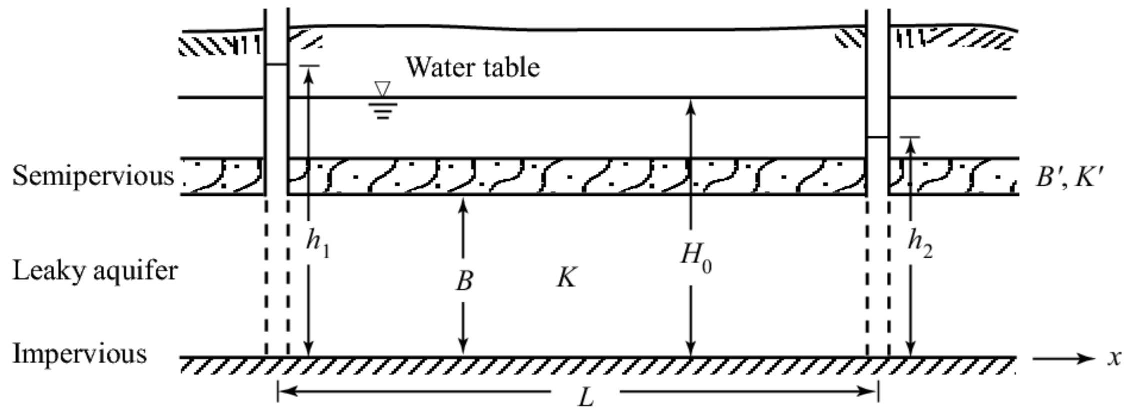


Figure 8.3.1: Flow in a leaky aquifer.

Solution Strategy

- Approximate solution

$$h \approx \hat{h} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

- Solved by collocation, subdomain, and Galerkin method
- Integration performed exactly

Solution Error

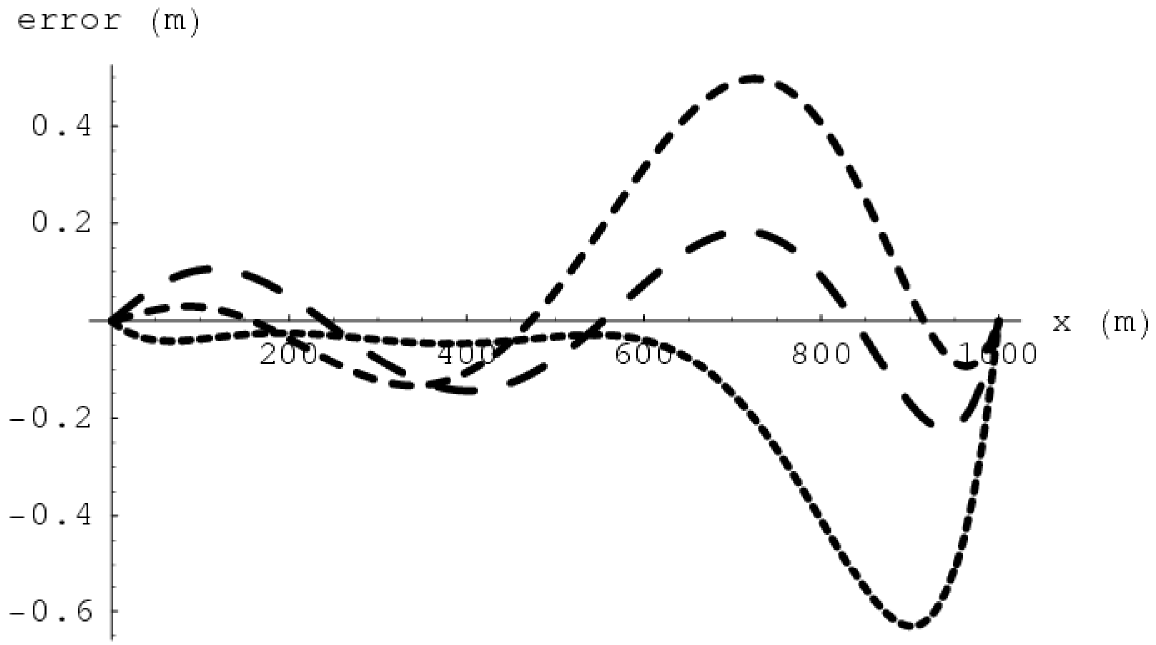


Figure 8.3.3: Comparison of error of the approximate three solutions. Short dash line: collocation method; medium dash line: subdomain method; and long dash line: Galerkin method.

Lessons Learned

- We observe that Galerkin method is the most accurate, and collocation the least.
- Integration distributes the error and point collocation concentrates the error.
- So why do we claim the point collocation has the highest accuracy?

FEM "MISTAKES"

FEM "Mistakes"

- FEM interpolate "physical variables"

$$u(x) \approx \sum_{e=1}^{n} \sum_{i=1}^{k} u_i^e N_i$$

- It is only feasible to interpolate physical variable locally, not globally
- Elements are introduced
- Elements approximate geometry; hence loses accuracy

FEM Mistakes

- Low degree polynomials are used for interpolation.
- Poor continuity between elements
- FEM error is

$$\varepsilon \sim O\left(h^k\right)$$

POWER OF EXPONENTIAL CONVERGENCE

What is Exponential Convergence?

$$\varepsilon \sim O\left(\lambda^{h^{-k}}\right), \quad 0 < \lambda < 1$$

$$\varepsilon \sim O\left(e^{-h^{-k}}\right)$$

$$\varepsilon \sim O\left(\lambda^{N}\right), \quad 0 < \lambda < 1$$

Power of Exponential Convergence

- Its accuracy is impossible for FEM, FDM, or any other methods that uses local, rather than global interpolation, to match.
- Using Trefftz method, Li solved Motz problem to an accuracy of 10^{-8} using about 30 terms
- Using RBF collocation, a Poisson equation was solved to an accuracy of 10^{-15} using a 20x20 grid.

How Accurate is That?

- Assume that using a modest mesh, FEM/FDM can solve a problem to an accuracy of 0.1%.
- Using a quadratic element or central difference, the error estimate is h^2.
- To reach an accuracy of 10^{-15}, the FEM result needs to be improved 10^{-12} fold.
- h needs to be refined 10^6 fold in linear dimension, meaning that between any two nodes, one million nodes need to be inserted.

- In a 3D problem, this means 10^{18} fold more degrees of freedom.
- The effort of solution will be between 10^{36} to 10^{54} fold
- The fastest computer in the world (not yet delivered) has just reached petaflop (10^{12} flops per second).
- If the original problem requires 1 floating point operation (CPU=10^{-12} sec), the CPU time needed will be between 10^{17} to 10^{35} years
- The age of universe is 2×10^{10} years!

COLLOCATION METHODS (SIMPLICITY)

Intuitive Derivation

- Governing equation

$$\mathcal{L}(u) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$$

- Boundary condition

$$\mathscr{B}(u) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma$$

Approximate Solution

- Assume approximate solution is given by

$$\hat{u}(\boldsymbol{x}) = \sum_{i=1}^{n} \alpha_i \phi_i(\boldsymbol{x})$$

where $\phi_i(\boldsymbol{x})$ are basis functions and α_i are constants to be determined.

$\phi(x)$

Choices of Basis Functions

- Monomial
- Chebyshev polynomial
- Fourier series
- Wavelet
- Fundamental solutions (MFS)
- Non-singular general solution (Trefftz)
- Radial basis function (RBF)

COLLOCATION TREFFTZ METHOD

Continuous Solution

- Trial functions:
  - harmonic polynomials,
  - translation of harmonic function
- Weighted residual formulation: Error bounded by quadrature error
- Point collocation: Exponential error bound

$$\varepsilon \sim O\left(\lambda^N\right), \quad 0 < \lambda < 1$$

- Condition number

$$\text{Cond} \sim O\left(\beta^N\right), \quad \beta > 1$$

- Effective condition number much better

Weakly and Strongly Singular Problem

- Weakly singular problem: No treatment of singularity, using only local mesh refinement, logarithmic error convergence
- Strongly singular problem: No treatment of singularity, no convergence.

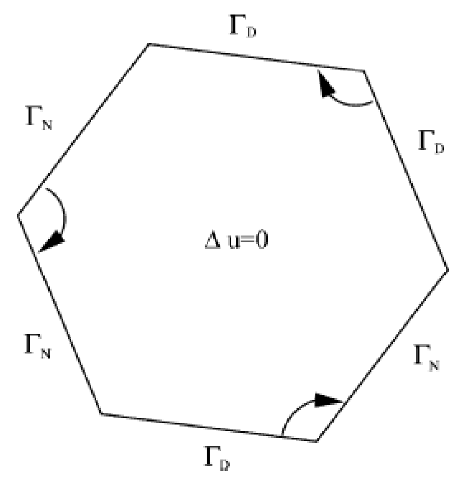


Fig. 4. A polygon.

- Harmonic polynomial is not suitable when singularity is present (polygonal region).
- Should use local particular solution.

Li, Z.-C., Lu, T.-T., Hu, H.-Y. and Cheng, A.H.-D., "Particular solutions of Laplace's equations on polygons and new models involving mild singularities," ENGINEERING ANALYSIS WITH BOUNDARY ELEMENTS, Vol. 29, No. 1, pp. 59–75, 2005.

METHOD OF FUNDAMENTAL SOLUTION

Comparison with Trefftz

- As $R \to \infty$ the sources behave as harmonic polynomials.
- Error bound of MFS can only be at best as good as Trefftz method. (Bogomolny, Schabck, J.T. Chen, Zi-Cai Li)
- Condition number of MFS is much worse that Trefftz

RBF COLLOCATION

Example: Multiquadric

- Inverse multiquadric

$$\hat{u}(\boldsymbol{x}) = \sum_{i=1}^{n} \alpha_i \frac{1}{\sqrt{r_i^2 + c^2}}$$

where

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$$

Point Collocation

- Select n_i points, $\{x_1, x_2, \cdots, x_{n_i}\} \in \Omega$, on which the governing equation is satisfied.

$$\mathscr{L}\left(\hat{u}(\boldsymbol{x}_j)\right) = \mathscr{L}\left(\sum_{i=1}^{n} \alpha_i \phi_i(\boldsymbol{x}_j)\right)$$

$$= \sum_{i=1}^{n} \alpha_i \mathscr{L}\left(\phi_i(\boldsymbol{x}_j)\right) = f(\boldsymbol{x}_j); \quad \text{for} \quad j = 1, \ldots, n_i$$

each is a linear equation in α_i

- Select n_b points, $\{x_{n_i+1}, x_{n_i+2}, \cdots, x_n\} \in \Gamma$, on which the boundary conditions are satisfied.

$$\mathscr{B}\left(\hat{u}(\boldsymbol{x}_j)\right) = \mathscr{B}\left(\sum_{i=1}^{n} \alpha_i \, \phi_i(\boldsymbol{x}_j)\right)$$

$$= \sum_{i=1}^{n} \alpha_i \mathscr{B}\left(\phi_i(\boldsymbol{x}_j)\right) = g(\boldsymbol{x}_j); \quad \text{for} \quad j = n_i + 1, \ldots, n$$

- Linear solution system

$$[\mathbf{A}]\{\boldsymbol{\alpha}\} = \{\mathbf{b}\}$$

- Once {α} is solved, the solution is a continuous function

$$u(x) = \sum \alpha_i \frac{1}{\sqrt{r_i^2 + c^2}}$$

- The function is infinitely smooth

Test Problem

$$\nabla^2 u(x,y) = -\frac{751\pi^2}{144}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4} + \frac{7\pi^2}{12}\cos\frac{\pi x}{6}\cos\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4} + \frac{15\pi^2}{8}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\cos\frac{3\pi y}{4}\cos\frac{5\pi y}{4}, \quad (x,y)\in[0,1]^2, \tag{7}$$

subject to the Dirichlet type boundary conditions

$$u(0,y) = 0, \tag{8a}$$

$$u(1,y) = \sin\frac{\pi}{6}\sin\frac{7\pi}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4}, \tag{8b}$$

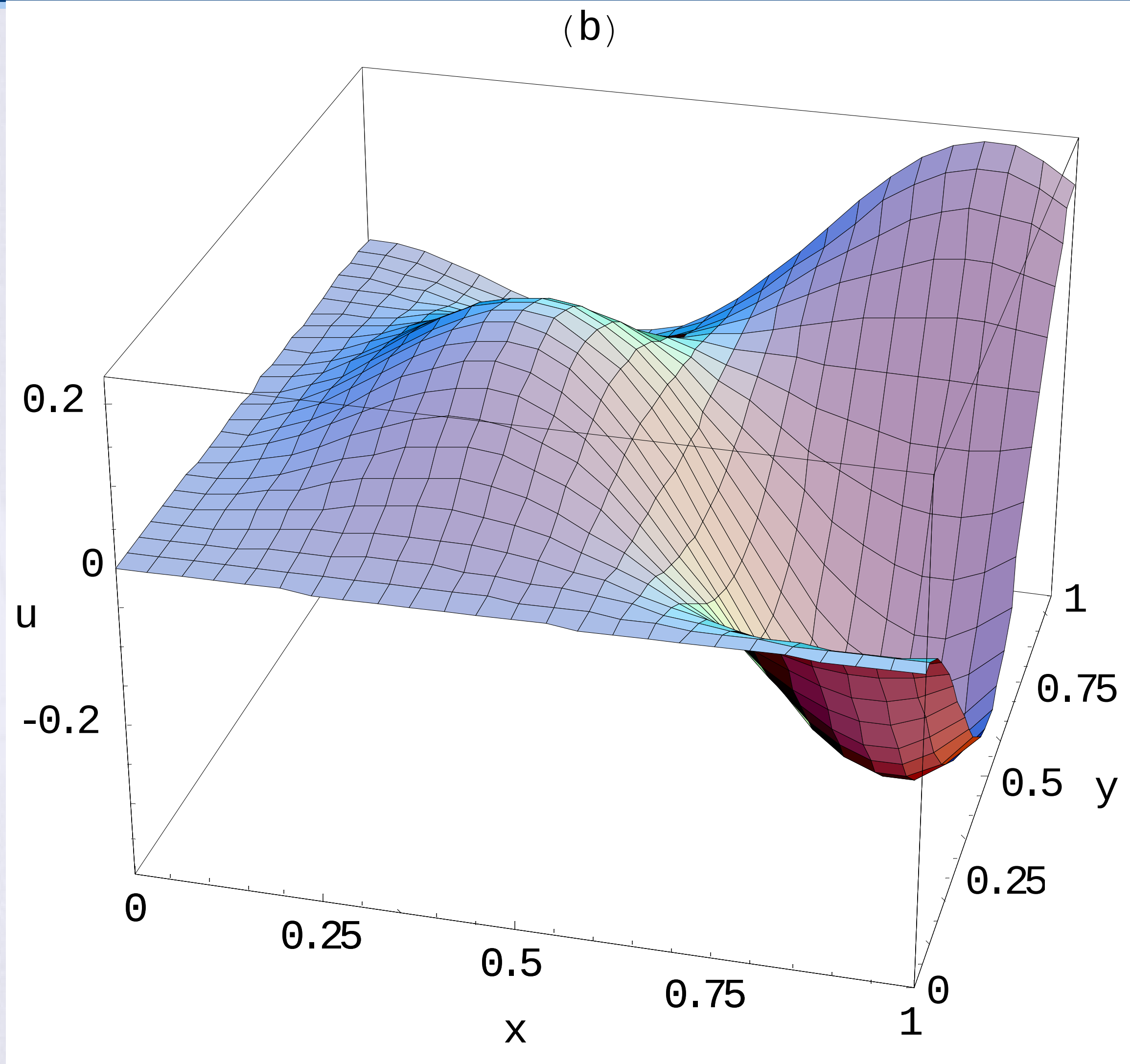
$$u(x,0) = 0, \tag{8c}$$

$$u(x,1) = \sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi}{4}\sin\frac{5\pi}{4}. \tag{8d}$$

The exact solution of this problem is

$$u(x,y) = \sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4}. \tag{9}$$

Exact Solution



Solution method

- Approximation by inverse multiquadric

$$\hat{u} = \sum_{i=1}^{n} \alpha_i \frac{1}{\sqrt{r_i^2 + c^2}}$$

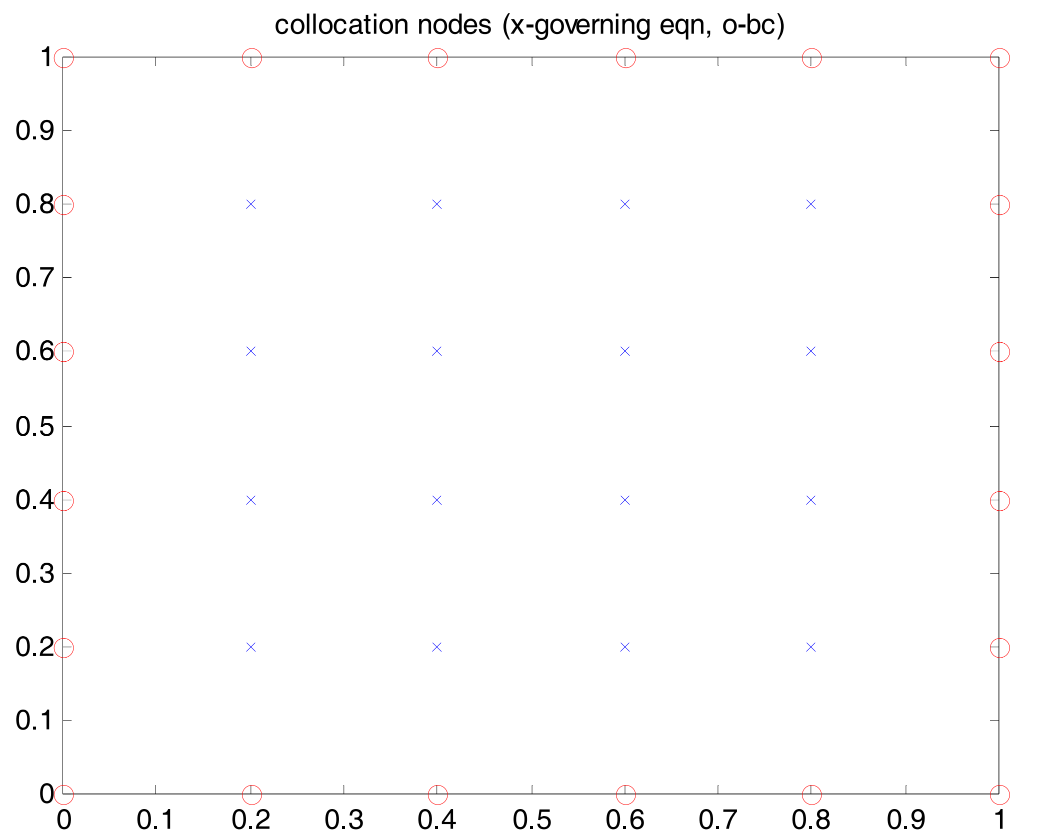
Watch out for the “c”

What Is the Role of *c*?

- People observe that as c increases, error decreases
- It is generally believed that as $c \to \infty$, $\varepsilon \to 0$
- If this is true, we have a *dream method*: higher and higher precision without paying a price
- However, matrix ill-condition gets in the way; the dream cannot be fulfilled.

- What if we can compute with infinite precision?
- Then, is it true that as $c \longrightarrow \infty$, $\varepsilon \longrightarrow 0$?
- (Or, is it true that for MFS, as $R \longrightarrow \infty$, $\varepsilon \longrightarrow 0$?)
- We can find out about these by using the infinite (arbitrary) precision computation capability of Mathematica and high precision capability of Fortran

- Use 6x6 mesh (h = 0.2, 4x4 interior collocation)



Result: h = 1/5

h	c	$\varepsilon_{\max}$	ε_{rms}	Condition Number
0.2	0.1	4.36×10^{-01}	1.40×10^{-01}	$4.95 \times 10^{+02}$
0.2	1.1	2.49×10^{-02}	9.08×10^{-03}	$8.89 \times 10^{+07}$
0.2	$1.2^{\dagger}$	1.92×10^{-02}	6.93×10^{-03}	$2.94 \times 10^{+08}$
0.2	1.3	1.94×10^{-02}	5.12×10^{-03}	$9.22 \times 10^{+08}$
0.2	$1.4^{\dagger}$	1.99×10^{-02}	4.24×10^{-03}	$2.76 \times 10^{+09}$
0.2	1.5	2.08×10^{-02}	4.94×10^{-03}	$7.92 \times 10^{+09}$
0.2	2.0^{*}	3.37×10^{-02}	1.85×10^{-02}	$8.49 \times 10^{+11}$
0.2	3.0	9.64×10^{-02}	5.84×10^{-02}	$1.09 \times 10^{+15}$
0.2	10.0	6.10×10^{-01}	4.19×10^{-01}	$6.38 \times 10^{+24}$
0.2	100.0	1.11×10^{0}	7.82×10^{-01}	$9.15 \times 10^{+42}$

Result: h = 1/10

h	c	ε_{max}	ε_{rms}	Condition Number
0.1	0.1	8.67×10^{-02}	2.89×10^{-02}	$2.19 \times 10^{+03}$
0.1	2.5	6.88×10^{-06}	1.74×10^{-06}	$2.88 \times 10^{+27}$
0.1	$4.0^{\dagger}$	1.88×10^{-06}	6.23×10^{-07}	$6.40 \times 10^{+34}$
0.1	$4.1^{\dagger,*}$	2.21×10^{-06}	6.09×10^{-07}	$1.57 \times 10^{+35}$
0.1	10.0	1.5×10^{-04}	1.11×10^{-04}	$4.82 \times 10^{+49}$
0.1	100.0	6.24×10^{0}	4.56×10^{0}	$3.49 \times 10^{+87}$

Result: h = 1/20

c	$\varepsilon_{\max}$	$\varepsilon_{\mathrm{rms}}$
7.0	2.22×10^{-15}	7.86×10^{-16}
7.5	1.91×10^{-15}	9.60×10^{-16}
7.7	2.37×10^{-15}	9.26×10^{-16}
8.0	2.88×10^{-15}	8.87×10^{-16}
8.5	3.58×10^{-15}	1.06×10^{-15}
9.0	3.75×10^{-15}	41.2×10^{-15}

Find Error Estimate Constants by Data Fitting

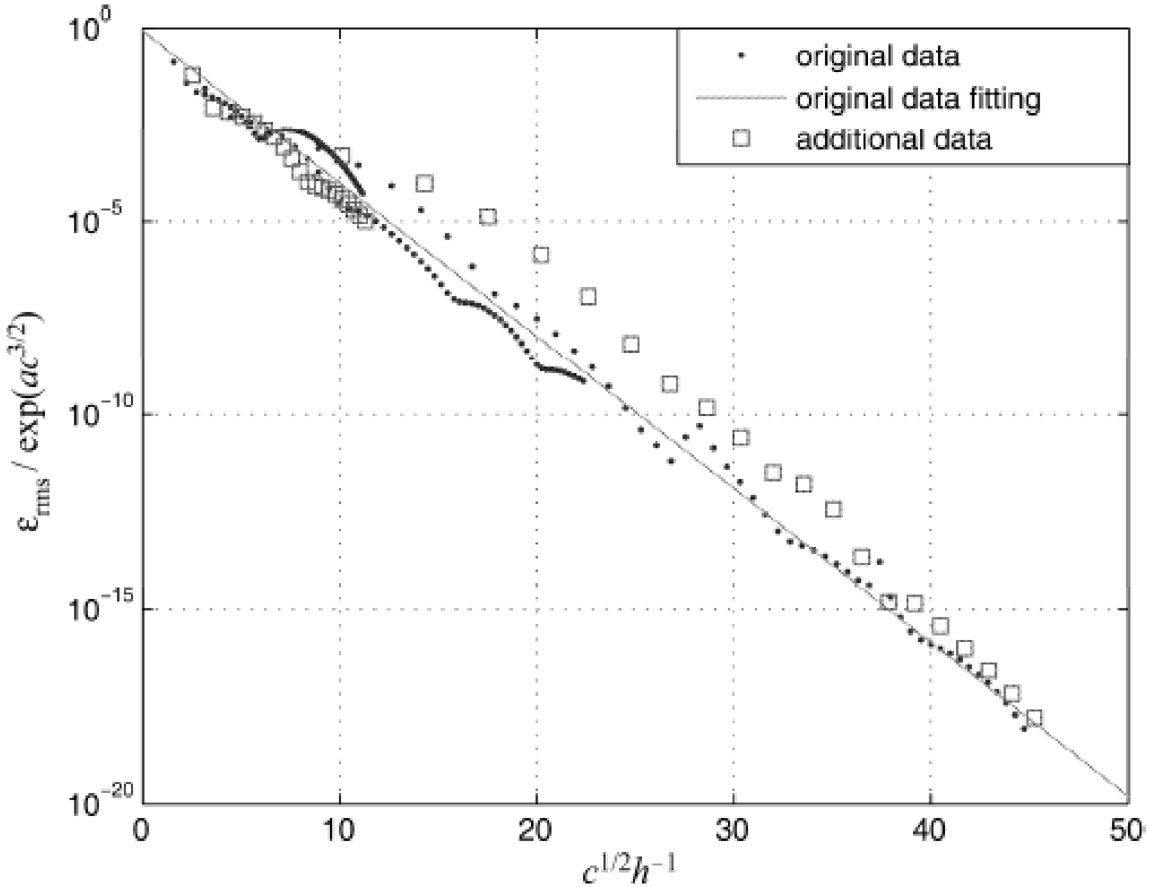


Fig. 2. Fitting for error estimate for IMQ solution of Poisson equation: composite plot of a large number of cases with different h and c values.

Error Estimate

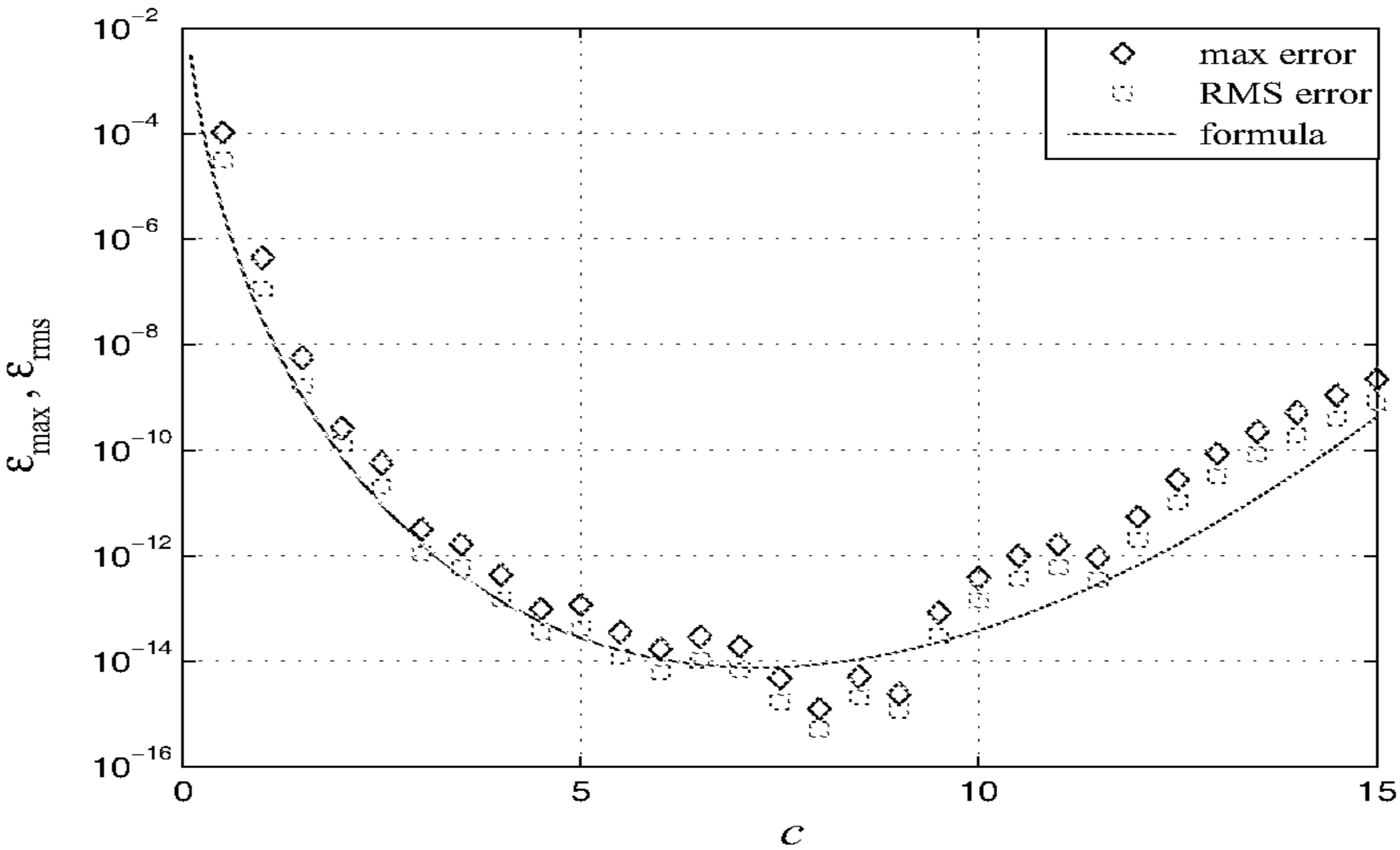


Fig. 5. Validating (15), second example.

Our Findings: Error Estimate

$$\varepsilon \sim \mathcal{O}(e^{ac^{3/2}} \lambda^{c^{1/2}h^{-1}}).$$

- $0 < \lambda < 1$, $a > 0$

Optimal *c*

- If the error estimate

$$\varepsilon \sim \mathcal{O}(e^{ac^{3/2}} \lambda^{c^{1/2}h^{-1}}).$$

is true, then there exists an optimal *c* where error is minimum

$$c_{\max} = -\frac{\ln \lambda}{3ah},$$

Revised Error Estimate

- If we can always use optimal *c* with a given mesh, what is the new error estimate?

$$\varepsilon \sim \mathcal{O}(\gamma^{h^{-3/2}}),$$

$$\gamma = \left(\lambda\, e^{-\ln\lambda/3}\right)^{\sqrt{-\ln\lambda/3a}}$$

$$0 < \gamma < 1$$

Effective Error Estimate If c_{max} Is Used

- $h = 1/5, \varepsilon \sim 10^{-2}$
- $h = 1/10, \varepsilon \sim 10^{-6}$
- $h = 1/20, \varepsilon \sim 10^{-15}$

THEORETICAL RESULT

Madych

- Madych (1992): For the interpolation of a class of "essentially analytic functions", which are "band limited", *using a class of interpolants that include the multiquadric, Gaussian, …, he proved*

$$\varepsilon = O\left(e^{ac}\lambda^{c/h}\right); \quad 0<\lambda<1, \quad a>0$$

- *This means, as* $c \to \infty, \quad \varepsilon \to 0$

- Madych also stated that for a "non-band-limited" function,

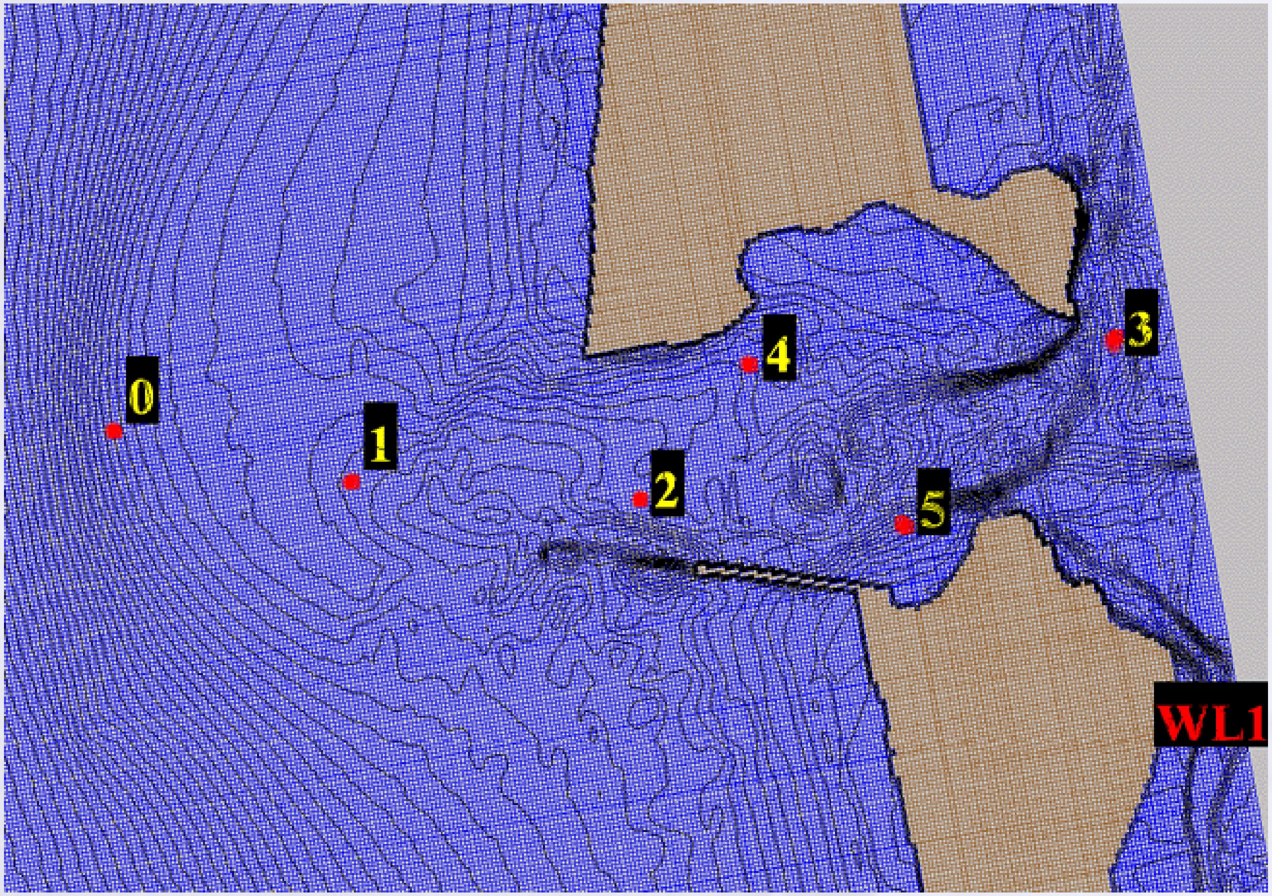
$$\varepsilon = O\left(e^{ac^2}\lambda^{c/h}\right); \quad 0 < \lambda < 1, \quad a > 0$$

- In this case, there exist a $c_{opt} = -\dfrac{\ln \lambda}{2ah}$ where $\varepsilon = \varepsilon_{\min}$
- If we can use the c_{opt} then $\varepsilon \sim O(\lambda^{1/h^2})$

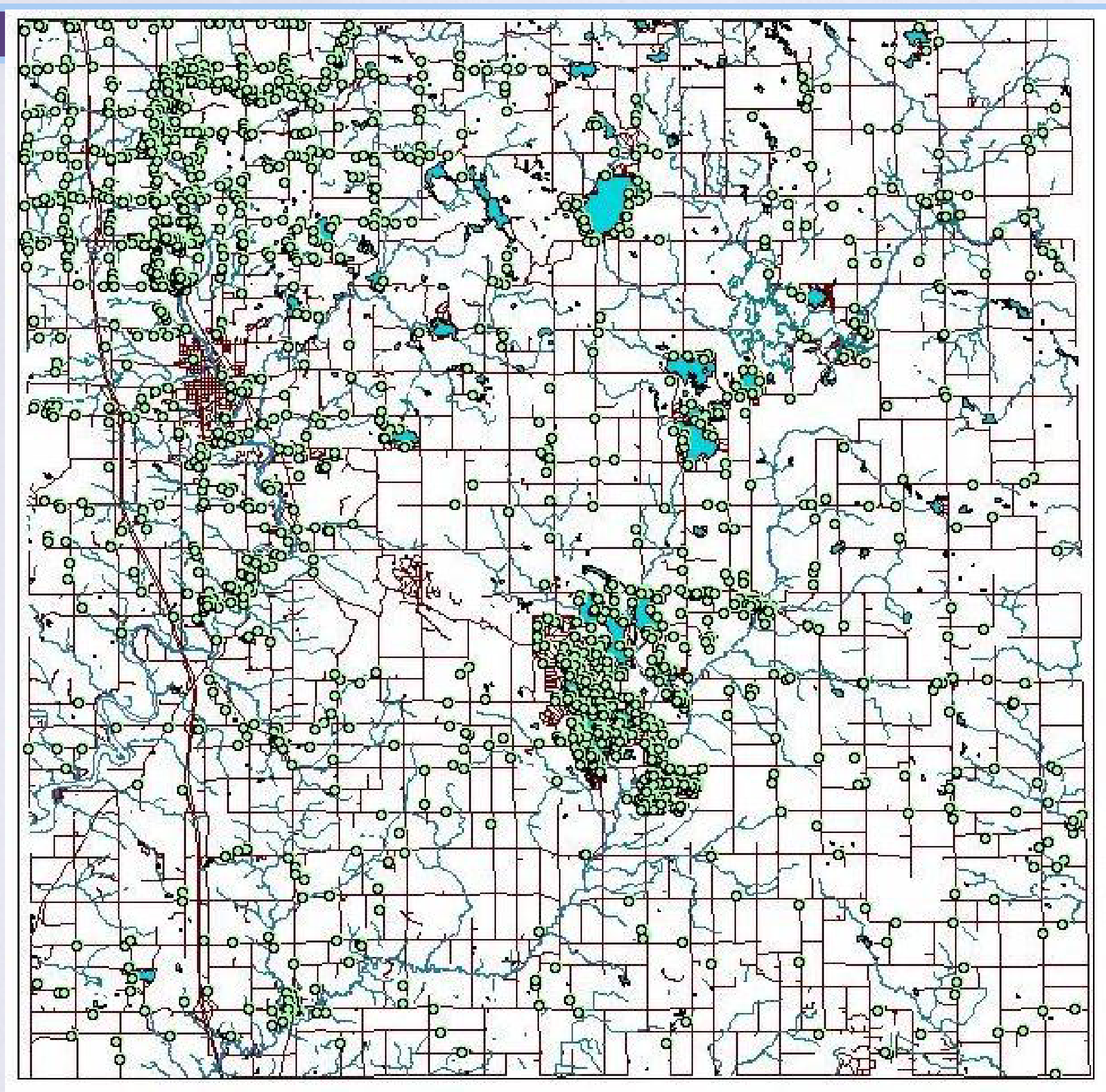
ILL-POSED PROBLEM

Examples of Ill-Posed Problems

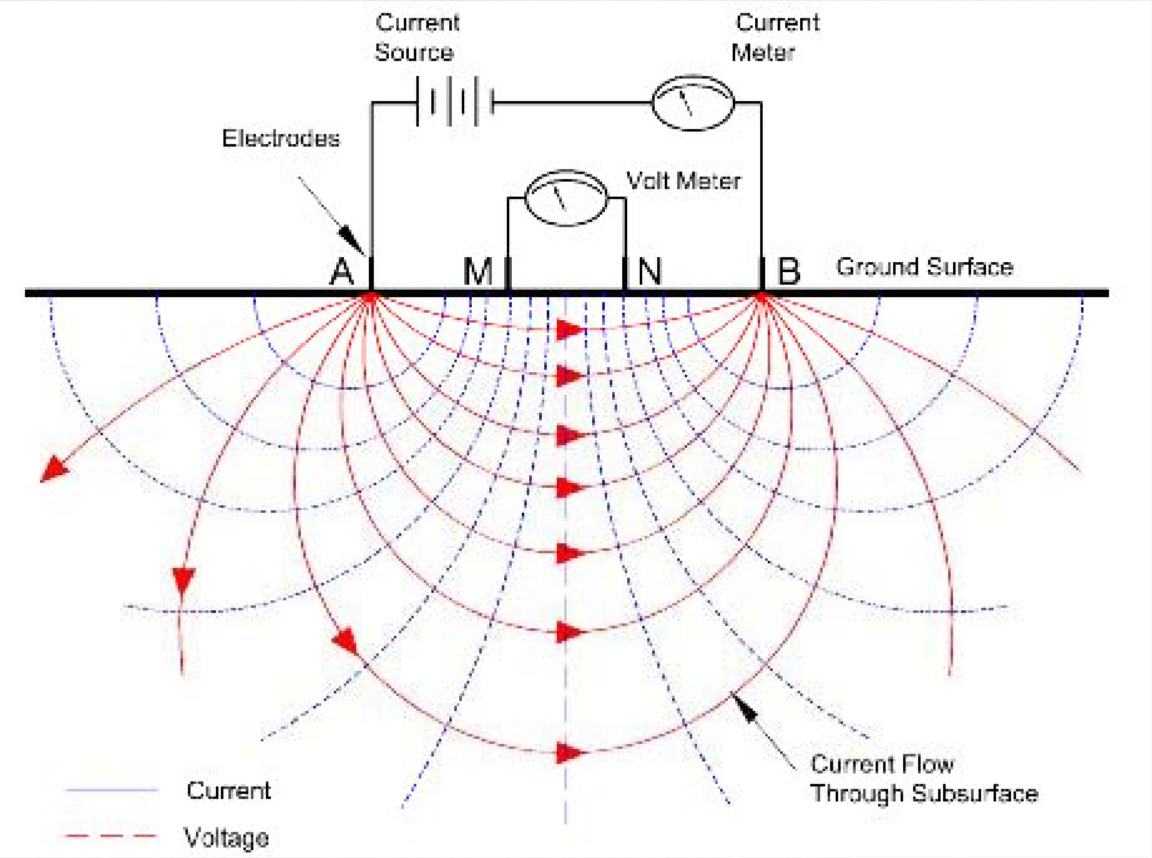
- Harbor wave field



- Groundwater field



Geoprospecting



- Non-Destructive Testing

Well- and Ill-Posed Boundary Value Problems

- Governing equation

$$\nabla^2 u = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$$

- Boundary conditions

$$u = g_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_D$$

$$\frac{\partial u}{\partial n} = g_2(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_N$$

- Interior condition

$$u(\boldsymbol{x}_j) = \bar{u}_j, \quad j = 1, \ldots, m, \quad \boldsymbol{x}_j \in \Omega$$

Difference between well-posed and ill-posed problems

- Well-posed problem

$$\Gamma_D \cup \Gamma_N = \Gamma \qquad \Gamma_D \cap \Gamma_N = \varnothing$$

$$\Gamma_D \neq \varnothing \qquad m = 0$$

- Ill-posed problem

$$\Gamma_D \cup \Gamma_N \neq \Gamma \qquad \Gamma_D \cap \Gamma_N \neq \varnothing$$

$$m \neq 0$$

Problem 1: Onishi (1995) FEM

- Governing equation

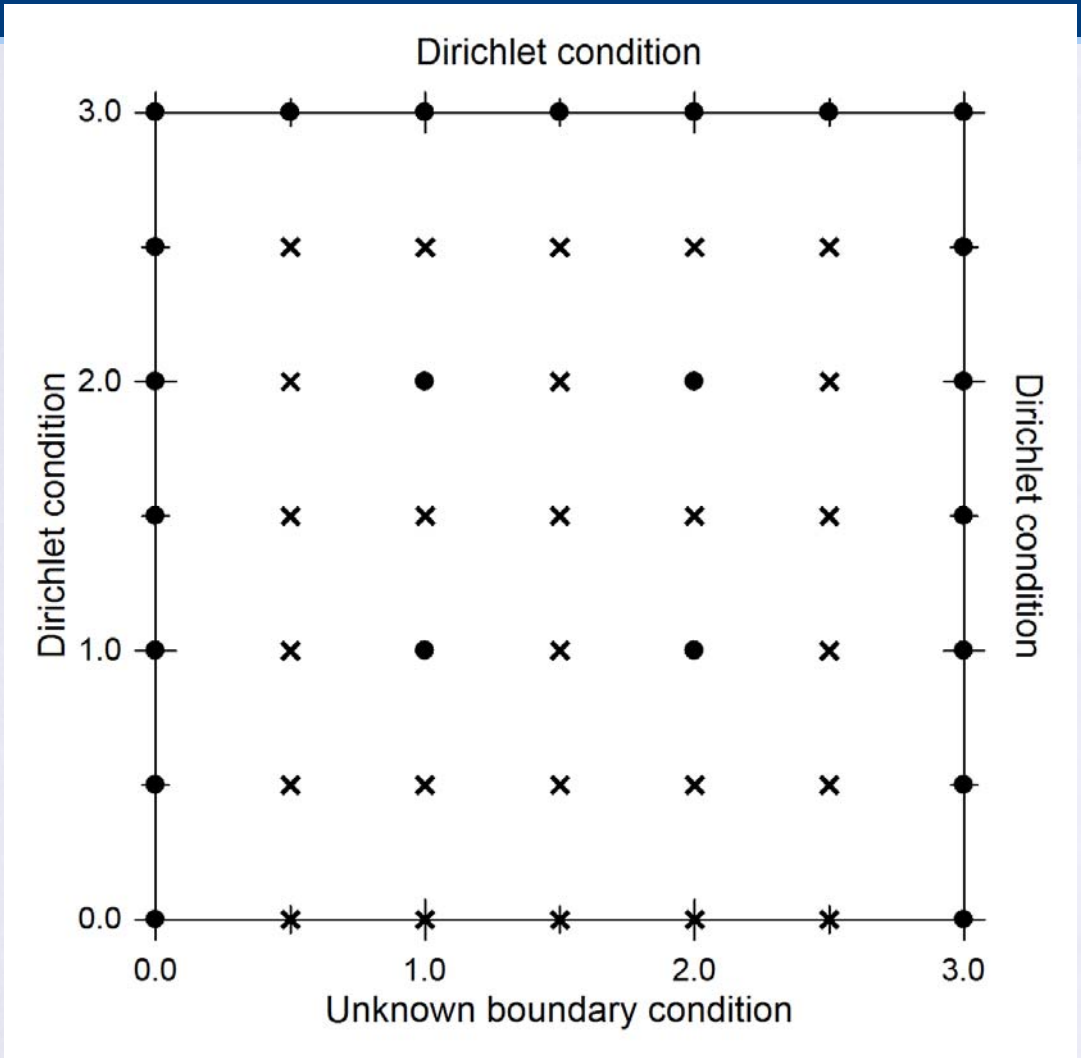
$$\nabla^2 u(x,y) = 0; \quad y \geq 0$$

- BC

$$u(x,3) = x^2 - 9; \quad u(0,y) = -y^2; \quad u(3,y) = 9 - y^2$$

- Internal values: u(1,1) = 0, u(2,1) = 3, u(1,2) = -3, u(2,2) = 0

Collocation Nodes



Result

Method	Potential value	Percent error (%)
Exact solution	2.250	0
Onishi, FEM 36 elements	2.323	3.2
Onishi, FEM 144 elements	2.341	4.0
RBF, 49 collocation nodes ($c = 3$)	2.296	2.0
RBF, 49 collocation nodes ($c = 4$)	2.251	0.04

Table 1: Comparison of error of potential at the point $(1.5, 0)$.

Problem 2: Lesnic (1998) BEM

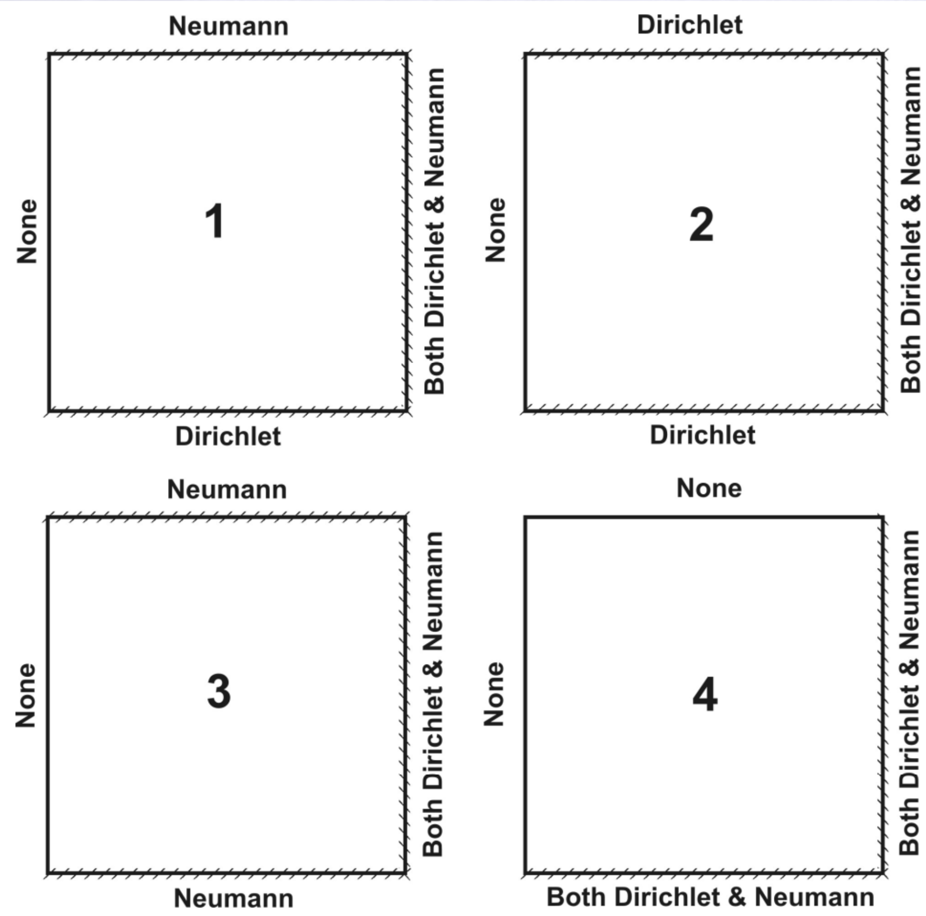


Figure 4: Four cases of Cauchy problems for steady state heat conduction with different boundary conditions (refer to Table 2 for boundary values).

Lesnic Result

	Case 1	Case 2	Case 3	Case 4
Number of elements	40	160	160	160
Number of iterations	100	1000	1000	10000
Error in temperature (%)	0.4	0.5	0.3	13
Error in heat flux (%)	2	6	1.5	50

Table 3: Percentage error of Lesnic's iterative BEM solution at the middle point of left side boundary, $(0, 0.5)$.

RBF Result

Grid	c	Case 1 (%)	Case 2 (%)	Case 3 (%)	Case 4 (%)
6×6	3	0.04	0.1	1.0	0.5
6×6	4	0.5	0.6	1.7	0.4
8×8	3	0.2	0.02	0.03	0.1
8×8	4	0.1	0.009	0.009	0.06
10×10	3	0.05	0.009	0.009	0.02
10×10	4	0.009	0.000	0.009	0.02

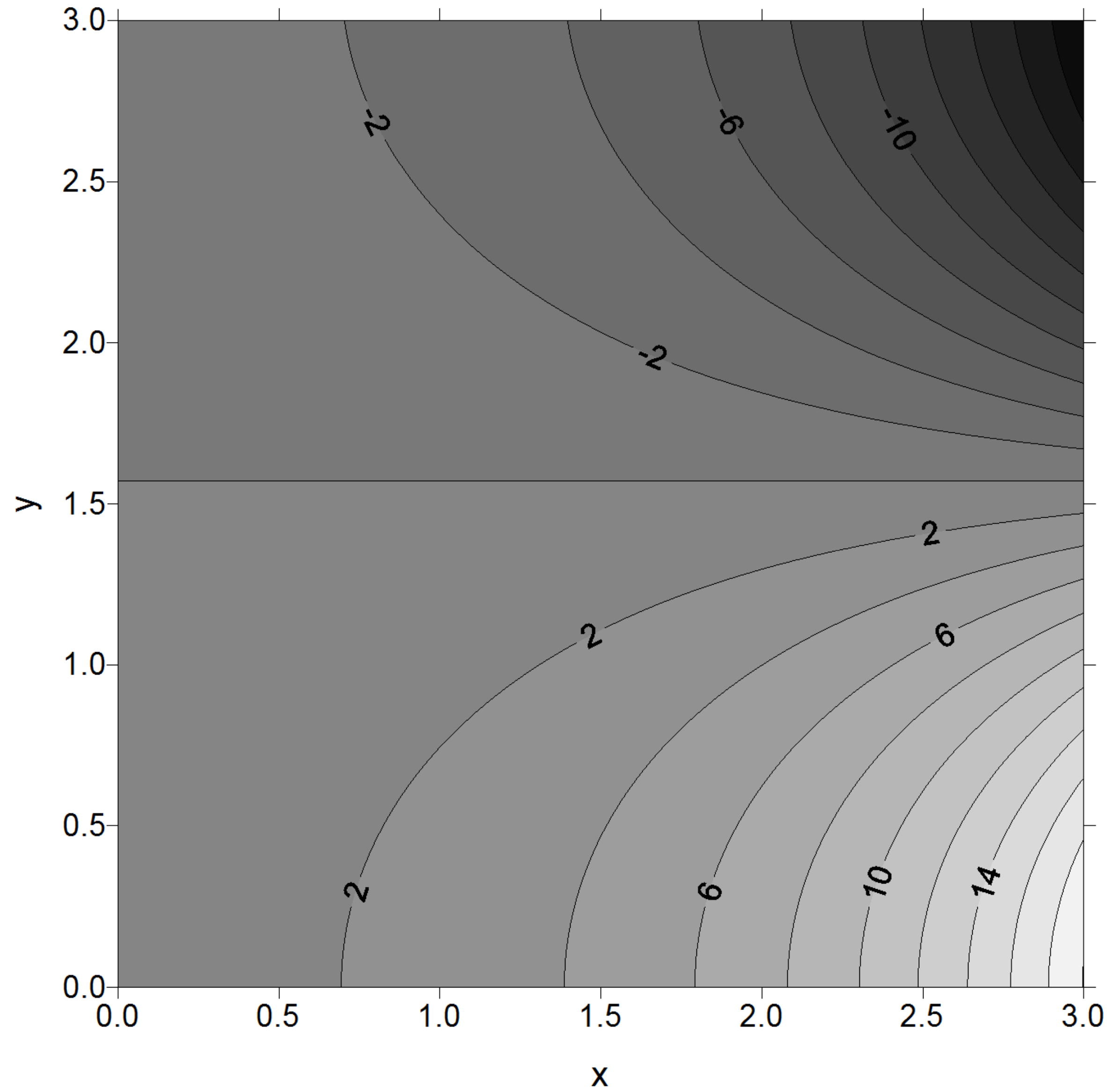
Table 4: Percentage error of RBF collocation solution for temperature at the middle point of left side boundary, $(0, 0.5)$, for different grids and c values.

Grid	c	Case 1 (%)	Case 2 (%)	Case 3 (%)	Case 4 (%)
6×6	3	0.02	0.8	2.8	6.2
6×6	4	3.2	0.2	3.4	4.0
8×8	3	1.5	0.06	0.1	0.6
8×8	4	0.6	0.06	0.06	0.06
10×10	3	0.6	0.04	0.02	0.1
10×10	4	0.1	0.02	0.2	0.2

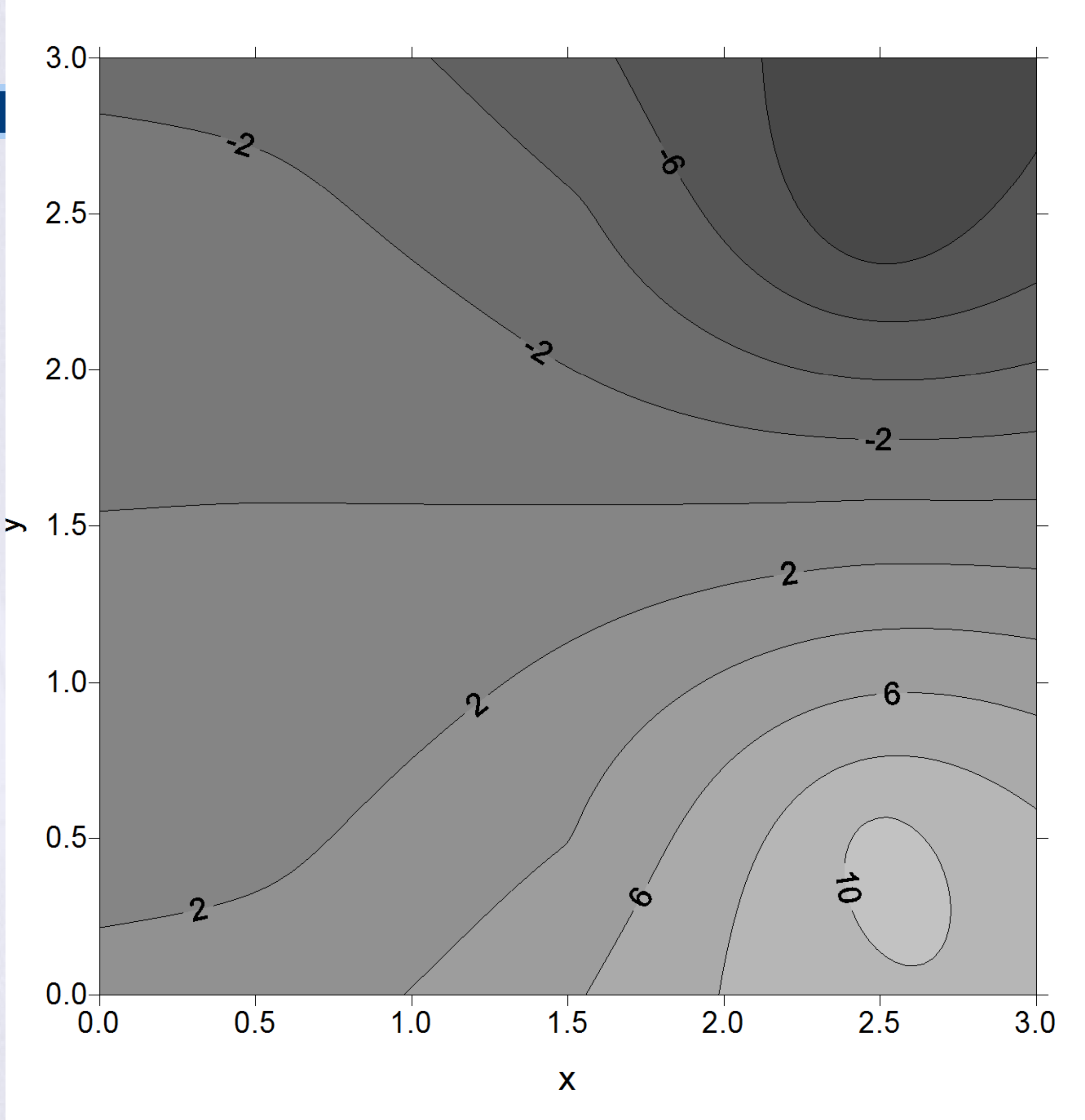
Table 5: Heat flux percentage error of RBF collocation solution at the middle point of left side boundary, $(0, 0.5)$, for different grids and c values.

Problem 3: "Groundwater"

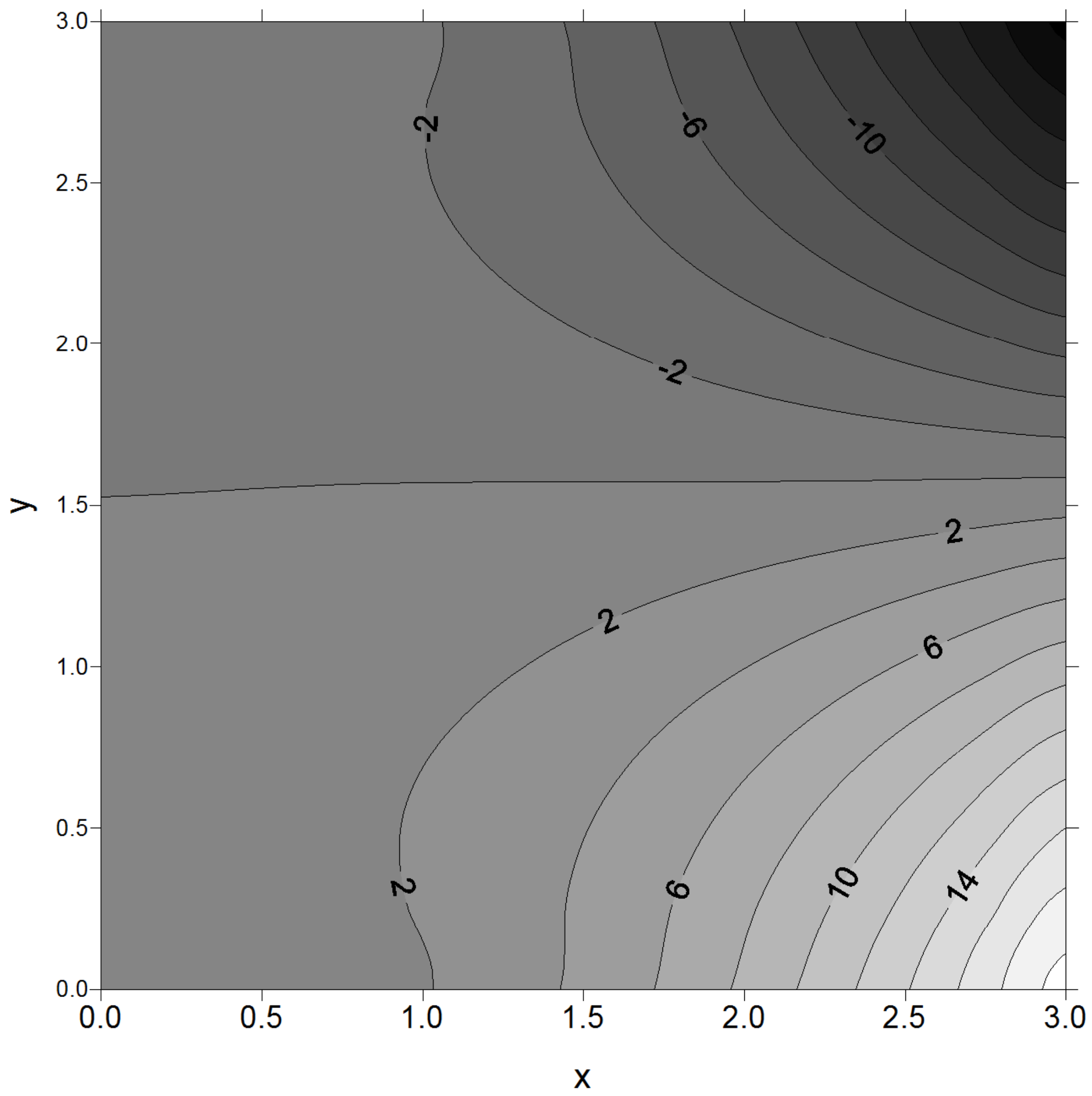
Exact
potential
9 data given



Kriging



Solving Ill-Posed Problem



SUMMARY OF POINT COLLOCATION

Point Collocation

- No geometric approximation error.
- No quadrature error.
- Exponential error convergence,

$$\varepsilon \sim O\left(\lambda^{1/h^k}\right); \quad 0 < \lambda < 1$$

- Convergence is the best if we make the interpolants as flat as possible.
- Meshless
- Solve ill-posed problem without iteration
- Solve n-D problem