Analytical Study of Rotating Hollow Cylinders of Strain-Hardening Viscoplastic Materials

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NSC PROJECT: NSC -96-2221-E-157-003

ABSTRACT

The paper analytically deals with plastic limit angular velocities of rotating hollow cylinders of nonlinear isotropic strain-hardening viscoplastic materials. Particularly, analytical solutions of plastic limit angular velocities and the onset of instability were derived and compared with numerical results. The corresponding stability condition was also obtained explicitly while the implicit form of the onset of instability was solved by the fixed point iteration. It is found that the analytical results have made it possible for rigorous comparison of its counterpart of numerical effort.

Keywords: plastic limit angular velocity, rotating cylinder, von Mises criterion, nonlinear strain-hardening, viscoplasticity, fixed point iteration.

1. INTRODUCTION

Plastic limit angular velocities of cylinders are useful information requested frequently for structure optimal design and safety evaluation. Much effort [1-4] made to such important topics by investigating the elastic-plastic behavior and the fully plastic state. For investigating such problems of optimization features, the author and his coworker [5] have investigated analytically and numerically the rotating problems involving nonlinear isotropic hardening materials. Similar attention [6-10] is also paid to the limit angular velocities of disks.

Based on the previously successful work [5, 11-13], the paper extends to investigate analytically the rotating cylinders problems involving nonlinear isotropic hardening viscoplastic materials. It is noted that such problems feature in involving hardening material properties and weakening behavior corresponding to the strain-rate sensitivity in addition to widening deformation [14]. Thus, the applicability of the CSSA algorithm is to be validated by its counterpart of analytical studies of thick-walled cylinders involving materials of the von Mises model with viscoplastic nonlinear isotropic hardening. Novelly, corresponding to the specific normalization condition adopted in the paper, the onset of instability and the existence of hardening phenomena before the weakening behavior are to be investigated analytically and explicitly.

In the following sections, the paper is based on the concept of sequential limit analysis to deal with the rotating hollow cylinders of the von Mises materials with viscoplastic nonlinear isotropic strain-hardening. Particularly, analytical solutions of plastic limit angular velocity, the onset of instability and the stability condition are to be derived for rigorous comparisons and validation.

2. PROBLEM STATEMENT AND RESULTS

2.1 Analytical Solution

We consider a plane-strain problem with a rotating hollow cylinder made of strain-hardening viscoplastic materials simulated by the von Mises model. The initial interior and exterior radii of the cylinder are denoted by a_0 and b_0. Also, its current interior and exterior radii are denoted by a and b. The behavior of viscoplastic, nonlinear isotropic hardening is as adopted by Haghi and Anand [15]

$$\bar{\sigma} = \left[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})\right] \left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m \tag{1}$$

where σ_0 is the initial yield strength, σ_∞ is the saturation stress and h is the hardening exponent, $\bar{\varepsilon}$ is the equivalent strain and $\dot{\bar{\varepsilon}}$ the equivalent strain rate. $\dot{\bar{\varepsilon}}_0$ and m are positive valued material parameters called the reference strain rate and strain rate sensitivity, respectively.

Similar to the procedures adopted by the previous work of Leu [11-13], Leu and Chen [5], we derive the analytical solutions as follows.

In the cylindrical coordinate system, the incompressibility condition requires that

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{2}$$

where v is the radial velocity at a point (r,θ). Accordingly, the radial velocity can be expressed as

$$v = \frac{a\dot{a}}{r} \tag{3}$$

where a, $\dot{a}$ are the interior radius and interior velocity, respectively. Accordingly, we can express the strain rates as

$$\dot{\varepsilon}_r = \frac{\partial v}{\partial r} = -\frac{a\dot{a}}{r^2} \tag{4}$$

$$\dot{\varepsilon}_\theta = \frac{v}{r} = \frac{a\dot{a}}{r^2} \tag{5}$$

$$\dot{\varepsilon}_z = 0 \tag{6}$$

and from Eqs. (4)~(6) we obtain the equivalent strain rate

$$\dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}(\dot{\varepsilon}_r^2 + \dot{\varepsilon}_\theta^2 + \dot{\varepsilon}_z^2)} = \frac{2}{\sqrt{3}}\frac{a\dot{a}}{r^2} \tag{7}$$

Accordingly, the equivalent strain is obtained as

$$\bar{\varepsilon} = \int \dot{\bar{\varepsilon}}\, dt = \frac{1}{\sqrt{3}}\ln\frac{r^2}{r_0^2} \tag{8}$$

where r_0 is the initial radius to the location concerned. The components of the stress deviator, s_r, s_θ, s_z, can be obtained by considering the flow rule and satisfying the yield condition. Thus, we obtain

$$s_r = -\frac{1}{\sqrt{3}}\left[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})\right]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m \tag{9}$$

$$s_\theta = \frac{1}{\sqrt{3}}\left[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})\right]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m \tag{10}$$

$$s_z = 0 \tag{11}$$

Thus, the stresses are given as

$$\sigma_r = s + s_r \tag{12}$$

$$\sigma_\theta = s + s_\theta \tag{13}$$

$$\sigma_z = s + s_z \tag{14}$$

where s is the mean normal stress. Substituting Eqs. (12)~(14) into the following equilibrium equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = -\rho\omega^2 r \tag{15}$$

Therefore, we obtain

$$\frac{\partial \sigma_r}{\partial r} = -\frac{\sigma_r - \sigma_\theta}{r} - \rho\omega^2 r = \frac{2}{\sqrt{3}r}\left[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})\right]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m - \rho\omega^2 r \tag{16}$$

Note that $h = \sqrt{3}$ is used in the derivation. Thus, with the boundary conditions $\sigma_r(r = a) = 0$ and $\sigma_r(r = b) = 0$, the limit value of the angular velocity factor $\rho\omega^2$ at the current radii a, b is given by

$$\rho\omega^2 = \frac{2}{b^2 - a^2}\left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2a\dot{a}}{\dot{\bar{\varepsilon}}_0}\right)^m \bullet \left\{\frac{\sigma_0}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right) - \frac{(\sigma_\infty - \sigma_0)}{m+1}(a_0^2 - a^2)\left(\frac{1}{a^{2m+2}} - \frac{1}{b^{2m+2}}\right)\right\} \tag{17}$$

If the angular velocity factor $\rho\omega^2$ is normalized by σ_0 / b_0^2, then we have the normalized angular velocity factor $\rho\omega^2 b_0^2 / \sigma_0$ in the form as

$$\frac{\rho\omega^2 b_0^2}{\sigma_0} = \frac{2b_0^2}{b^2 - a^2}\left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2a\dot{a}}{\dot{\bar{\varepsilon}}_0}\right)^m \bullet \left\{\frac{1}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right) - \frac{(\sigma_\infty/\sigma_0 - 1)}{m+1}(a_0^2 - a^2)\left(\frac{1}{a^{2m+2}} - \frac{1}{b^{2m+2}}\right)\right\} \tag{18}$$

For the case with $m = 0$

$$\lim_{m \to 0}\frac{a^{-m} - a^m b^{-2m}}{m} = \ln\left(\frac{b^2}{a^2}\right) \tag{19}$$

Thus, we reduce the viscoplasticity problems to rate independent plasticity problems [5] with the strain rate sensitivity $m = 0$, such that

$$\frac{\rho\omega^2 b_0^2}{\sigma_0} = \frac{2b_0^2}{b^2 - a^2}\left\{\frac{1}{\sqrt{3}}\ln\frac{b^2}{a^2} - \frac{(\sigma_\infty/\sigma_0 - 1)}{\sqrt{3}}\left(\frac{a_0^2}{a^2} - \frac{b_0^2}{b^2}\right)\right\} \tag{20}$$

For the case with $\sigma_\infty = \sigma_0$, we reduce to non-hardening power-law viscoplasticity problems such that

$$\frac{\rho\omega^2 b_0^2}{\sigma_0} = \frac{2b_0^2}{b^2 - a^2}\left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2a\dot{a}}{\dot{\bar{\varepsilon}}_0}\right)^m\left\{\frac{1}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right)\right\} \tag{21}$$

2.2 Onset of Instability

To consider instability and then the existence of the maximum value of the limit angular velocity during the whole widening process, we apply the necessary condition for the maximum of $\rho\omega^2 b_0{}^2/\sigma_0$, namely the following mathematical expression with the current interior radius a

$$\frac{\partial(\rho\omega^2 b_0{}^2/\sigma_0)}{\partial a}=0 \tag{22}$$

Note that, we have $a^2-a_0{}^2=b^2-b_0{}^2=r^2-r_0{}^2$, namely $a\dot{a}=b\dot{b}=r\dot{r}$, due to the incompressibility condition inherent in the von Mises model. Particularly, we recall the normalization condition $\int_D \bar{u}\cdot\bar{r}\,dA=1$ adopted in the computational procedure as detailed in [14] or later in the paper. Accordingly, it implies that $a\dot{a}=b\dot{b}=r\dot{r}$ is a constant in the computations in the paper. Thus, $\partial(\rho\omega^2 b_0{}^2/\sigma_0)/\partial a=0$ we have

$$\begin{aligned}&\frac{-1}{a^{2m+1}}+\frac{a}{b^{2m+2}}+\frac{\sigma_\infty/\sigma_0-1}{m+1}\left(\frac{1}{a^{2m+1}}-\frac{a}{b^{2m+2}}\right)\\&+(\sigma_\infty/\sigma_0-1)\frac{a_0{}^2-a^2}{a^{2m+3}}-(\sigma_\infty/\sigma_0-1)\frac{a_0{}^2a-a^3}{b^{2m+4}}=0\end{aligned} \tag{23}$$

We can reorganize the equation in the form as

$$\begin{aligned}&\frac{(m\sigma_\infty/\sigma_0+1)}{m+1}\frac{(b^{2m+2}-a^{2m+2})}{b^{2m+2}}\\&=(\sigma_\infty/\sigma_0-1)\frac{a_0{}^2}{a^2}-(\sigma_\infty/\sigma_0-1)\frac{a^{2m+2}b_0{}^2}{b^{2m+4}}\end{aligned} \tag{24}$$

To solve the nonlinear equation, we apply the method of fixed point iteration [16] to acquire the onset of instability in terms of a/a_0. Thus, the nonlinear equation is reorganized as

$$\frac{a^2}{a_0{}^2}=\frac{(\sigma_\infty/\sigma_0-1)b^{2m+4}}{\frac{m\sigma_\infty/\sigma_0+1}{m+1}\left(b^{2m+4}-a^{2m+2}b^2\right)+(\sigma_\infty/\sigma_0-1)a^{2m+2}b_0{}^2} \tag{25}$$

And get the solution of a/a_0 in the form ready for the method of fixed point iteration [16]

$$\frac{a}{a_0}=\sqrt{\frac{(\sigma_\infty/\sigma_0-1)b^{2m+4}}{\frac{m\sigma_\infty/\sigma_0+1}{m+1}\left(b^{2m+4}-a^{2m+2}b^2\right)+(\sigma_\infty/\sigma_0-1)a^{2m+2}b_0{}^2}} \tag{26}$$

Finally, we come to consider the condition of stability, namely the existence of hardening phenomena before the weakening behavior. Mathematically, it is to consider the case expressed in the form

$$\frac{\partial(\rho\omega^2 b_0{}^2/\sigma_0)}{\partial a}>0 \tag{27}$$

Certainly, the condition expressed by Eq. (27) is equivalent to see if there is the solution $a/a_0>1$ to Eqs. (26). Therefore, corresponding to the viscoplastic strain-hardening behavior with the hardening exponent $h=\sqrt{3}$, we can get the stability condition by Eqs. (26) as

$$\frac{\sigma_\infty}{\sigma_0}>m+2 \tag{28}$$

Therefore, if the viscoplastic strain-hardening behavior adopted by Haghi and Anand [15] with the hardening exponent $h=\sqrt{3}$, then there exists strengthening phenomenon if the rotating cylinders are made of hardening materials with $\sigma_\infty/\sigma_0>m+2$.

2.3 Numerical Formulation

Limit analysis is a conventional but yet convenient and comparable tool [17-29], especially in conjunction with finite element methods [30] and computational optimization techniques [31]. Providing efficiently the plastic limit loads with simple input data, limit analysis plays the role of a snapshot look at the structural performance. Furthermore, sequential limit analysis is to conduct a sequence of limit analysis problems with updating local yield criteria in addition to the configuration of the deforming structures. Accordingly, it has been illustrated widely that sequential limit analysis is an accurate and efficient tool for the large deformation analysis [32-38].

The hollow cylinder is considered to rotate about its axis at a constant angular velocity ω. It is assumed that the angular acceleration is negligible. In the beginning, we consider a general plane-strain problem with the domain D consisting of the static boundary ∂D_s and the kinematic boundary ∂D_k. The problem is then to search for the maximum allowable angular velocity factor $\rho\omega^2(\sigma)$ subjected to constraints of static and constitutive admissibility such that

$$\begin{aligned}&\text{maximize} \quad \rho\omega^2(\sigma)\\&\text{subject to} \quad \nabla\cdot\sigma+\rho\omega^2\bar{r}=0 \ \text{ in } D\\&\qquad\qquad \|\sigma\|_\vee\le\sigma_0 \quad \text{ in } D\end{aligned} \tag{29}$$

where ρ is the constant material density of the rotating hollow cylinders, ω is the angular velocity, $\rho\omega^2\bar{r}$ is the centrifugal force with $\bar{r}$ the position vector, $\|\sigma\|_\vee$ denotes the von Mises primal norm on stress tensor σ and σ_0 is a material constant. Therefore, this constrained problem is simply to maximize the angular velocity factor $\rho\omega^2(\sigma)$ representing the magnitude of the driving load. Obviously, the problem statement leads naturally to the lower bound formulation seeking the extreme solution under constraints of static and constitutive admissibility. The statically admissible solutions satisfy the equilibrium equation and the static boundary condition. The constitutive admissibility is stated by the yield criterion in an inequality form.

As detailed by the author [14], we can transform the lower bound formulation to the upper bound formulation stated in the form of a constrained minimization problem as

$$\begin{aligned} &\text{minimize} && \rho\bar{\varpi}^2(\bar{u}) \\ &\text{subject to} && \rho\bar{\varpi}^2(\bar{u}) = \sigma_0 \int_D \|\dot{\varepsilon}\|_{-\vee}\, dA && (30) \\ & && \int_D \bar{u}\cdot\bar{r}\, dA = 1 \quad \text{in } D \\ & && \nabla\cdot\bar{u} = 0 \quad \text{in } D \\ & && \text{kinematic boundary conditions on } \partial D_k \end{aligned}$$

where $\int_D \bar{u}\cdot\bar{r}\, dA = 1$ is the normalization condition and $\nabla\cdot\bar{u} = 0$ is the incompressibility constraint inherent in the von Mises model.

Therefore, the upper bound formulation seeks sequentially the least upper bound for each step on kinematically admissible solutions. Accordingly, the primal–dual problems (29) and (30) are convex programming problems following the work of Huh and Yang [24] and Yang [38] and as demonstrated by Yang [27-28, 39]. Thus, for each step, there exist a unique maximum and minimum to problems (29) and (30), respectively.

As detailed in the previous work [14], rigorous upper bounds are then computed sequentially and effectively based on a combined smoothing and successive approximation (CSSA) algorithm incorporated with an inner and outer iterative sequence. The CSSA algorithm is comparable for its simple implementation, unconditional convergence. All the abovementioned procedures have been summarized as the flowchart shown in the previous work by Leu and Chen [5].

3. ILLUSTRATIVE COMPARISONS

The paper is based on the concept of sequential limit analysis to investigate the plastic limit angular velocity of hollow cylinders involving strain-hardening viscoplastic materials in plane-strain conditions. Analytical solutions for the limit angular velocity have been derived. To be complete, the onset of the instability and the stability condition are also investigated analytically. Comparisons between analytical solutions and numerical results are then to be made.

In the numerical formulation, the centrifugal force associated with the angular velocity is the driving load to cause the rotating cylinders fully plastic. In the computations, the behavior of viscoplastic, nonlinear isotropic hardening is as adopted by Haghi and Anand [15] as shown in Eq. (1).

In the illustrated examples, the initial inner and outer radii are denoted as a_0 and b_0, respectively. The angular velocity required to keep the deforming cylinder fully plastic is then computed sequentially by using the CSSA algorithm [14]. In the following case studies, we adopt the following parameters of consistent dimensions: $a_0 = 5.0$, $b_0 = 10.0$, $h = \sqrt{3}$, $\dot{\bar{\varepsilon}}_0 = 1.0$ and a constant step size $\Delta t = 1.0$.

Firstly, we consider the rotating cylinder with various values of the strain-rate sensitivity m. For the case with $m = 0$, the problem is then reduced to a rate independent plasticity problem involving strain-hardening materials as investigated in the previous work [5]. Parametric studies are performed with various values of the strain rate sensitivity m together with the value of the yield strength ratio $R = \sigma_\infty / \sigma_0 = 2.0$. The results of the normalized plastic limit angular velocity factor $\rho\omega^2 b_0{}^2 / \sigma_0$ are summarized in Figure 1. All the computed upper bounds agree very well with the analytical solutions.

Secondly, parametric studies are performed with various values of the yield strength ratio $R = \sigma_\infty / \sigma_0$ associated with the values of the strain rate sensitivity $m = 0.1$. The results of the normalized plastic limit angular velocity factor $\rho\omega^2 b_0{}^2 / \sigma_0$ are summarized in Figure 2. All the computed upper bounds match very well with the analytical solutions.

On the other hand, there may be a strengthening phenomenon before the weakening phenomenon as shown in Figures 1~2 depending on the values of the hardening exponent h and the strain rate sensitivity m. For some values of the hardening exponent h and the strain rate sensitivity m, rotating hollow cylinders are strengthened due to the strain-hardening until the onset of instability. Following that, however, the weakening phenomenon is observed while the effect of strain-rate sensitivity and widening deformation counteracts that of the strain-hardening. Note that, the onset of instability concerned is about the plastic instability marked by the rotating speed maximum while dealing with thick-walled cylinders, see the work by Rimrott [4], Chakrabarty [40]. Namely, the strengthening due to material hardening is exceeded by the weakening resulting from the widening deformation and viscoplasticity.

The onset of instability is then acquired in terms of the inner radius a / a_0 by using fixed point iteration. Figure 3 shows the relationship between the onset of instability and the yield strength ratio $R = \sigma_\infty / \sigma_0$ with various values of the strain-rate sensitivity m. Again, the computed results for the onset of instability are in good agreement with the analytical solutions as shown in Figures 1~3. On the other hand, it is found that the strengthening phenomena exist only for the cases with $\sigma_\infty / \sigma_0 > m + 2$. Note that, considering the viscoplastic strain-hardening behavior with the hardening exponent $h = \sqrt{3}$, the stability condition for the widening problem of rotating hollow cylinders is obtained analytically as $\sigma_\infty / \sigma_0 > m + 2$.

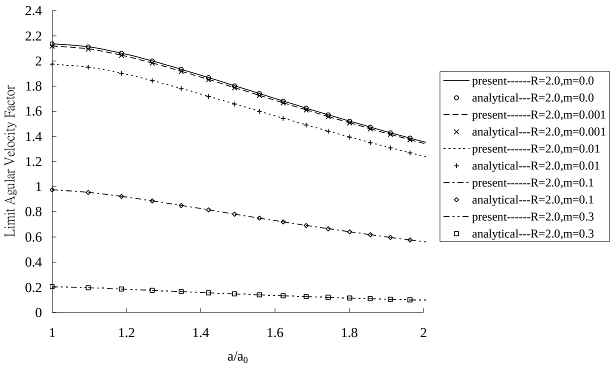

Figure 1 Effect of strain-rate sensitivity m on the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ with yield strength ratio $R = \sigma_\infty/\sigma_0 = 2$

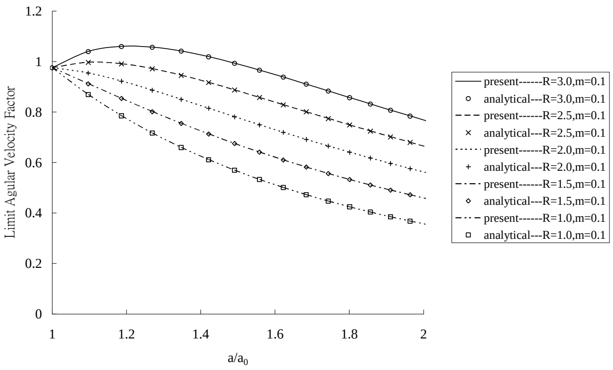

Figure 2 Effect of yield strength ratio $R = \sigma_\infty/\sigma_0$ on the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ with strain-rate sensitivity $m = 0.1$

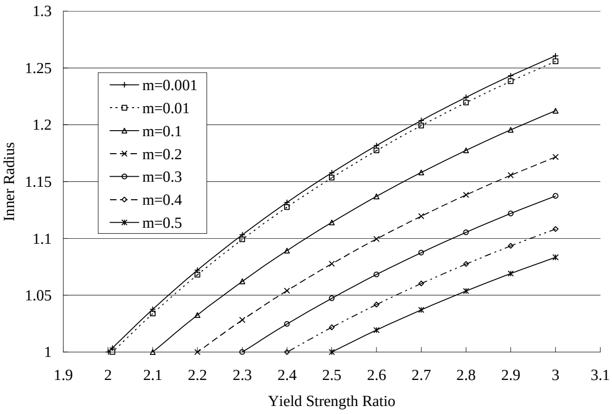

Figure 3 Effect of strain-rate sensitivity m and yield strength ratio $R = \sigma_\infty/\sigma_0$ on the onset of instability in terms of the inner radius a/a_0

4. CONCLUSION

Plastic limit angular velocities of cylinders are useful information for structure optimal design or safety evaluation. The paper is based on the concept of sequential limit analysis to investigate analytically the plastic limit angular velocity of rotating hollow cylinders made of nonlinear isotropic strain-hardening viscoplastic materials.

Particularly, analytic solutions of the plastic limit angular velocity as well as the onset of instability and the stability condition corresponding to the hardening exponent $h = \sqrt{3}$ were also derived in the paper for rigorous comparisons. The onset of instability was novelly acquired in terms of the inner radius a/a_0 by using fixed point iteration. The computed upper-bound results are in good agreement with analytical solutions.

Especially, it is found numerically and analytically that the strengthening phenomena exist only for the cases with $\sigma_\infty/\sigma_0 > m+2$ considering the viscoplastic strain-hardening behavior with the hardening exponent $h = \sqrt{3}$ and the strain-rate sensitivity m.

5. ACKNOWLEDGEMENT

This research is being partially supported by the National Science Council through Grant NSC 96-2221-E-157-003.

6. REFERENCES

[1] E.A. Davis and F.M. Connelly, “Stress Distribution and Plastic Deformation in Rotating Cylinders of Strain-Hardening Material,” J. Appl. Mech., vol. 26, pp. 25-30, 1959.
[2] J. Lenard and J.B. Haddow, “Plastic Collapse Speeds for Rotating Cylinders,” Int. J. Mech. Sci., vol. 14, pp. 285-292, 1972.
[3] A. Nadai, *Theory of Flow and Fracture of Solids*, McGraw-Hill, New York, 1950.
[4] F.P.J. Rimrott, “On the Plastic Behavior of Rotating Cylinders,” J. Appl. Mech., vol. 27, pp. 309-315, 1960.
[5] S.-Y. Leu and J.T. Chen, “Sequential Limit Analysis of Rotating Hollow Cylinders of Nonlinear Isotropic Hardening,” CMES-Comp. Model. Eng. Sci., vol. 14, pp. 129-140, 2006.
[6] N. Alexandrova and S. Alexandrov, “Elastic-Plastic Distribution in a Rotating Annular Disk,” Mech. Based Des. Struct. Mech., vol. 32, pp. 1-15, 2004.
[7] A.N. Eraslan and H. Argeso, “Limit Angular Velocity of Variable Thickness Rotating Disks,” Int. J. Solids Struct., vol. 39, pp. 3109-3130, 2002.
[8] U. Güven, “The Fully Plastic Rotating Disk with Rigid Inclusion,” ZAMM, vol. 77, pp. 714-716, 1997.
[9] G. Ma, H. Hao and Y. Miyamoto, “Limit Angular Velocity of Rotating Disk with Unified Yield Criterion,” Int. J. Mech. Sci., vol. 43, pp. 1137-1153, 2001.
[10] Y. Orcan and A.N. Eraslan, “Elastic-Plastic Stresses in Linearly Hardening Rotating Solid Disks of Variable Thickness,” Mech. Res. Commun., vol. 29, pp. 269-281, 2002.
[11] S.-Y. Leu, “Limit Analysis of Viscoplastic Flows Using an Extended General Algorithm Sequentially:

Convergence Analysis and Validation," Comput. Mech., vol. 30, pp. 421-427, 2003.

[12] S.-Y. Leu, "Convergence Analysis and Validation of Sequential Limit Analysis of Plane-Strain Problems of the von Mises Model with Nonlinear Isotropic Hardening," Int. J. Numer. Methods Engrg., vol. 64, pp. 322-334, 2005.

[13] S.-Y. Leu, "Analytical and Numerical Investigation of Strain-Hardening Viscoplastic Thick-Walled Cylinders under Internal Pressure by Using Sequential Limit Analysis," Comput. Methods Appl. Mech. Engrg., vol. 196, 2713-2722, 2007.

[14] S.-Y. Leu, "Investigation of Rotating Hollow Cylinders of Strain-Hardening Viscoplastic Materials by Sequential Limit Analysis," Comput. Methods Appl. Mech. Engrg., under review, 2007.

[15] M. Haghi and L. Anand, "Analysis of Strain-Hardening Viscoplastic Thick-Walled Sphere and Cylinder under External Pressure," Int. J. Plasticity, vol. 7, pp. 123-140, 1991.

[16] P.G. Ciarlet, *Introduction to Numerical Linear Algebra and Optimisation*, Cambridge University Press, Cambridge, 1989.

[17] E. Anderheggen and H. Knöpfel, "Finite Element Limit Analysis Using Linear Programming," Int. J. Solids Struct., vol. 8, pp. 1413-1431, 1972.

[18] A. Bottero, R. Negre, J. Pastor and S. Turgeman, "Finite Element Method and Limit Analysis Theory for Soil Mechanics Problems," Comput. Methods Appl. Mech. Engrg., vol. 22, pp. 131-149, 1980.

[19] A. Capsoni and L. Corrad, "A Finite Element Formulation of the Rigid-Plastic Limit Analysis Problem," Int. J. Numer. Methods Engrg., vol. 40, pp. 2063-2086, 1997.

[20] E. Christiansen, "Computation of Limit Loads," Int. J. Numer. Methods Engrg., vol. 17 pp. 1547-1570, 1981.

[21] L. Corradi, L. Luzzi and P. Vena, "Finite Element Limit Analysis of Anisotropic Structures," Comput. Methods Appl. Mech. Engrg., vol. 195, pp. 5422-5436, 2006.

[22] N. Dang Hung, "Direct Limit Analysis via Rigid-Plastic Finite Elements," Comput. Methods Appl. Mech. Engrg., vol. 8, pp. 81-116, 1976.

[23] P.G. Hodge and T. Belytschko, "Numerical Methods for the Limit Analysis of Plates," J. Appl. Mech., vol. 35, pp. 796-802, 1968.

[24] H. Huh and W.H. Yang, "A General Algorithm for Limit Solutions of Plane Stress Problems," Int. J. Solids Struct., vol. 28, pp. 727-738, 1991.

[25] J. Pastor, T-H. Thai and P. Francescato, "New Bounds for the Height Limit of a Vertical Slope," Int. J. Numer. Anal. Methods Geomech., vol. 24, pp. 165-182, 2000.

[26] S.W. Sloan, "Upper Bound Limit Analysis Using Finite Elements and Linear Programming," Int. J. Numer. Anal. Methods Geomech., vol. 13, pp. 263-282, 1989.

[27] W.H. Yang, "Minimization Approach to Limit Solutions of Plates," Comput. Methods Appl. Mech. Engrg., vol. 28, pp. 265-274, 1981.

[28] W.H. Yang, "Admissibility of Discontinuous Solutions in Mathematical Models of Plasticity," *in: W.H. Yang Ed., Topics in Plasticity*, AM Press, MI, 1991.

[29] Y.G. Zhang, P. Zhang and W.M. Xue, "Limit Analysis Considering Initial Constant Loadings and Proportional Loadings," Comput. Mech., vol. 14, pp. 229-234, 1994.

[30] J.N. Reddy, *An Introduction to the Finite Element Method*, McGraw-Hill, New York, 1993.

[31] D.G. Luenberger, *Linear and Nonlinear Programming*, Addison-Wesley, MA, 1984.

[32] L. Corradi, N. Panzeri and C. Poggi, "Post-Critical Behavior of Moderately Thick Axisymmetric Shells: a Sequential Limit Analysis Approach," Int. J. Struct. Stabil. Dynam., vol. 1, pp. 293-311, 2001.

[33] L. Corradi and N. Panzeri, "A Triangular Finite Element for Sequential Limit Analysis of Shells," Adv. Engng. Software, vol. 35, pp. 633-643, 2004.

[34] H. Huh and C.H. Lee, "Eulerian Finite-Element Modeling of the Extrusion Process for Working-Hardening Materials with the Extended Concept of Limit Analysis," J. Mater. Process. Technol., vol. 38, pp. 51-62, 1993.

[35] H. Huh, C.H. Lee and W.H. Yang, "A General Algorithm for Plastic Flow Simulation by Finite Element Limit Analysis," Int. J. Solids Struct., vol. 36, pp. 1193-1207, 1999.

[36] C.L. Hwan, "An Upper Bound Finite Element Procedure for Solving Large Plane Strain Deformation," Int. J. Numer. Methods Engrg., vol. 40, pp. 1909-1922, 1997.

[37] K.P. Kim and H. Huh, "Dynamic Limit Analysis Formulation for Impact Simulation of Structural Members," Int. J. Solids Struct., vol. 43, pp. 6488-6501, 2006.

[38] W.H. Yang, "Large Deformation of Structures by Sequential Limit Analysis," Int. J. Solids Struct., vol. 30, pp. 1001-1013, 1993.

[39] W.H. Yang, "A Duality Theorem for Plastic Plates," Acta Mech., vol. 69, pp. 177-193, 1987.

[40] J. Chakrabarty, *Theory of Plasticity*, McGraw-Hill, New York, 1987.