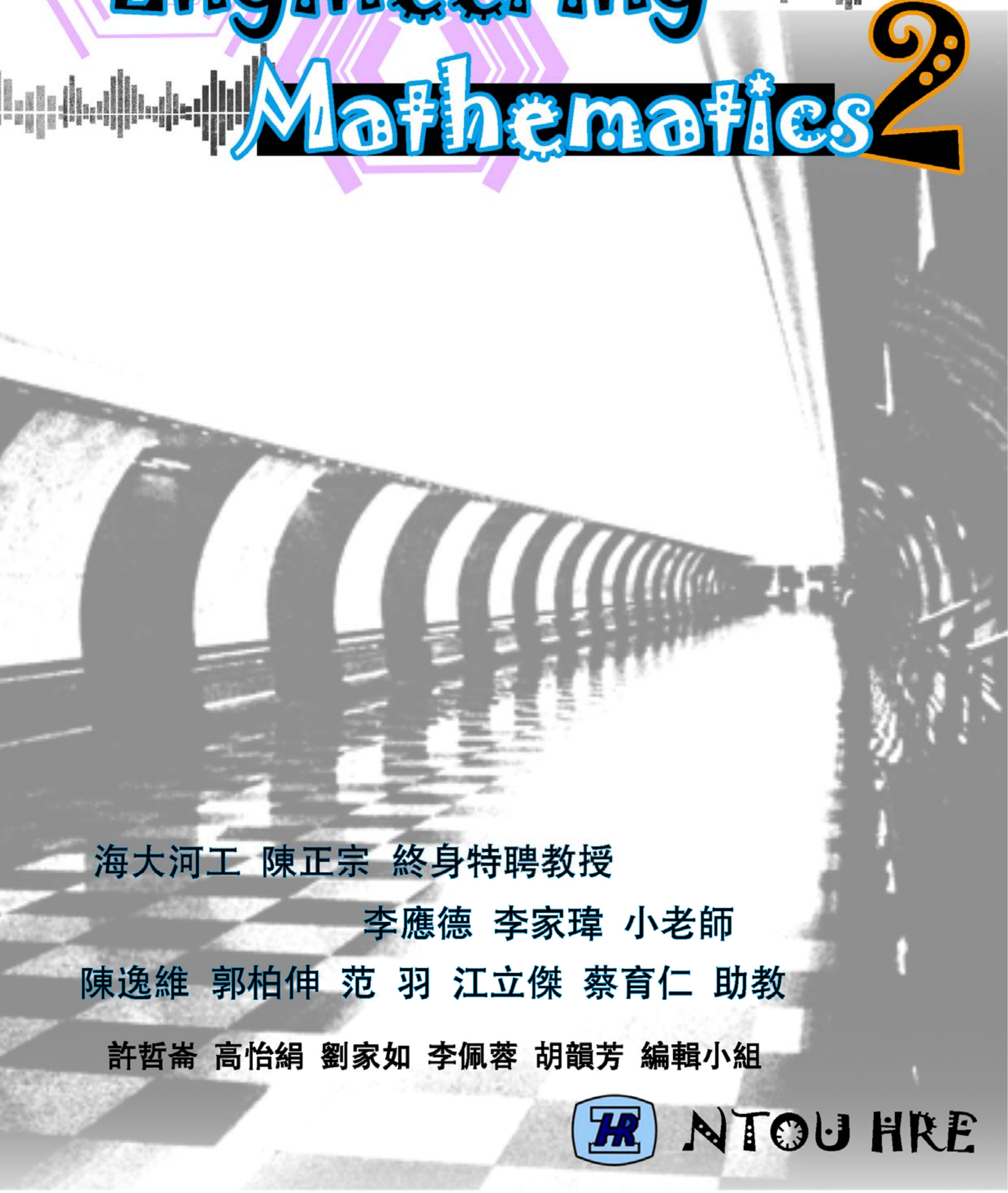
Engineering Mathematics 2



海大河工 陳正宗 終身特聘教授

李應德 李家瑋 小老師

陳逸維 郭柏伸 范 羽 江立傑 蔡育仁 助教

許哲崙 高怡絹 劉家如 李佩蓉 胡韻芳 編輯小組

NTOU HRE

力學聲響振動實驗室 工程數學(二)2012,02~2012,06 陳正宗終身特聘教授

第零章 工數修課相關資料

100年度第2學期行事曆--1
工數二修課名單--2
寫在前面--4
歷屆資訊1994-2012 --5
海大河工系數學相關課程規劃--6
海大工學院基礎數學課程教學內容--------------------------------------7
河工系2B上課日程與內容---8
李家瑋領班心得--9
編輯小組同學心得---10

國立臺灣海洋大學100學年度第2學期行事曆

（99學年度第2學期第2次行政會議通過；教育部100年4月28日臺高（一）字第1000068100號函同意備查；100年12月27日海人字第10006200161號函修正）

年	月	星期 週次	星期日	星期一	星期二	星期三	星期四	星期五	星期六	舉辦事項
中華民國一〇一年	二月					1	2	3	4	（1）第二學期開始
			5	6	7	8	9	10	11	2月中旬預官考選成績發放（9）復學生註冊
			12	13	14	15	16	17	18	（16-24）1002第三階段電腦選課（18）寒假結束
		一	19	20	21	22	23	24	25	（19）各學制學生開始上課（20-21）註冊（21）就學貸款申辦截止日（22-23）准假學生補註冊（2/25-3/2）1002 人工特殊加選及超修學分申請
		二	26	27	28	29				（27）調整放假日（28）和平紀念日（放假）
	三月						1	2	3	（3）補行上班、該日課程由教師自行擇期補課（1-30）校內獎學金申請
		三	4	5	6	7	8	9	10	3月中旬預官選填志願申請，研發替代役申請（9）選課上網確認截止日
		四	11	12	13	14	15	16	17	（12-19）舊生床位保留申請（17）中、英文會考
		五	18	19	20	21	22	23	24	
		六	25	26	27	28	29	30	31	
	四月	七	1	2	3	4	5	6	7	（4）兒童節、民族掃墓節（放假）（5-6）敦親活動（教職員彈性放假、該日課程由教師自行擇期補課）
		八	8	9	10	11	12	13	14	
		九	15	16	17	18	19	20	21	（15-21）期中考試（20）學分費繳交截止日（20-30）碩士生學位考試申請
		十	22	23	24	25	26	27	28	（23）期中預警輸入開始日（23-27）轉系申請（23-30）舊生住宿電腦抽籤申請（4/23-5/4）期中退選作業，教育學程申請
		十一	29	30						（4/30-5/11）下學期學雜費減免申請
	五月				1	2	3	4	5	（3-4）變更轉系申請
		十二	6	7	8	9	10	11	12	5月中旬預官錄取公告（7）期中預警輸入截止日
		十三	13	14	15	16	17	18	19	（14-18）100暑修第一期登記（18）轉系考試（18-24）1011第一階段電腦選課
		十四	20	21	22	23	24	25	26	（20-26）畢業生學期考試（21-28）學生宿舍暑期住宿申請
		十五	27	28	29	30	31			（5/30-6/1）電腦選課抽籤作業（31）博士生學位考試申請截止日，水上運動會
	六月							1	2	
		十六	3	4	5	6	7	8	9	（9）畢業典禮（全校正常上班上課）
		十七	10	11	12	13	14	15	16	（11）補假一日（畢業典禮）（12-18）1011第二階段電腦選課（15）1002學期申請休學截止日（15-22）研究所新生暑期住宿申請
		十八	17	18	19	20	21	22	23	（17-23）學期考試（23）端午節（適逢例假日，不補假）
			24	25	26	27	28	29	30	（24）暑假開始（29-30）學生宿舍關閉（30）暑期住宿宿舍開放
	七月		1	2	3	4	5	6	7	
			8	9	10	11	12	13	14	（9）100暑修第一期開始上課（9-13）100暑修第二期登記（13-20)研究所新生床位保留申請
			15	16	17	18	19	20	21	
			22	23	24	25	26	27	28	
			29	30	31					（31）第二學期結束，研究生畢業論文繳交截止日

備註：

1 具原住民身分者 其各該原住民族歲時祭儀放假日期 依行政院原住民族委員會公告日期放假 （紀念日及節日實施

001	電機工程學系 2A	00053202	周信佑	
002	河海工程學系 2B	09952101	黃子翔	
003	河海工程學系 2B	09952102	劉家如	
004	河海工程學系 2B	09952103	余柏宏	
005	河海工程學系 2B	09952104	張永霖	
006	河海工程學系 2B	09952105	張力	
007	河海工程學系 2B	09952106	藍舒平	
008	河海工程學系 2B	09952107	吳旭強	
009	河海工程學系 2B	09952108	陳弘彝	
010	河海工程學系 2B	09952109	邱駿豪	
011	河海工程學系 2B	09952110	陳峙霖	
012	河海工程學系 2B	09952111	邱揚凱	
013	河海工程學系 2B	09952112	黃仲安	
014	河海工程學系 2B	09952113	吳振豪	
015	河海工程學系 2B	09952114	高怡絹	
016	河海工程學系 2B	09952115	王藝潔	
017	河海工程學系 2B	09952116	王昱璁	
018	河海工程學系 2B	09952117	張恩慈	
019	河海工程學系 2B	09952118	郭哲瑋	
020	河海工程學系 2B	09952119	李柏霖	
021	河海工程學系 2B	09952120	謝晴杰	
022	河海工程學系 2B	09952121	李佩蓉	
023	河海工程學系 2B	09952122	顏立翔	
024	河海工程學系 2B	09952123	許少陵	
025	河海工程學系 2B	09952125	李佩諭	
026	河海工程學系 2B	09952126	戴　薇	
027	河海工程學系 2B	09952127	林竹恩	
028	河海工程學系 2B	09952128	林翊翔	
029	河海工程學系 2B	09952129	楊晉	
030	河海工程學系 2B	09952130	許哲崙	
031	河海工程學系 2B	09952131	黃昱豪	
032	河海工程學系 2B	09952132	陳振鈞	
033	河海工程學系 2B	09952133	胡韻芳	

034	河海工程學系 2B	09952135	張育仁	
035	河海工程學系 2B	09952136	劉宗瑋	
036	河海工程學系 2B	09952137	徐兆緯	
037	河海工程學系 2B	09952138	彭櫳漢	
038	河海工程學系 2B	09952139	文軍強	
039	河海工程學系 2B	09952140	陳堯夫	
040	河海工程學系 2B	09952141	林昱汶	
041	河海工程學系 2B	09952142	詹承諭	
042	河海工程學系 2B	09952143	張哲嘉	
043	河海工程學系 2B	09952144	王思晴	
044	河海工程學系 2B	09952146	陳韋翰	
045	河海工程學系 2B	09952147	鄭宇宏	
046	河海工程學系 2B	09952148	楊瑾	
047	河海工程學系 2B	09952149	黃舜揚	
048	河海工程學系 2B	09952150	羅文韋	
049	河海工程學系 2B	09952151	朱佩欣	
050	河海工程學系 3B	09952207	楊政霖	
051	電機工程學系 2A	09953016	陳靖文	
052	電機工程學系 2A	09972142	林奕穎	
053	河海工程學系 4B	B97520125	李利淳	
054	河海工程學系 4B	B97520150	賴威宇	
055	河海工程學系 4B	B97520151	李哲	
056	電機工程學系 4A	B97530046	周立榮	
057	機械與機電工程學系 4A	B97720053	林暐崧	
058	系統工程暨造船學系 3A	B98510060	王智弘	
059	河海工程學系 2B	B98520143	游令婉	
060	河海工程學系 5A	B98523004	何奎德	
061	輪機工程學系能源應用組 3A	B986A0007	許孝民	
062	機械與機電工程學系 3B	B98720124	高煜凱	
063	機械與機電工程學系 3B	B98720144	蘇青	
064	機械與機電工程學系 3B	B98720154	楊宗穎	

寫在前面

本講義係根據本人在海大河工系教授工程數學多年 累積的手稿, 除了包括工數內容的介紹外, 亦含蓋了歷年的作業、小考、大考及一些參考解答。希望對工數有興趣的同學、老師、學者專家或工程師們能有參考的價值。作者將多年來教學的心得與實務工作經驗融入, 並加入習題, 儘量朝向適合當教科書的方向來努力。同時, 也將最近研究過程所用到的工數併入, 希望初學者在學習的過 程中, 除了能入門外, 也能 體會到學工數眞正的目的與樂趣。當然, 我們也希望經由興趣的培養, 能對工數有更深一層的体認, 而 不僅流於應付考試的庸俗想法。本講義的資料, 多數取材於現今幾本知名的工數教科書, 當然也含蓋我們在教學研究多年所 獲得的心得。除了感謝作者的老師、同事、同好、研究生與修 課同學們的貢獻外, 過往學者專家對本講義的 諸多建議, 也一併予以考慮。然而再有的暇疵, 當然是由於作者的疏忽所致。若有任何批評與指教, 煩請告知。

陳正宗 Jeng-Tzong Chen, Ph.D., Prof.
海洋大學河工系教授
E mail:B0209@ntou66.ntou.edu.tw
E mail:jtchen@ind.ntou.edu.tw

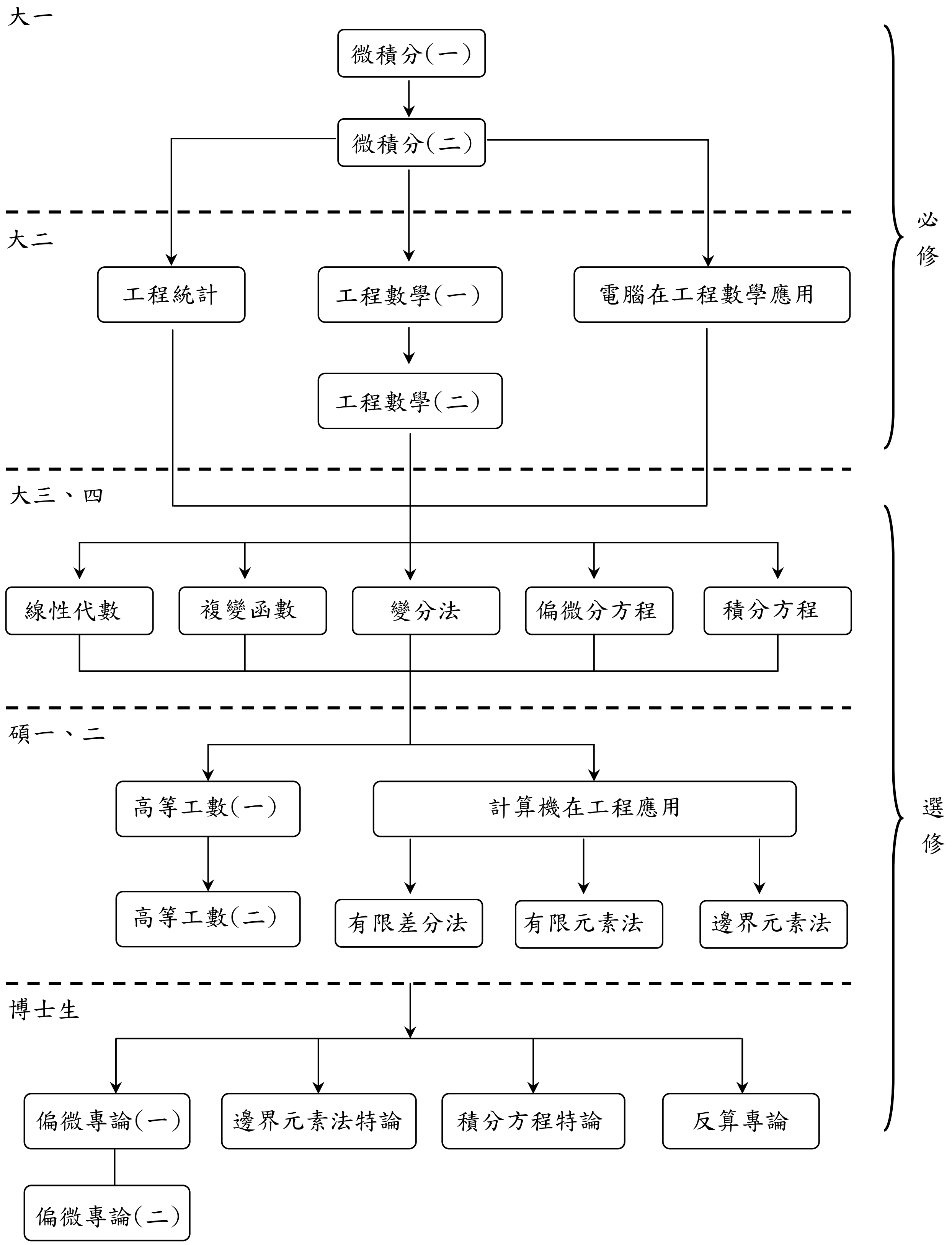
西元一九九九年九月二十八日

台北郵政 23 之 36 號信箱
基隆郵政 7 之 59 號信箱

年 代	教 師	教 課 書/助教	學 生
1994	陳正宗(工數三 2,8、四 21)	Churchill, Farlow(游大偉與楊森翔)	陳桂鴻
1995	梁明德、陳正宗(B)	Grossman(陳桂鴻與翁煥昌)	林建華
1996	梁明德、陳正宗(B)	Whylie (陳鈺文與鄭超明)	李慶鋒
1997	梁明德、葉為忠	Kreyszig	劉立偉
1997	陳正宗(工數三 58、四 19)	Churchill, Farlow(丘宜平與黃川夏)	李慶鋒
1998	梁明德、陳正宗(A)	Greenberg(林建華,程永正,鍾渝隆)	林盛益
1999	林三賢、陳正宗(B)	Kreyszig(李慶鋒,陳韋誌,張銘翰)	葉雅婷
2000	梁明德、陳正宗(B)	Riley(劉立偉與林書睿)	朱雅雯
2001	梁明德、林三賢	Lopez	蔡孟蓉
2001	陳正宗(工數三 42、四 15)	Riley(林盛益與林宗衛)	李應德
2002	陳正宗(B)、張景鐘 、曹登皓	Oneil(吳清森與李應德)	林智凱
2003	陳正宗(B)、張景鐘、曹登皓	Oneil(蕭嘉俊,沈文成,陳佳聰)	林坤生
2004	陳正宗(工數三 68)	Oneil(李應德與高政宏)	吳建和
2004	陳桂鴻(工數一、二) 2A	Oneil(沈文成與吳安傑)	吳國綸
2004	呂學育(工數一、二) 2B	Oneil(蕭嘉俊與陳柏源)	謝紹恆
2005	陳正宗(工數四 23)	Oneil(李應德與陳佳聰)	高聖凱
2005	陳桂鴻(工數一、二) 2A	Zill(高政宏)	林裕桀/余尚儒
2005	呂學育(工數一、二) 2B	Zill(柯佳男與廖奐禎)	曹大發
2006	陳正宗(複變 19)	Brown and Churchil(李應德)	吳國綸
2007	陳正宗(偏微分方程 11)	Riley(廖奐禎與高聖凱)動畫	周克勳
2007	葉為忠(2A) 陳正宗(2B/65)	Kreyszig(周克勳與李家瑋)	陳聖詒/賴芝亭
2007	徐文信(複變)	Churchill	林羿州/謝祥志
2011	陳正宗(B)、張景鐘、呂秋水	Kreyszig(陳逸維與郭柏伸)	李佩諭 高怡絹

(File:book-2012-math2.doc date:Jan.14/2012) 2005 08-2006 07 休假 2008-2010 停開

海大河工系數學相關課程規劃　　陳正宗終身特聘教授



Mathplan-HRE-2007.doc　by Chen J T Aug. 20, 2007.

數理程度 每下愈況 大家加油

海洋大學工學院基礎數學課程教學內容

微積分(上): 1. 函數、極限與連續函數 2. 微分的運算與基本函數的微分 3. 微分的性質與應用 4. 定積分、不定積分與微積分基本定理 5. 積分應用 6. 超越函數。

微積分(下): 1. 超越函數 2. 積分技巧 3. 瑕積分 4. 無窮級數 5. 多變數函數微分 6. 極坐標 7. 重積分。

工程數學(一): 1. 一階常微分方程 2. 高階常微分方程 3. 矩陣。

工程數學(二): 1. 向量分析 2. Fourier 級數及轉換 3. Laplace 轉換。

海洋大學河工系工程數學教學評鑑小組 陳正宗終身特聘教授與葉為忠教授

【file:mathcontent2007.doc by Chen Aug.29/2007】

工程數學 2B 班(2012,2-2012,6)上課日程及內容

陳正宗終身特聘教授(海大河工系)E-mail:jtchen@mail.ntou.edu.tw

週數	日期	預定進度		備註
1	Feb.22	Introduction		檢定考試
2	Feb.29	Vector analyis	空間曲線描述法	小考(1)
3	Mar.07	Gradient, curl	Frenet formula	
4	Mar.14	Divergence		小考(2)
5	Mar.21	Green theorem		小考(3)
6	Mar.28	Fourier series		
7	Apr.11	Gibbs phenomenon		期中考
8	Apr.18	Parseval theorem		
9	Apr.25	Stokes 轉換		小考(4)
10	May 02	Fourier transform		
11	May 09	Hilbert transform		小考(5)
12	May 16	Laplace transform		
13	May 23	Initial and final theorem		
14	May 30	Convolution		
15	Jun.6	Applications of LT		
16	June13	Applications (II)		
17	Jun.13			期末考
				公佈成績
				June 9 畢業典禮

海洋大學河海工程學系　陳正宗終身特聘教授 math2-2012-schedule.doc,Jan.26/12)

2012 工數(二)講義編輯緣由

工程數學是一門數學與物理結合的科目，是身為一個工程師處理實際工程問題時所該具備的基礎數學能力。我相信要學好工程數學，1.不僅需要老師賣力的演出(相信大家上過課，都可以感受到陳老師的用心)，2.自己付出的努力與時間，3.講義也是一個相當重要的輔助工具。我也相信上學期修習過陳老師工數(一)的同學們，都渴望有一本井然序有的工數講義，因為這對學習工數有著正面的幫助，若再搭配上自己努力的付出與投入的時間，絕對可以達到事半功倍的效果。

然而陳老師的工數講義，是陳老師在海大河工系教授工數十多年來，所累積的資產，內容是相當豐富的，此外有些內容，更是在一般的教科書裡所找不到的，因此更能突顯出這份講義的不凡，也代表了陳老師在這方面付出了多少的心力與人力。因此如何編列出一套井然有序的工數講義，是我們 NTOU/MSV 即刻所面臨的艱鉅課題。而趁著這次寒假，我們 NTOU/MSV 編列了一個工數講義的編輯計畫，就短期目標而言，就是把這學期工數(二)的講義，重新排版一次並附上目錄與一些考試內容，就長期目標而言，則是希望把所有工數的相關檔案，系統性的分類與建檔，編製出一本完全屬於 NTOU/MSV 的工數講義。

因此我們對外招募有興趣且志同道合的學弟妹們，來一起執行這次的工數講義編輯計畫與學習。來 NTOU/MSV 學習，我不認為彼此是雇主與員工的關係，而是一起學習與同甘共苦的夥伴。然而這次的計畫雖有許多學弟妹的報名，但礙於經費的有限，我們從中錄取了五位河工 2B 班的學弟妹們，來執行這次的工數編輯計畫，為各位手上所看到的這份工數講義，盡一點棉薄之力。或許這本講義還不是那麼的盡善盡美，但在這過程中，我看到了這五位夥伴的認真付出與努力，
還請各位同學給這五位編輯同學一點掌聲。但若各位還有看到這本講義瑕疵的地方，煩請各位不吝嗇地告訴我們，如此一來只會使本講義更加完美。

最後感謝河工 2B 許哲崙、高怡絹、劉家如、李佩蓉與胡韻芳，這五位學弟妹在這次寒假的幫忙。

工數講義編輯班

領班:李家瑋

許哲崙

相信修過陳老師工數(一)的人都知道，工數(一)的講義編排很凌亂，而且沒有系統，所以時常聽到以下的對話:「某甲:『啊!我工數講義忘了帶耶!』 某乙:『阿~沒差啦！上陳老師的課不用講義，帶紙跟筆記下上課筆記就好了。』」由此得證，工數講義並未被善加利用。有鑑於此，老師組成了工數講義編輯小組，要我們來編排講義內容的順序。所以，你們現在手上的這本講義，是我們精心編排出來的，認為這樣子的編排順序最能夠和老師上課的內容有所呼應，也最容易讓同學們讀懂，所以希望同學們能善加利用，當然若同學們認為講義的編排還有任何缺點，煩請告知，我們會繼續努力。

小故事

記得剛進研究室時，老師跟我說:「你們編輯這些資料要小心，就是說，要是你們一不小心把資料用不見的話，我就要退休了。」當時我漫不經心的笑了笑，後來發現原來我自己弄一張講義要他 XX 的兩個多小時，我才知道他說的都是真的。我也還記得老師問過我說:「你以前有沒有常常上我們研究室的網站去看看資料、歷屆考題甚麼的」我說:「沒有，只有偶爾上去考考試時間和地點而已」老師說:「厚！難怪你成績那麼差」也是後來才發現原來老師網站上的資料一直都還保留得很齊全，也很豐富，所以若有需求或疑問的同學，也可上老師研究室的網站去看看，相信會有更進一步的收穫。

高怡絹

這次的講義編輯，我負責的部分是向量微積分，在編輯的過程中，對向量微積分有初步的認識，除此之外，也培養了一些文書處理的能力，在剛開始的前幾個禮拜，大多只是做點簡單的雜事，看看講義的內容，在要把講義裝訂成一本時，把先前的工作量通通補回來，感覺挺充實的，相信經過這次的講義編排工作，可以讓我對下學期的工數有更深的了解及吸收。

劉家如

這次寒假到工數研究室幫忙編輯講義，發現真的學習到很多。除了懂得架設網站、排列講義順序之外，更學習到如何用 Mathematica 畫圖形，連上學期學到的 Matlab 也更熟悉更懂得運用了。雖然才一個寒假，卻過得特別充實，沒事的時候還可以讀讀書，先預習下學期會學到的課程，相信下學期開始上課時，一定能很快就進入狀況的。而且當大家一起攜手完成工數(二)的講義時，看到自己名字出現在講義上時，真的覺得努力辛苦是值得的。所以希望之後還可以有機會繼續在研究室學習幫忙，讓自己變得更充實更有實力！

李佩蓉

一開始到研究室什麼都不熟悉，但慢慢學習後，很多以前不曾接觸的事物也學習到了，例如：護貝文件，修改網頁、軟體的應用還有工數(二)的部分也接觸了一點，雖然在研究室的時間沒有很長，但每天忙忙碌碌感覺很充實，對學術上的事物也有一點認識，當完成工數(二)講義最後步驟的時候，覺得這一段時間的努力都值得了，團隊的感覺真好。

胡韻芳

我負責整理的是一些有的沒的，雖然沒有特別學到什麼工數二的內容，但是我想在這寒假參與的過程當中，我學到不少東西，像是更新網頁、使用 pdf、作資料的搜尋語彙整，最重要的學習是與老師、學長還有共同編輯的同學互相溝通，參與合作免不了每個人各有意見的情形發生，但是如何在中間取得共識與協調，是我看到而且還要繼續學習的，我想這必定是受用一生的智慧。

第一章　向量微積分

1.1 微積分基本內容

工程數學－向量微積分--11

1.2 面積相關

Area computation--12
最簡單的邊界元素法－求積儀--13
場論--14
由向量微積分算面積--15
Divergence of position vector and gradient of radial function-------16

1.3 平面曲線與空間曲線

平面、空間曲線--17
空間曲線表示(Mathematica) --18
Exact form-parameter--19
工程數學－空間曲線描述法--23

1.4 曲率相關

曲率與曲率半徑--25
Radius of curvature for 2-D plane--26
2D 例題--27
3D 例題--29
Flowchart--32
Frenet formula--33
Radius of curvature for plane curve--35
曲率兩種表示法關係(曲率工數到材力.doc) --36
曲率兩種表示法關係(曲率材力到工數.doc) --37

1.5 梯度、散度、旋度、方向導微

Gradient-梯度∇--38
散度與旋度--39
散度與旋度－簡例--40
保守力場與路徑無關積分--41
保守力場與路徑無關(2-D 與 3-D) --42
向量微積分基本物理量--43
靜力、動力與材力--44
工程數學－Vector operators for radial basis functions--45
工程數學(三)三角函數積分--46
Gradient-方向導微--47
Taylor 多項式逼近--48
Taylor’s expansion--53
方向導微(directional derivatives) --54

工數(二)大考 1---55
Directional derivative--56
工數(二)大考 1---57

1.6 Green 定理、Gauss 定理、Stokes 定理

Green theorem---58
Green 恆等式--59
Two views of Green’s theorem--------------------------------------61
面積分轉線積分---62
雙重積分變成單重積分---65
重積分轉線積分---67
Gauss-stokes--71
Stokes’ Theorem---72

1.7 小考

小考與解答---73

1.8 大考

大考與解答---84

Vector calculus 向量分析(林琦焜教授的觀點)

- 向量 微積分基本內容

- 面積相關

面積計算 (最簡單的 BEM) (李家瑋示範求積儀 VCR、2012 求積儀) 場論
由向量微積分算面積
Divergence of position vector and gradient of radial function
從醉月湖的面積談起:向量微積分簡介(蔡聰明,數學傳播 21 卷 2 期,民 86 年)

- 平面曲線與空間曲線

平面曲線	時空參數表示法
空間曲線	空間曲線描述法(mathematica 指令) 十種例子：Mathematica 動畫模擬專區(包刮 nb.與 avi.)
	空間曲線參數表示
	空間曲線描述法 I
	空間曲線描述法 II

- 曲率相關

曲率與曲率半徑
平面曲率半徑(三種方法的比較) (2D 例題) (3D 例題)
Flowchart
Frenet formula
Radius of curvature for plane curve
曲率兩種表示法的關係 (工數到材力) (材力到工數)

由修課到研究：(2D 戴志豪)(3D 施佑勳)暑期計畫

梯度、旋度、散度、方向導微

梯度 散度 旋度	一二三維梯度　　散度與旋度
	散度與旋度-簡例　保守力場與路徑無關積分　保守力場(2D & 3D)
	vector calculus
	散度定理(教學影片 1,2,3,4)
	散度定理應用：靜力、動力與材力
	Vector operators for radial basis functions
	散度定理補充：三角函數積分
方向導微	gradient-方向導微　(泰勒展開複習)(一二三變數泰勒展開)(方向導微)(例題)
	3 月 27 方向導微 上課講義　例題

Green 定理、Gauss 定理　、Stokes 定理

Green 定理	Green theorem 管用嗎
	格林 3 個恆等式(ppt 檔)(李家瑋整理 2007/07/24)
	Green theorem(教學影片 1,2)
Gauss 定理	面積分轉線積分　雙重積分轉單重積分(方法 1&2, 3)
Stokes 定理	Gauss → Stokes　Stokes 練習題

工程數學—向量積分

⑯ line integrals
⑰ gradient and path independence
⑱ double integrals
⑲ Green's theorem
⑳ surface integrals and flux
㉑ divergence and curl
㉒ Stokes' theorem
㉓ divergence theorem
㉔ Jacobian
㉕ applications

⑯向量基本運算與空間幾何

①純量三重積、向量三重積、純量四重積、向量四重積等公式之證明及應用

②空間中點、線、平面等之距離、法線、夾角

⑰ ∇ 算子之相關公式

①方向導數、梯度、散度、旋度之定義、物理意義及演算

② ∇ 算子之相關公式之證明

⑱向量微分：空間曲線

①空間曲線之表示法及弧長計算

②空間曲線切線、法線、曲率、扭率等計算與公式證明

⑲向量微分：坐標系與尺度因子

①直角、圓柱及球面坐標之轉換關係

②正交坐標系之條件，及尺度因子的意義

③圓柱及球面坐標下的梯度、散度、旋度與 Laplace 算子等公式及證明

⑳純量與向量函數之線積分

①直接線積分：參數法

② Green's 定理之應用於求平面曲線線積分

③可由 Green's 定理推衍的其他恆等式之證明

④ Stokes' 定理之應用於求封閉曲線線積分

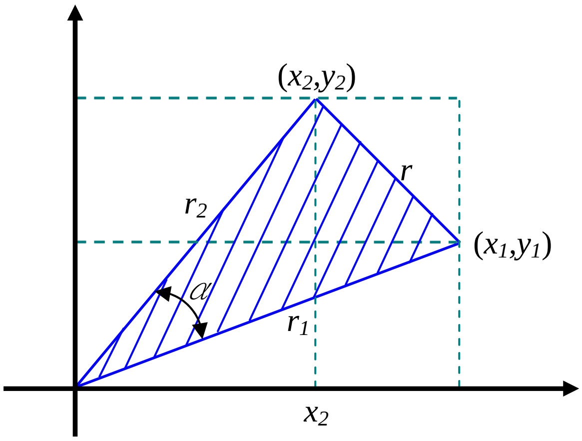
⑳純量與向量函數之曲面積分

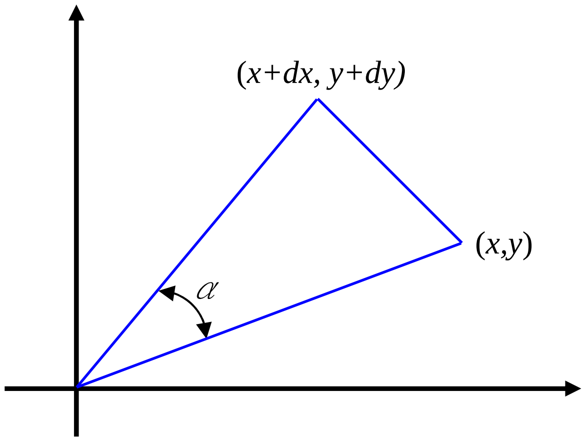
①直接面積分

②散度定理之應用於求曲面積分

③可由散度定理推衍的其他恆等式之證明

Area computation





(1). $A = \frac{1}{2} r_1 r_2 \sin\theta$

(2). $A = \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}(x_2 y_1 - x_1 y_2)$(用幾何來談)

(3). $dA = \frac{1}{2}\begin{vmatrix} x+dx & y+dy \\ x & y \end{vmatrix} = \frac{1}{2}(ydx - xdy)$

(4). $A = \frac{1}{2}\oint ydx - xdy$

(5). $A = \frac{1}{2}\oint \underset{\sim}{r} \bullet \underset{\sim}{n}\, ds$ (應用在求積儀)

(6). $\int Pdx + Qdy = \iint (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$

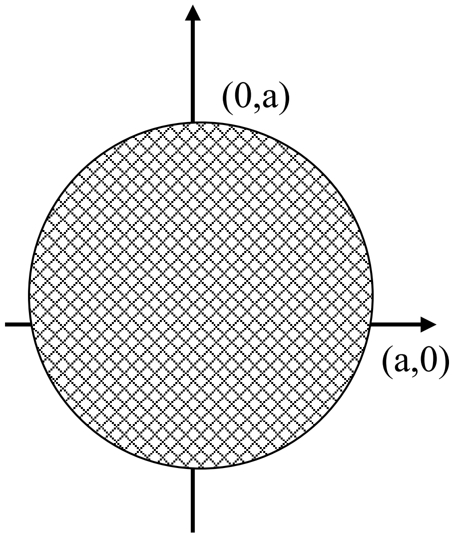
(7). 求積儀繞邊走一圈就知道面積(BEM)

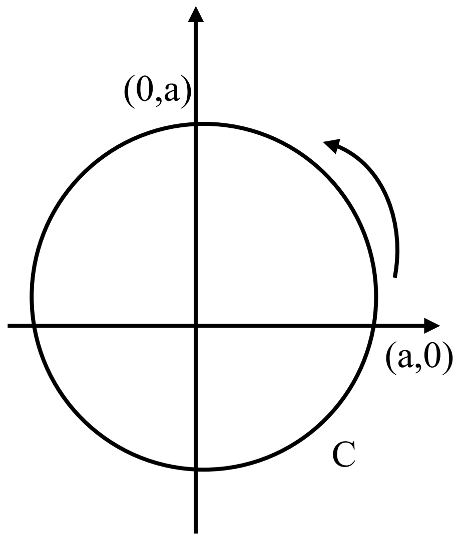
(8). 算方格紙面積(FEM)

AREA2008.doc 羿州 製

最簡單的邊界元素法—求積儀

$$\iint_A dA = \frac{1}{2}\oint_C \underset{\sim}{r}\bullet\underset{\sim}{n}\, ds = \frac{1}{2}\oint_C xdy - ydx$$





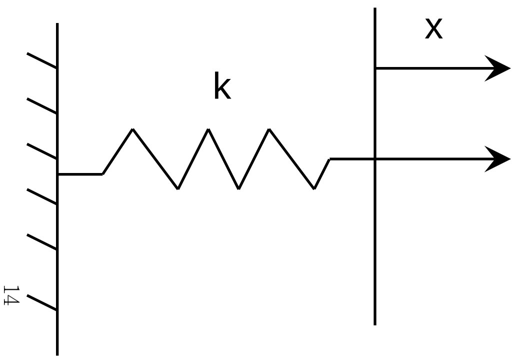
面積分(FEM)　　　　　　線積分(BEM)

方格紙　　　　　　　　　求積儀

$$\iint r dr d\theta = \pi a^2 \quad = \quad \frac{1}{2}\int_0^{2\pi} a a d\theta = \pi a^2$$

場論

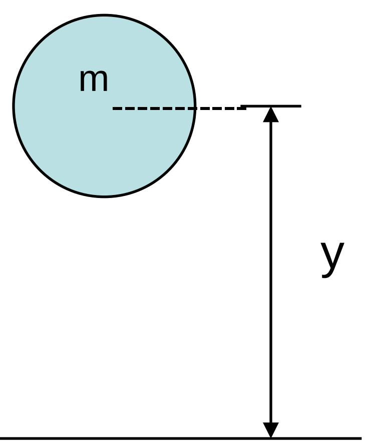
彈簧力場



$$V(x) = \frac{1}{2}kx^2$$

$$-\nabla V(x) = -kx$$

地表重力場



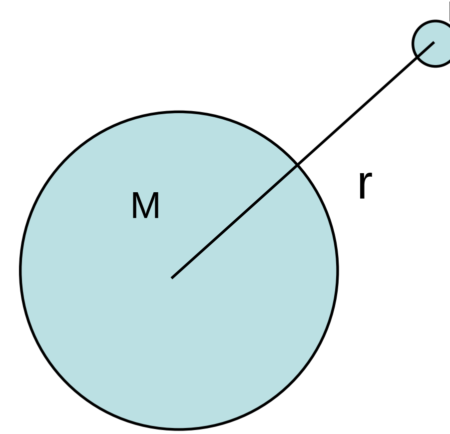
$$V(y) = mgy$$

$$-\nabla V(y) = -mg$$

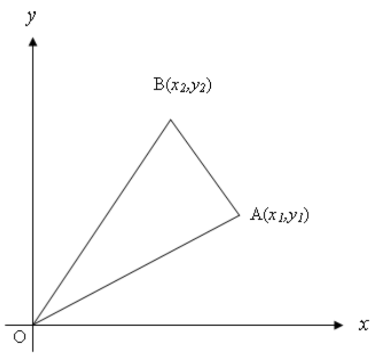
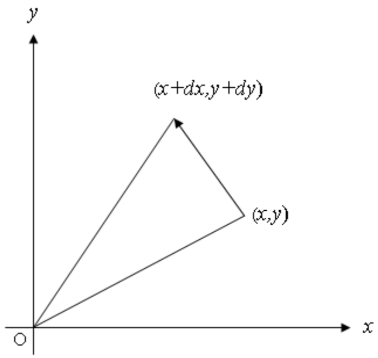
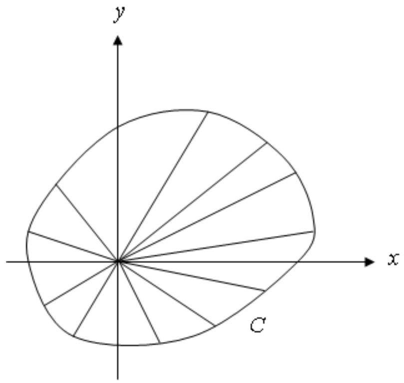
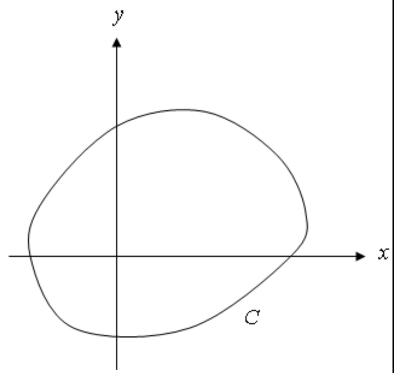
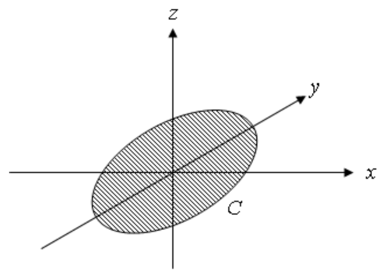
大尺度重力場

$$V(r) = \frac{-GMm}{r}$$

$$-\nabla V(r) = \frac{-GMm}{r^2}\hat{e}_r$$



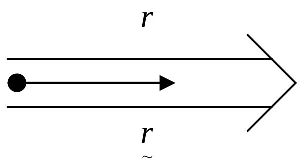
由向量微積分算面積

				
外積(行列式)	neighborhood	Green theorem (1793~1841)	Divergence theorem Gauss (1777~1855)	Stokes theorem (1819~1903)
$\frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_1 & y_1 \end{vmatrix}$	$A=\frac{1}{2}\oint xdy-ydx$	$A=\oint_C Pdx+Qdy$ $=\iint(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dA$ $P=y$ $Q=-x$	$\oint_C \underset{\sim}{v}\cdot\underset{\sim}{n}\, ds=\iint\nabla\cdot v\, dA$ $\underset{\sim}{v}=(Q,-P)$	$\oint_C \underset{\sim}{v}\cdot\underset{\sim}{t}\, ds=\iint_S(\nabla\times\underset{\sim}{v})\cdot\underset{\sim}{n}\, dS$ $\underset{\sim}{v}=(P,Q,0)$ $\underset{\sim}{t}=(\frac{dx}{ds},\frac{dy}{ds},0)$

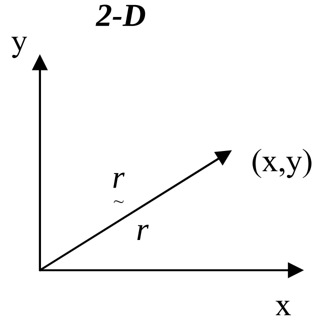
Divergence of position vector and gradient of radial function

1-D		2-D	3-D
r	$\sqrt{x^2}$	$\sqrt{x^2+y^2}$	$\sqrt{x^2+y^2+z^2}$
$\underset{\sim}{r}$	$x\underset{\sim}{i}$	$x\underset{\sim}{i}+y\underset{\sim}{j}$	$x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$
∇r	$\frac{\underset{\sim}{r}}{r}$	$\frac{\underset{\sim}{r}}{r}$	$\frac{\underset{\sim}{r}}{r}$
$\nabla\cdot\underset{\sim}{r}$	1	2	3
$\nabla\times\underset{\sim}{r}$	$\underset{\sim}{0}$	$\underset{\sim}{0}$	$\underset{\sim}{0}$

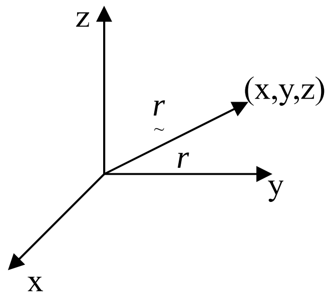
1-D



2-D



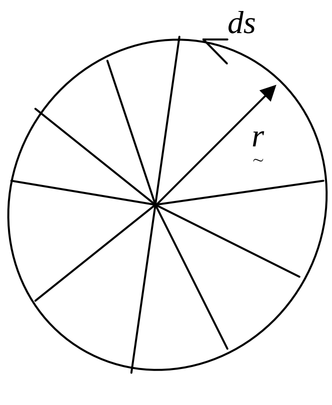
3-D

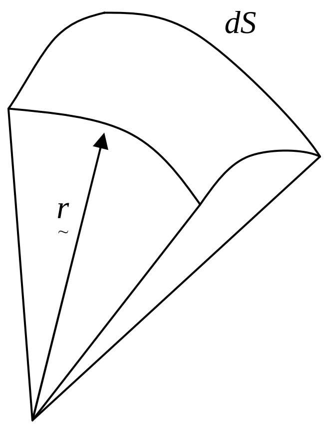


$$\iint \underset{\sim}{r}\cdot\underset{\sim}{n}\,dS = \iiint \nabla\cdot\underset{\sim}{r}\,dV = 3V$$

$$\oint \underset{\sim}{r}\cdot\underset{\sim}{n}\,ds = \iint \nabla\cdot\underset{\sim}{r}\,dA = 2A$$

$$\oint \underset{\sim}{r}\cdot\underset{\sim}{t}\,ds = \iint \nabla\times\underset{\sim}{r}\cdot dA = 0$$





平面、空間曲線

時間、空間的參數表示法轉換

$$\begin{pmatrix} x(t)=t \\ y(t)=3t \end{pmatrix} \Rightarrow \begin{pmatrix} x(s)=? \\ y(s)=? \end{pmatrix}$$

$$(ds)^2=(dx)^2+(dy)^2$$

$$(\frac{ds}{dt})^2=(\frac{dx}{dt})^2+(\frac{dy}{dt})^2 \longrightarrow \text{關鍵}$$

$$\frac{ds}{dt}=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}$$

$$\frac{dx}{dt}=1 \qquad \frac{dy}{dt}=3$$

$$\therefore \frac{ds}{dt}=\sqrt{10}$$

$$\Rightarrow s=\sqrt{10}t$$

$$\therefore t=\frac{s}{\sqrt{10}}+C \qquad C=0$$

$$\therefore \begin{pmatrix} x(t)=t \\ y(t)=3t \end{pmatrix} \Rightarrow \begin{pmatrix} x(s)=\frac{s}{\sqrt{10}} \\ y(s)=\frac{3s}{\sqrt{10}} \end{pmatrix}$$

三維以此類推

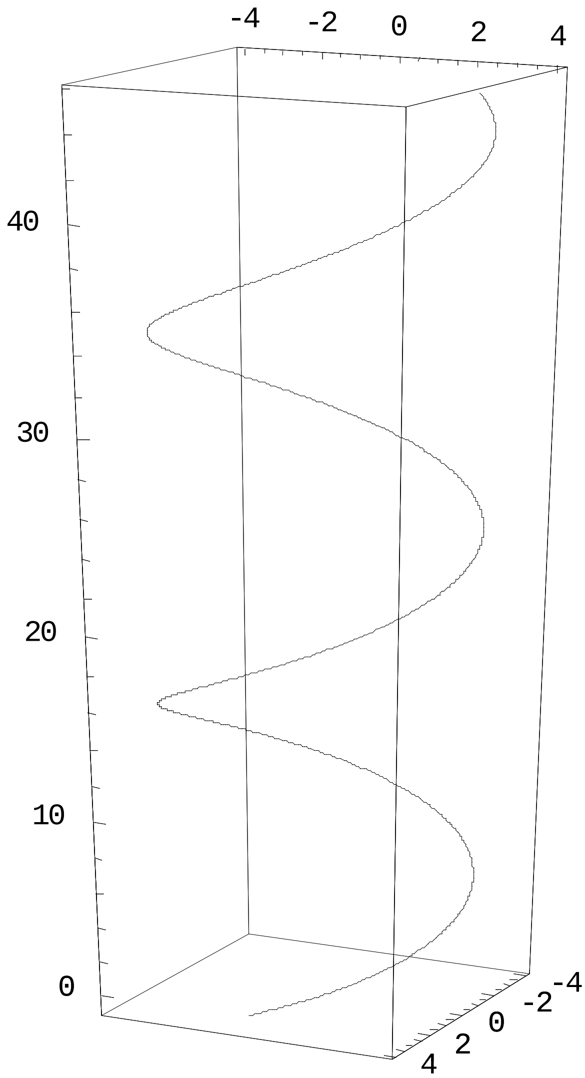
時空參數表示法.doc 2008/03/17 小江製表

空間曲線表示(Mathematica)

STEP 1 指令

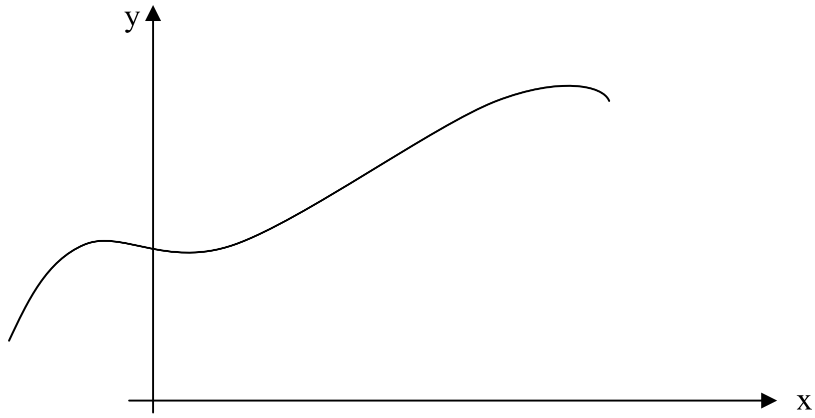
ParametricPlot3D[{4Cos[t],4Sin[t],3t},{t,0,15},PlotPoints➔1000,ViewPoint   {2,1,1},AspectRatio➔1/0.5]

STEP 2 圖形



. Graphics3D.

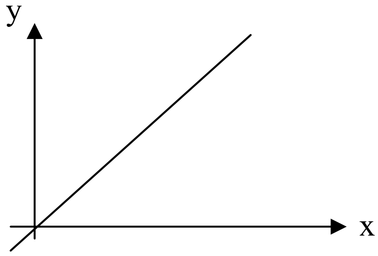
在 **x-y** 平面



Plane　curve (平面參數表示法)

[x(t) , y(t)]

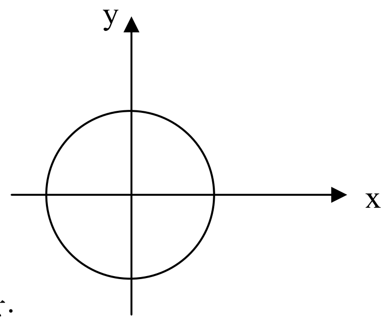
$$\begin{cases} x(t)=t \\ y(t)=t \end{cases}$$ 可以得到圖形如下:



當 x=t 時，

y=x 則爲一條直線

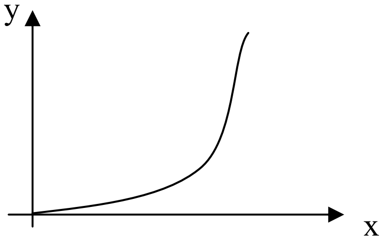
$$\begin{cases} x(t)=\cos t \\ y(t)=\sin t \end{cases}$$ 可以得到圖形如下:



可以知道$x^2+y^2=1$

則爲一單位圓

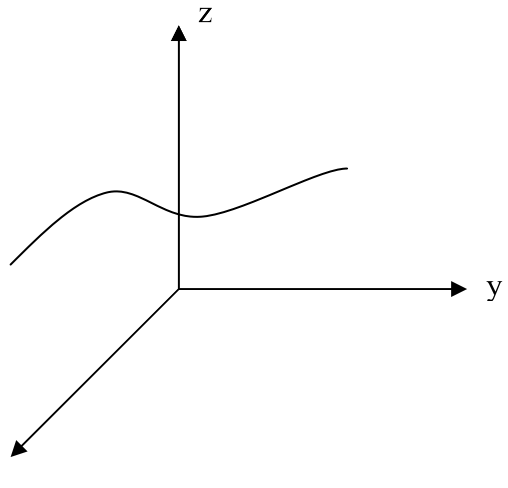
$$\begin{cases} x(t)=t \\ y(t)=t^2 \end{cases}$$ 可以得到圖形如下:



當 x=t 時，

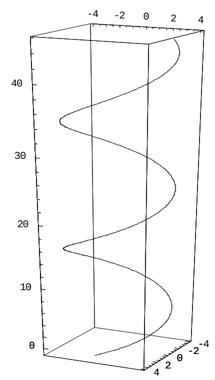
y=x^2則爲一條拋物線

在 **x-y-z** 平面



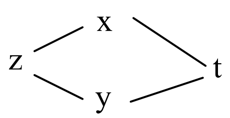
Spatial curve (空間參數表示法)

[x(t) , y(t) , z(t)]

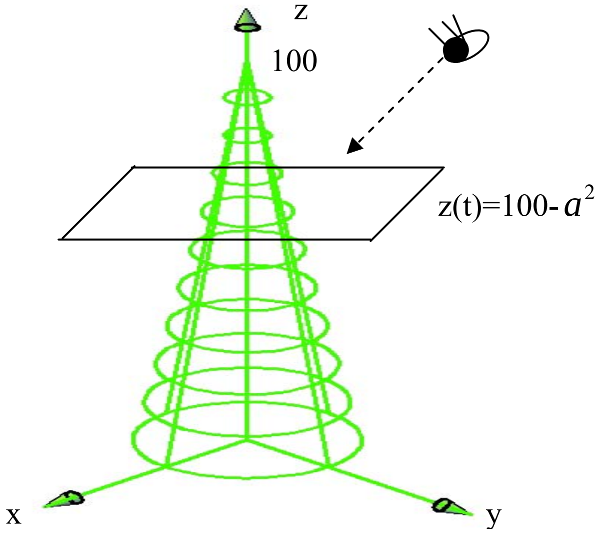


$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = 3 \end{cases}$$ 可以得到圖形:

$z = 100 - x^2 - y^2$ → z(x , y) ,

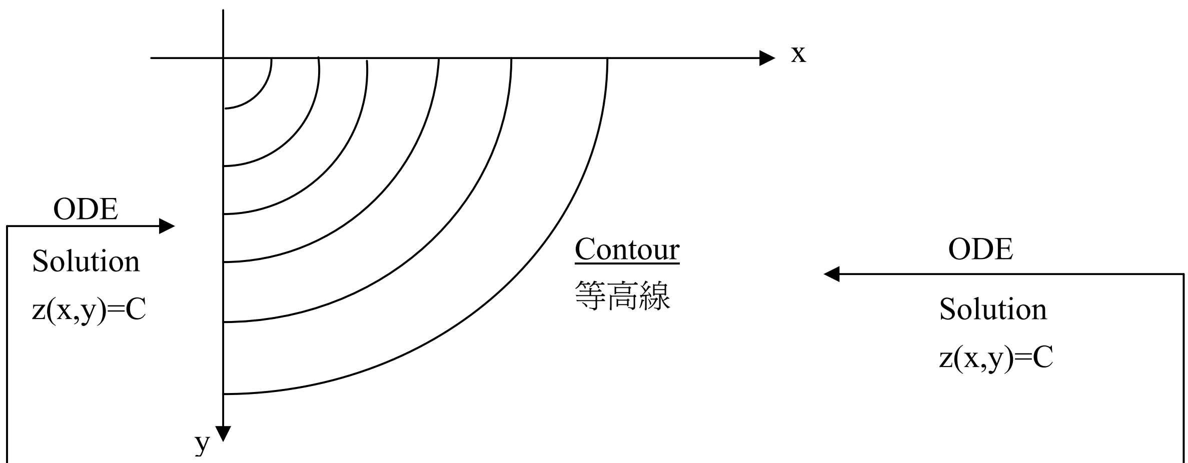


可以畫出下面的圖形:



$$\begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \\ z(t) = 100 - a^2 \end{cases}$$

在 **z=**$100-a^2$ 上有一個平面經過此圖形,切出一個半徑為 **a** 的圓。



我們可以得到 z(t)=z[x(t),y(t)]

令 z=C

C=z[x(t), y(t)]

$$\frac{dC}{dt} = \frac{dz}{dt} = 0$$

$$C = z(t) = 100 - x^2(t) - y^2(t)$$

$$\frac{dC}{dt} = 0 = -2x(t)\frac{dx(t)}{dt} - 2y\frac{dy(t)}{dt}$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$xdx + ydy = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$\Longleftrightarrow$

$$\frac{dz(t)}{dt} = 0 = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$\frac{dy}{dx} = \frac{-(\frac{\partial z}{\partial x})}{(\frac{\partial z}{\partial y})}$$

Exact form-parameter

If $\dfrac{dy}{dx}=\dfrac{M(x,y)}{N(x,y)}$

則判別式爲➔ $\dfrac{\partial M}{\partial y}+\dfrac{\partial N}{\partial x}=0$

工程數學—空間曲線描述法

Given

$$\begin{Bmatrix} \dot{\tau} \\ \dot{\nu} \\ \dot{\beta} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{\rho} & 0 \\ \frac{-1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & \frac{-1}{\sigma} & 0 \end{bmatrix} \begin{Bmatrix} \tau \\ \nu \\ \beta \end{Bmatrix} \tag{1}$$

(1) Set $\mathbf{x} = \begin{Bmatrix} \tau \\ \nu \\ \beta \end{Bmatrix}$,

$$\begin{Bmatrix} \dot{\tau}_1 \\ \dot{\tau}_2 \\ \dot{\tau}_3 \\ \dot{\nu}_1 \\ \dot{\nu}_2 \\ \dot{\nu}_3 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ \frac{-1}{\rho} & \frac{-1}{\rho} & \frac{-1}{\rho} & 0 & 0 & 0 & \frac{1}{\sigma} & \frac{1}{\sigma} & \frac{1}{\sigma} \\ \frac{-1}{\rho} & \frac{-1}{\rho} & \frac{-1}{\rho} & 0 & 0 & 0 & \frac{1}{\sigma} & \frac{1}{\sigma} & \frac{1}{\sigma} \\ \frac{-1}{\rho} & \frac{-1}{\rho} & \frac{-1}{\rho} & 0 & 0 & 0 & \frac{1}{\sigma} & \frac{1}{\sigma} & \frac{1}{\sigma} \\ 0 & 0 & 0 & \frac{-1}{\sigma} & \frac{-1}{\sigma} & \frac{-1}{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sigma} & \frac{-1}{\sigma} & \frac{-1}{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sigma} & \frac{-1}{\sigma} & \frac{-1}{\sigma} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \nu_1 \\ \nu_2 \\ \nu_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} \tag{2}$$

Reduce the Eq. (2) to simple case

$$\begin{Bmatrix} \dot{\tau}_1 \\ \dot{\nu}_1 \\ \dot{\beta}_1 \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{\rho} & 0 \\ \frac{-1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & \frac{-1}{\sigma} & 0 \end{bmatrix} \begin{Bmatrix} \tau_1 \\ \nu_1 \\ \beta_1 \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} \tau_1 \\ \nu_1 \\ \beta_1 \end{Bmatrix} = e^{As} \begin{Bmatrix} \tau_1(0) \\ \nu_1(0) \\ \beta_1(0) \end{Bmatrix} \tag{4}$$

If ρ and σ are constants and initial conditions $\begin{Bmatrix} \tau(s) \\ \nu(s) \\ \beta(s) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, what is the curve determined by Eq. (3)?

Eq.(3) can be reformulated to $\dot{\mathbf{x}}$=W $\mathbf{x}$=$\omega\times\mathbf{x}$, find matrix of W and vector of ω.

工程數學—空間曲線描述法

Given

$$\left\{\begin{array}{c}\dot{\tau}\\ \dot{\nu}\\ \dot{\beta}\end{array}\right\}=\left[\begin{array}{ccc}0 & \frac{1}{\rho} & 0\\ \frac{-1}{\rho} & 0 & \frac{1}{\sigma}\\ 0 & \frac{-1}{\sigma} & 0\end{array}\right]\left\{\begin{array}{c}\tau\\ \nu\\ \beta\end{array}\right\} \tag{1}$$

(1) Set $\mathbf{x}=\left\{\begin{array}{c}\tau\\ \nu\\ \beta\end{array}\right\}$,

$$\left\{\begin{array}{c}\dot{\tau_1}\\ \dot{\tau_2}\\ \dot{\tau_3}\\ \dot{\nu_1}\\ \dot{\nu_2}\\ \dot{\nu_3}\\ \dot{\beta_1}\\ \dot{\beta_2}\\ \dot{\beta_3}\end{array}\right\}=\left[\begin{array}{ccccccccc}0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0\\ \frac{-1}{\rho} & \frac{-1}{\rho} & \frac{-1}{\rho} & 0 & 0 & 0 & \frac{1}{\sigma} & \frac{1}{\sigma} & \frac{1}{\sigma}\\ \frac{-1}{\rho} & \frac{-1}{\rho} & \frac{-1}{\rho} & 0 & 0 & 0 & \frac{1}{\sigma} & \frac{1}{\sigma} & \frac{1}{\sigma}\\ \frac{-1}{\rho} & \frac{-1}{\rho} & \frac{-1}{\rho} & 0 & 0 & 0 & \frac{1}{\sigma} & \frac{1}{\sigma} & \frac{1}{\sigma}\\ 0 & 0 & 0 & \frac{-1}{\sigma} & \frac{-1}{\sigma} & \frac{-1}{\sigma} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{-1}{\sigma} & \frac{-1}{\sigma} & \frac{-1}{\sigma} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{-1}{\sigma} & \frac{-1}{\sigma} & \frac{-1}{\sigma} & 0 & 0 & 0\end{array}\right]\left\{\begin{array}{c}\tau_1\\ \tau_2\\ \tau_3\\ \nu_1\\ \nu_2\\ \nu_3\\ \beta_1\\ \beta_2\\ \beta_3\end{array}\right\} \tag{2}$$

Determine ρ and σ for the following cases.

(1). Line — $(s, 0, 0)$

(2). Circle — $(cos(s), sin(s), 0)$

(3). Circle — $(0, cos(s), sin(s))$

(4). Circular helix— $(cos(s/\sqrt{2}), sin(s/\sqrt{2}), s/\sqrt{2})$

(5). Helix— $(a\,cos(s\alpha/a), a\,sin(s\alpha/a), ssin(\alpha))$

(6). Curve — $(e^t cos(2t), e^t sin(2t), e^t)$

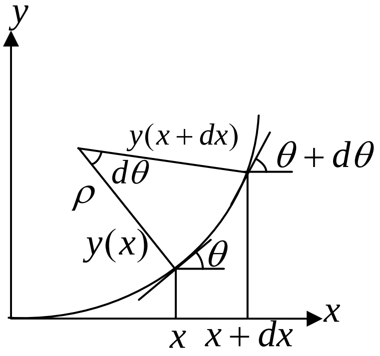
(7). Curve— $(1, t, t^2)$

(8). Plane curve by $xy = 1$ at (1, 1).

(9). Plane curve by $y = x^2$ at (1, 1).

(10). Plane curve by $y = \sqrt{1-x^2}$ at (0, 1).

曲率與曲率半徑



$$\tan\theta = \frac{dy}{dx}$$

$$\frac{d\tan\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{(1+\tan^2\theta)d\theta}{dx} = \sec^2\theta\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\therefore \frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{(1+\tan^2\theta)} = \frac{\frac{d^2y}{dx^2}}{\left[1+(\frac{dy}{dx})^2\right]}$$

$$\rho d\theta = ds \qquad ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+(\frac{dy}{dx})^2}dx = \left[1+(\frac{dy}{dx})^2\right]^{\frac{1}{2}}dx$$

$$\therefore \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{\left[1+(\frac{dy}{dx})^2\right]^{\frac{1}{2}}dx} = \frac{\frac{d^2y}{dx^2}}{\left[1+(\frac{dy}{dx})^2\right]^{\frac{3}{2}}}$$

曲率.doc　小江製表

Radius of curvature for 2-D plane

Method1 (Frenet formula → 2D)

$$\rho = \frac{\dot{X}^2 + \dot{Y}^2}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}$$

$\because \tau(s) = (\dot{X}(s), \dot{Y}(s), 0)$ 為單位長度

$\therefore\ \dot{X}^2(s) + \dot{Y}^2(s) = 1$

故 $\rho = \dfrac{1}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}$

特別注意：$(X(s), Y(s), 0)$ 為弧長表示法。

Method2 (微積分想法)

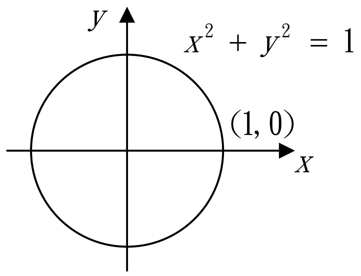
$$\rho = \frac{(1 + y'^2)^{3/2}}{\|y''\|} \Rightarrow \frac{(\dot{X}^2 + \dot{Y}^2)^{3/2}}{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}$$

特別注意：此時彈性比較大，只要是參數表示式均可用。

$$\begin{cases} x(s) \\ y(s) \end{cases}, \begin{cases} x(t) \\ y(t) \end{cases}$$

Radius of curvature for 2-D plane			
	$\rho = \dfrac{\dot{X}^2 + \dot{Y}^2}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}} = \dfrac{1}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}$	$\rho = \dfrac{(\dot{X}^2 + \dot{Y}^2)^{3/2}}{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}$	$\rho = \dfrac{(1 + y'^2)^{3/2}}{\|y''\|}$
優點	計算上比較快速，但參數式必需為弧長表示法。	所有參數式均可利用此公式計算。	題式為函數型 $y = y(x)$ 時，比較適用。
算例	若函數為 $x^2 + y^2 = 1$，其中可知其參數式 $\begin{cases} x = \cos(s) \\ y = \sin(s) \end{cases}$，故利用此方法計算曲率半徑會比較迅速	若函數為 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$，其中可知其參數式 $\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases}$，故可直接利用此法計算曲率半徑，不用轉換為弧長表示法。	若函數為 $y = x^2$ 為函數型式，故可直接利用此法計算曲率半徑。

Radius of curvature for 2-D plane .yu 製 2008/04/04



1. Find the parameter form for the curve.
2. Find the total arc length of the curve.
3. Use the Green theorem to calculate the area bounded by the curve.

   (Hint: $2A = \oint xdy - ydx$)
4. Find the radius of curvature for the curve at the point $(1, 0)$.

(1) $\because \cos^2\theta + \sin^2\theta = 1, \therefore$ set $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

(2) Set $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{d\theta} = -\sin\theta \\ \dfrac{dy}{d\theta} = \cos\theta \end{cases}$

$$ds = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2}\,d\theta = d\theta$$

$$s = \int ds = \int_0^{2\pi} 1 d\theta = 2\pi$$

(3) $A = \frac{1}{2}\oint(xdy - ydx)$

$$A = \frac{1}{2}\oint(\cos\theta\cos\theta d\theta + \sin\theta\sin\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

(4) <u>Method1</u>

$\because$ *the point* $(1, 0) = (\cos\theta, \sin\theta)$ $\therefore$ *we konw* $\theta = 0^\circ$

$$ds = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2}\,d\theta = d\theta \Rightarrow s = \theta \Rightarrow \begin{cases} x = \cos(s) \\ y = \sin(s) \end{cases}$$

$$\begin{cases} \dfrac{dx(s)}{ds} = -\sin(s) \\ \dfrac{dy(s)}{ds} = \cos(s) \end{cases} \Rightarrow \begin{cases} \dfrac{d^2x(s)}{ds^2} = -\cos(s) \\ \dfrac{d^2y(s)}{ds^2} = -\sin(s) \end{cases}$$

$$\rho = \frac{(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2}{\left|\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{d^2x}{ds^2}\frac{dy}{ds}\right|}$$

$$= \frac{(\cos s)^2 + (\sin s)^2}{\left|(-\sin s)(-\sin s) - \cos(s)(-\cos(s))\right|} = 1$$

So $we\ konw\ \theta = 0^\circ, \rho = 1$

Method2

$$\rho = \frac{1}{\sqrt{(\frac{d^2x}{ds^2})^2 + (\frac{d^2y}{ds^2})^2}} = \frac{1}{\sqrt{(-\cos(s))^2 + (-\sin(s))^2}} = 1$$

So $we\ konw\ \theta = 0^\circ, \rho = 1$

Method3

$$\begin{cases} x(\theta) = \cos\theta \\ y(\theta) = \sin\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dx(\theta)}{d\theta} = -\sin\theta \\ \dfrac{dy(\theta)}{d\theta} = \cos\theta \end{cases} \Rightarrow \begin{cases} \dfrac{d^2x(\theta)}{d\theta^2} = -\cos\theta \\ \dfrac{d^2y(\theta)}{d\theta^2} = -\sin\theta \end{cases}$$

$$\rho = \frac{\left[(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2\right]^{\frac{3}{2}}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\frac{dy}{d\theta}\right|} = \frac{\left[(\cos\theta)^2 + (\sin\theta)^2\right]^{\frac{3}{2}}}{\left|(-\sin\theta)(-\sin\theta) - \cos\theta(-\cos\theta)\right|} = 1$$

So $we\ konw\ \theta = 0^\circ, \rho = 1$

Given a spatial curve described by time-like parameter as follows：

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = t \end{cases} \Rightarrow \begin{cases} X(s) \\ Y(s) \\ Z(s) \end{cases} ?$$

Try to find the following solutions:

（1）Try to transform the time-like into space-like parameter. ($t \to s$, where s is arc length)

（2）Find the unit tangential vector, normal vector and bi-normal vector of the curve, $\tau(s), \upsilon(s)$ *and* $\beta(s)$.

（3）Please determine the radius of curvature for ρ and σ, try to plot the curve.

（4）If $\underset{\sim}{r}(t) = \left(x(t), y(t), z(t)\right) \Rightarrow R(s) = \left(X(s), Y(s), Z(s)\right)$, please determine $\left(\frac{d\underset{\sim}{R}}{ds} \times \frac{d^2\underset{\sim}{R}}{ds^2}\right) \cdot \frac{d^3\underset{\sim}{R}}{ds^3}$.

Sol:

(1)

$$\begin{cases} \dot{x}(t) = -\sin(t) \\ \dot{y}(t) = \cos(t) \\ \dot{z}(t) = 1 \end{cases}$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(-\sin(t)\right)^2 + \left(\cos(t)\right)^2 + 1^2 = 2$$

$$ds = \sqrt{2}dt \to s = \sqrt{2}t$$

$$\begin{cases} X(s) = \cos\left(\frac{s}{\sqrt{2}}\right) \\ Y(s) = \sin\left(\frac{s}{\sqrt{2}}\right) \\ Z(s) = \frac{s}{\sqrt{2}} \end{cases}$$

(2)

$$\underset{\sim}{\tau}(s)=\left(\frac{dX(s)}{ds},\frac{dY(s)}{ds},\frac{dZ(s)}{ds}\right)$$

$$=\left[-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{i}+\left[\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{j}+\left[\frac{1}{\sqrt{2}}\right]\underset{\sim}{k}$$

$$\underset{\sim}{\upsilon}(s)//\frac{d}{ds}\underset{\sim}{\tau}(s)$$

$$\frac{d}{ds}\underset{\sim}{\tau}(s)=\left[-\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{i}+\left[-\frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{j}+[0]\underset{\sim}{k}$$

$$\left\|\frac{d}{ds}\underset{\sim}{\tau}(s)\right\|=\sqrt{\frac{1}{4}}=\frac{1}{2}$$

$$\underset{\sim}{\upsilon}(s)=-\cos(\frac{s}{\sqrt{2}})\underset{\sim}{i}-\sin(\frac{s}{\sqrt{2}})\underset{\sim}{j}+[0]\underset{\sim}{k}$$

$$\underset{\sim}{\beta}(s)=\underset{\sim}{\tau}(s)\times\underset{\sim}{\upsilon}(s)$$

$$=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}\\ -\cos\left(\frac{s}{\sqrt{2}}\right) & -\sin\left(\frac{s}{\sqrt{2}}\right) & 0\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}\\ -\sin\left(\frac{s}{\sqrt{2}}\right) & 0\end{vmatrix}\underset{\sim}{i}-\begin{vmatrix}-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}\\ -\cos\left(\frac{s}{\sqrt{2}}\right) & 0\end{vmatrix}\underset{\sim}{j}+\begin{vmatrix}-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right)\\ -\cos\left(\frac{s}{\sqrt{2}}\right) & -\sin\left(\frac{s}{\sqrt{2}}\right)\end{vmatrix}\underset{\sim}{k}$$

$$=\left[\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{i}-\left[\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{j}+\left[\frac{1}{\sqrt{2}}\right]\underset{\sim}{k}$$

(3)

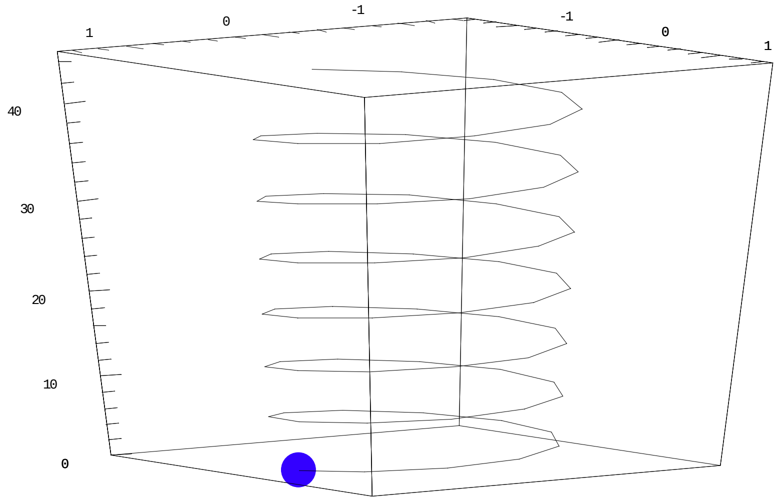
$$\frac{1}{\rho}=\left\|\frac{d}{ds}\underset{\sim}{\tau}(s)\right\|=\frac{1}{2}$$

$$\rho=2$$

$$\frac{d}{ds}\underset{\sim}{\beta}=\left[\frac{1}{2}\cos\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{i}+\left[\frac{1}{2}\sin\left(\frac{s}{\sqrt{2}}\right)\right]\underset{\sim}{j}+[0]\underset{\sim}{k}$$

$$\frac{1}{\sigma}=\left\|-\frac{d\underset{\sim}{\beta}}{ds}\right\|=\frac{1}{2}$$

$$\sigma=2$$



(4)

$$\left(\frac{d\underset{\sim}{r}}{ds}\times\frac{d^2\underset{\sim}{r}}{ds^2}\right)\cdot\frac{d^3\underset{\sim}{r}}{ds^3}=\left(\frac{d\underset{\sim}{R}}{ds}\times\frac{d^2\underset{\sim}{R}}{ds^2}\right)\cdot\frac{d^3\underset{\sim}{R}}{ds^3}$$

$$=\left(\underset{\sim}{\tau}\times\frac{1}{\rho}\underset{\sim}{\upsilon}\right)\cdot\left[\frac{1}{\rho}\left(\frac{d}{ds}\underset{\sim}{\upsilon}\right)\right]$$

$$=\frac{1}{\rho}\underset{\sim}{\beta}\cdot\left[\frac{1}{\rho}\left(-\frac{1}{\rho}\underset{\sim}{\tau}+\frac{1}{\sigma}\underset{\sim}{\beta}\right)\right]$$

$$=\frac{1}{\rho^2}\frac{1}{\sigma}$$

$$=\frac{1}{4}\cdot\frac{1}{2}$$

$$=\frac{1}{8}$$

Flowchart

(1). $\underset{\sim}{\tau}(s) = (\dot{x}(s), \dot{y}(s), \dot{z}(s))$

$\underset{\sim}{v}(s) = \rho \dot{\underset{\sim}{\tau}}(s)$

$\underset{\sim}{\beta} = \underset{\sim}{\tau} \times \underset{\sim}{v}$

1. Given a curve.
2. Determine $\rho = \dfrac{1}{|\dot{\tau}(s)|}, \sigma = \dfrac{1}{|\dot{\beta}(s)|}$
3. Frame （ 高聖凱學長已做出)

(2). ρ ,σ 給定初 Frame 及位置($x(0), y(0), z(0)$)

Solve 聯立 ODE

$$\begin{Bmatrix} \dot{\tau} \\ v \\ \beta \end{Bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{\rho} & 0 \\ \dfrac{-1}{\rho} & 0 & \dfrac{1}{\sigma} \\ 0 & \dfrac{-1}{\sigma} & 0 \end{bmatrix} \begin{Bmatrix} \tau \\ v \\ \beta \end{Bmatrix}$$

$$\begin{Bmatrix} \underset{\sim}{\tau}(0) \\ \underset{\sim}{v}(0) \\ \underset{\sim}{\beta}(0) \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix}$$

1. Given $\underset{\sim}{\tau}(0), \underset{\sim}{v}(0), \underset{\sim}{\beta}(0), x(0), y(0)$ and $z(0)$.
2. Given ρ and σ.
3. Find the curve.

Flowchart.doc 羿州製

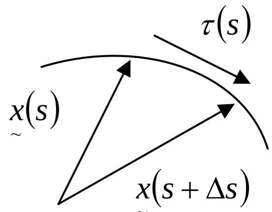
Frenet formula   Oct,13,2004

Step 1. Give a space curve of $(x(t), y(t), z(t))$

Step 2. Path coordinate $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

Spet 3. $((x(t), y(t), z(t))) \rightarrow X(s) = ((\tilde{x}(s), \tilde{y}(s), \tilde{z}(s)))$ transform t to s

Step 4. Tangent vector $\tau(s) = \dfrac{\dot{X}(s)}{\left\|\dot{X}(s)\right\|}$



Step 5. $\underset{\sim}{\tau}\cdot\underset{\sim}{\tau} = 1 \Rightarrow \underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\tau}} = 0 \quad \dot{\tau} \perp \tau$ (Note that $|\tau| = |\nu| = |\beta| = 1$)

Choose $\underset{\sim}{\nu} = \rho\dot{\tau}$ (f or simplicity)

Step 6. Why ρ is the radius of curvature

$$\tau(s)\cdot\tau(s+\Delta s) = |\tau(s)||\tau(s+\Delta s)|\cos(d\theta) = \cos(d\theta)$$

$$\tau(s)\cdot\tau(s+\Delta s) = \tau(s)\cdot\left[\tau(s) + \frac{1}{1!}\dot{\tau}(s)(ds) + \frac{1}{2!}\ddot{\tau}(s)(ds)^2 + \frac{1}{3!}\dddot{\tau}(s)(ds)^3 + \cdots\right]$$

$$= 1 + \tau(s)\cdot\dot{\tau}(s)(ds) + \frac{1}{2}\tau(s)\cdot\ddot{\tau}(s)(ds)^2 + \frac{1}{6}\tau(s)\cdot\dddot{\tau}(s)(ds)^3 + \cdots$$

$$= 1 + \frac{1}{2}\tau(s)\cdot\ddot{\tau}(s)(ds)^2 + \cdots$$

$$\cos(d\theta) = 1 - \frac{1}{2!}(d\theta)^2 + \frac{1}{4!}(d\theta)^4 - \frac{1}{6!}(d\theta)^6 + \cdots = 1 - \frac{1}{2!}(d\theta)^2 + \cdots$$

$$1 + \frac{1}{2}\tau(s)\cdot\ddot{\tau}(s)(ds)^2 = 1 - \frac{1}{2!}(d\theta)^2 \Rightarrow \quad \tau(s)\cdot\ddot{\tau}(s)(ds)^2 = -(d\theta)^2 \qquad (1)$$

$\tau\cdot\dot{\tau} = 0 \Rightarrow \dot{\tau}\cdot\dot{\tau} + \tau\cdot\ddot{\tau} = 0 \quad \therefore \tau\cdot\ddot{\tau} = -\dot{\tau}\cdot\dot{\tau}$ 代入(1)

$$-\dot{\tau}\cdot\dot{\tau}(ds)^2 = -(d\theta)^2 \Rightarrow |\dot{\tau}|^2 = \left(\frac{d\theta}{ds}\right)^2 = \frac{1}{\left(\frac{ds}{d\theta}\right)^2} = \frac{1}{\left(\frac{\rho d\theta}{d\theta}\right)^2} = \frac{1}{\rho^2} \quad \therefore \rho d\theta = ds$$

$$\therefore |\dot{\tau}| = \frac{1}{\rho}, \quad \rho = \frac{ds}{d\theta}$$

Step 7. $\underset{\sim}{\beta}\cdot\underset{\sim}{\beta} = 1 \Rightarrow \underset{\sim}{\dot{\beta}}\cdot\underset{\sim}{\beta} = 0$

$\therefore \underset{\sim}{\dot{\beta}} = p\underset{\sim}{\tau} + q\underset{\sim}{\nu} \quad \Rightarrow$

高聖凱提出

$$\underset{\sim}{\tau}\cdot\underset{\sim}{\beta}=0$$

$$\underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\beta}}+\underset{\sim}{\dot{\tau}}\cdot\underset{\sim}{\beta}=0$$

$$\underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\beta}}+\frac{1}{\rho}\underset{\sim}{\nu}\cdot\underset{\sim}{\beta}=0$$

$$\therefore\ \underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\beta}}=0$$

$$\underset{\sim}{\dot{\beta}}\cdot\tau=p\,\underset{\sim}{\tau}\cdot\underset{\sim}{\tau}+q\,\underset{\sim}{\nu}\cdot\underset{\sim}{\tau}$$

$$0=p$$

$$\therefore\ set\ \dot{\beta}=-\frac{1}{\sigma}\nu\ \text{(f or simplicity)}$$

Step 8. $\underset{\sim}{\nu}=\underset{\sim}{\beta}\times\underset{\sim}{\tau}$

$$\underset{\sim}{\dot{\nu}}=\underset{\sim}{\dot{\beta}}\times\underset{\sim}{\tau}+\underset{\sim}{\beta}\times\underset{\sim}{\dot{\tau}}$$

$$=-\frac{1}{\sigma}\underset{\sim}{\nu}\times\underset{\sim}{\tau}+\underset{\sim}{\beta}\times\frac{1}{\rho}\underset{\sim}{\nu}$$

$$=\frac{1}{\sigma}\underset{\sim}{\beta}-\frac{1}{\rho}\underset{\sim}{\tau}$$

∴we have

$$\begin{Bmatrix}\dot{\tau}\\ \dot{\nu}\\ \dot{\beta}\end{Bmatrix}=\begin{bmatrix}0 & \frac{1}{\rho} & 0\\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma}\\ 0 & -\frac{1}{\sigma} & 0\end{bmatrix}\begin{Bmatrix}\tau\\ \nu\\ \beta\end{Bmatrix}\ \rightarrow\qquad \underset{\sim}{\dot{P}}=A\underset{\sim}{P}$$

where $A=\begin{bmatrix}0 & \frac{1}{\rho} & 0\\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma}\\ 0 & -\frac{1}{\sigma} & 0\end{bmatrix}$ (反對稱)， $\underset{\sim}{P}=\begin{Bmatrix}\tau\\ \nu\\ \beta\end{Bmatrix}$

Radius of curvature for plane curve

Method 1：(Frenet formula)

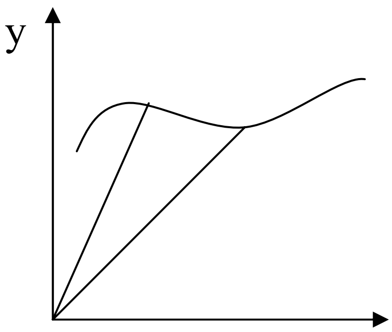
$$\underset{\sim}{X}(s) = (X(s), Y(s), 0)$$

$$\dot{\underset{\sim}{X}}(s) = (\dot{X}(s), \dot{Y}(s), 0)$$

$$\underset{\sim}{\tau}(s) = \frac{\dot{\underset{\sim}{X}}(s)}{\left\|\dot{\underset{\sim}{X}}(s)\right\|} = \left(\frac{\dot{X}(s)}{\sqrt{\dot{X}(s)^2 + \dot{Y}(s)^2}}, \frac{\dot{Y}(s)}{\sqrt{\dot{X}(s)^2 + \dot{Y}(s)^2}}, 0\right)$$

$$\left\|\dot{\underset{\sim}{\tau}}(s)\right\| = \frac{1}{\rho} \quad \rightarrow \quad \rho = \frac{(\dot{X}^2 + \dot{Y}^2)}{(\dot{X}\ddot{Y} - \ddot{X}\dot{Y})}$$

Method 2：(微積分想法)



$$\tan\theta = y'(x)$$

$$\tan(\theta + \Delta\theta) = y'(x + \Delta x)$$

$$\tan(\theta + \Delta\theta) - \tan(\theta) = y'(x + \Delta x) - y'(x)$$

$$\sec^2\theta d\theta = y''(x)dx$$

$$(1 + (y')^2)d\theta = y''\frac{ds}{\sqrt{1 + (y')^2}}$$

$$\frac{ds}{d\theta} = \frac{(1 + (y'(x))^2)^{3/2}}{y''}$$

$$\rho = \frac{(1 + (y')^2)^{3/2}}{\|y''\|}$$

曲率兩種表示法的關係

$$\tau(s)=\left(\frac{\dot{X}}{\sqrt{\dot{X}^2+\dot{Y}^2}},\frac{\dot{Y}}{\sqrt{\dot{X}^2+\dot{Y}^2}}\right)$$

$$\dot{\left(\frac{\dot{X}(s)}{\sqrt{\dot{X}^2(s)+\dot{Y}^2(s)}}\right)}=\frac{\ddot{X}\dot{Y}^2-\dot{X}\dot{Y}\ddot{Y}}{(\dot{X}^2(s)+\dot{Y}^2(s))^{3/2}}$$

$$\dot{\left(\frac{\dot{Y}(s)}{\sqrt{\dot{X}^2(s)+\dot{Y}^2(s)}}\right)}=\frac{\ddot{Y}\dot{X}^2-\dot{X}\dot{Y}\ddot{X}}{(\dot{X}^2(s)+\dot{Y}^2(s))^{3/2}}$$

$$\left\|\dot{\tau}(s)\right\|=\frac{\ddot{X}\dot{Y}-\dot{X}\ddot{Y}}{(\dot{X}^2+\dot{Y}^2)}$$

$$\rho=\frac{(\dot{X}^2+\dot{Y}^2)}{(\ddot{X}\dot{Y}-\dot{X}\ddot{Y})}\leftrightarrow\frac{(\dot{X}^2+\dot{Y}^2)^{3/2}}{(\dot{X}\ddot{Y}-\dot{Y}\ddot{X})}$$

奇怪，為何不一致!

曲率工數到材力.doc

曲率兩種表示法的關係

$$\rho = \frac{(1 + y'^2)^{3/2}}{\|y''\|} \rightarrow \rho = \frac{\dot{X}^2 + \dot{Y}^2}{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}$$

$$\begin{cases} x = X(s) \\ y = Y(s) \end{cases}$$

$$\frac{dy}{dx} = \frac{\dot{Y}}{\dot{X}}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{ds}\left(\frac{\dot{Y}}{\dot{X}}\right)\frac{ds}{dx}$$

$$= \frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{\dot{X}^2} \cdot \frac{1}{\dot{X}} \qquad \left(\frac{ds}{dx} = \frac{1}{\dot{X}}\right)$$

$$\rho = \frac{\left(1 + \left(\frac{\dot{Y}}{\dot{X}}\right)^2\right)^{3/2}}{\left(\frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{\dot{X}^3}\right)}$$

$$= \frac{(\dot{X}^2 + \dot{Y}^2)^{3/2}}{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}$$

評論:

1. 只管在 2D 平面曲線
2. 3-D space curve

曲率材力到工數.DOC

$v(\underset{\sim}{x})$ 是純量函數

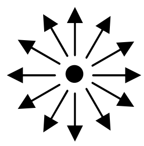
(1) 一維梯度 $v(x)$

$$\nabla v=\frac{\partial v}{\partial x}\underset{\sim}{i}$$

方向導微 $(1,0)\Rightarrow\frac{\partial v}{\partial x}$,$(-1,0)\Rightarrow\frac{-\partial v}{\partial x}$

(2) 二維梯度 $v(x,y)$

$$\nabla v=\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)=\frac{\partial v}{\partial x}\underset{\sim}{i}+\frac{\partial v}{\partial y}\underset{\sim}{j}$$



方向導微 $\frac{\partial v}{\partial n}=\nabla v\cdot\underset{\sim}{n}$ ($\underset{\sim}{n}$ 平面 360 度隨您選)

(3) 三維梯度 $v(x,y,z)$

$$\nabla v=\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial z}\right)=\frac{\partial v}{\partial x}\underset{\sim}{i}+\frac{\partial v}{\partial y}\underset{\sim}{j}+\frac{\partial v}{\partial z}\underset{\sim}{k}$$

方向導微 $\frac{\partial v}{\partial n}=\nabla v\cdot\underset{\sim}{n}$ ($\underset{\sim}{n}$ 空間 4π 體角任您選)

※ $\frac{\partial v}{\partial n}=\lim_{\varepsilon\to 0}\frac{v(\underset{\sim}{x}+\varepsilon\underset{\sim}{n})-v(\underset{\sim}{x})}{\varepsilon}$ (v 函數在 n 方向導微)，其中 $\underset{\sim}{n}$ 為單位向量。

物理量(可想成溫度場)在 $\underset{\sim}{x}$ 處與 $\underset{\sim}{x}+\varepsilon\underset{\sim}{n}$ 處(鄰居，neighborhood)的差值除以兩點的距離($\varepsilon=\underset{\sim}{x}+\varepsilon\underset{\sim}{n}-\underset{\sim}{x}$)，即有該物理量在 $\underset{\sim}{n}$ 方向導微。

※多變數函數的 Taylor 展開式：

$$v\left(\underset{\sim}{x}+\varepsilon\underset{\sim}{n}\right)=v\left(\underset{\sim}{x}\right)+\frac{\partial v}{\partial x_1}\varepsilon n_1+\frac{\partial v}{\partial x_2}\varepsilon n_2+\frac{\partial v}{\partial x_3}\varepsilon n_3+H.O.T.$$

$$\nabla\phi\,,\nabla\times\underset{\sim}{v}\,,\nabla\bullet\underset{\sim}{v}$$

If

$$\underset{\sim}{v}=\nabla\phi\rightarrow\nabla\times\underset{\sim}{v}=\underset{\sim}{0}$$

If

$$\underset{\sim}{p}=\nabla\times\underset{\sim}{v}\rightarrow\nabla\bullet\underset{\sim}{p}=0$$

$$\underset{\sim}{u}=\nabla\phi+\nabla\times\underset{\sim}{v}$$

$$\nabla\bullet(\nabla\times\underset{\sim}{v})=0$$

$$\nabla\times(\nabla\phi)=\underset{\sim}{0}$$

$$\nabla\times(\nabla\times\underset{\sim}{v})=\nabla(\nabla\bullet\underset{\sim}{v})-\nabla^2\underset{\sim}{v}$$

$$\nabla^2\underset{\sim}{v}=\nabla(\nabla\bullet\underset{\sim}{v})-\nabla\times(\nabla\times\underset{\sim}{v})$$

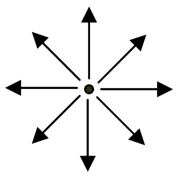
$$\nabla\bullet\nabla\phi=\nabla^2\phi$$

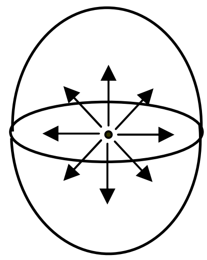
ϕ : *scalar function*

$\underset{\sim}{v}$ = *vector function*

散度與旋度—簡例

散度:

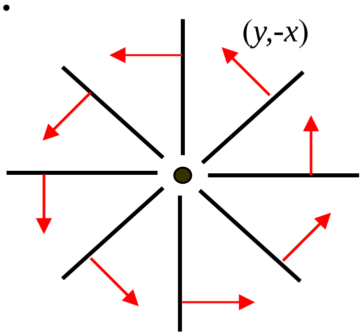


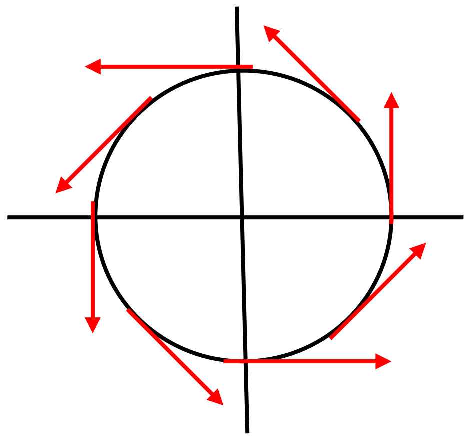


$\underset{\sim}{v}=\underset{\sim}{r}=(x,y)\quad,\nabla\bullet v=2\qquad(2\text{ 維})$

$\underset{\sim}{v}=\underset{\sim}{r}=(x,y,z)\quad,\nabla\bullet v=3\quad(3\text{ 維})$

旋度:



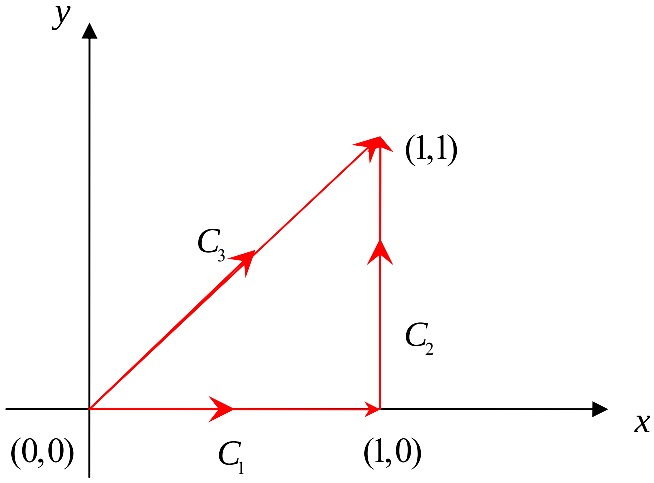


$$\underset{\sim}{v}=(y,-x,0)$$

$$\nabla\times\underset{\sim}{v}=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 1 \end{vmatrix}=-\underset{\sim}{k}-\underset{\sim}{k}$$

$$=-2\underset{\sim}{k}$$

保守力場與路徑無關積分



$$\oint Pdx + \oint Qdy = 0$$

$$\int_{C_1+C_2} Pdx + Qdy = \int_{C_3} Pdx + Qdy$$

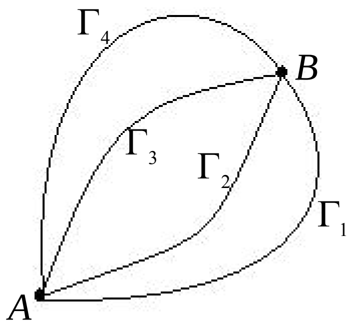
If there exist ϕ, *such that*

$$\nabla\phi = (P,Q)$$

$$i.e., \frac{\partial\phi}{\partial x} = P, \frac{\partial\phi}{\partial y} = Q$$

Green's theorem

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$$



若 $(P,Q) = \nabla\phi$ 是力場，則稱為保守力場。

此時 $\int Pdx + \int Qdy = \phi(B) - \phi(A)$ 為路徑無關積分，只在乎起始(A)與終點(B)的位置。

2-D

$$\int_{c_1}^{c_2} Pdx + Qdy$$

$$if\ (P,Q) = \nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y})$$

$$\int_{c_1}^{c_2} Pdx + Qdy = \int_{c_1}^{c_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

$$= \phi(x,y)\Big|_{(x,y)=c_1}^{(x,y)=c_2}$$

3-D

$$\int_{c_1}^{c_2} Pdx + Qdy + Rdz$$

$$if\ (P,Q,R) = \nabla\phi = \underset{\sim}{F}$$

$$\int_{c_1}^{c_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \phi(x,y,z)\Big|_{(x,y,z)=c_1}^{(x,y,z)=c_2}$$

$$\oint Pdx + Qdy + Rdz = \iint \nabla\times\underset{\sim}{F}\cdot\underset{\sim}{n}dS = 0$$

$$\underset{\sim}{F} = (P,Q,R) \to \nabla\times\underset{\sim}{F} = 0$$

Stokes' theorem

$$\oint \underset{\sim}{F}\cdot dr = \iint \nabla\times\underset{\sim}{F}\cdot\underset{\sim}{n}dS$$

If F is consevative, we have

$$\oint \underset{\sim}{F}\cdot dr = \phi\Big|_A^A = 0$$

向量微積分基本物理量

名稱	數學操作	張量階數
純量	ϕ	0
純量場	$\phi(\underset{\sim}{\chi})$	0
向量	$\underset{\sim}{F}$	1
向量場	$\underset{\sim}{F}(\underset{\sim}{x})$	1
散度	$\nabla\cdot F=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$	0
旋度	$\nabla\times F=\begin{pmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{pmatrix}$	1
梯度	$\nabla\phi(\underset{\sim}{x})=\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$	1
方向導微	$\frac{\partial\phi}{\partial v}=\nabla\phi\cdot\underset{\sim}{v}$	0

Vecyor Calculus.2008.doc 江宗翰製表

靜力、動力與材力

(1). $\iint_A 1\, dxdy$	轉線積分 →	$\frac{1}{2}\oint_{\Gamma} \frac{\partial y^2}{\partial n} d\Gamma$
(2). $\iint_A x\, dxdy = \overline{x}\ A$	轉線積分 →	$\frac{1}{2}\oint_{\Gamma} x^2 \frac{\partial x}{\partial n} d\Gamma$
(3). $\iint_A y\, dxdy = \overline{y}\ A$	轉線積分 →	$\frac{1}{2}\oint_{\Gamma} y^2 \frac{\partial y}{\partial n} d\Gamma$
(4). $\iint_A x^2\ dxdy$	轉線積分 →	$\frac{1}{6}\oint_{\Gamma} x^2 \frac{\partial x^2}{\partial n} d\Gamma$
(5). $\iint_A y^2\ dxdy$	轉線積分 →	$\frac{1}{6}\oint_{\Gamma} y^2 \frac{\partial y^2}{\partial n} d\Gamma$
(6). $\iint_A xy\ dxdy$	轉線積分 →	$\frac{1}{2}\oint_{\Gamma} x^2 y \frac{\partial x}{\partial n} d\Gamma = \frac{1}{2}\oint_{\Gamma} y^2 x \frac{\partial y}{\partial n} d\Gamma$

工程數學—Vector operators for radial basis functions

1. Given the radial position vector and radial basis function,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla\phi(r) = \frac{d\phi}{dr}\hat{\mathbf{r}}$$

$$\nabla \cdot [\phi(r)\mathbf{r}] = 3\phi(r) + r\frac{d\phi(r)}{dr}$$

$$\nabla^2\phi(r) = \frac{d^2\phi(r)}{dr^2} + \frac{2}{r}\frac{d\phi(r)}{dr}$$

$$\nabla \times [\phi(r)\mathbf{r}] = \mathbf{0}$$

$$\nabla r = \hat{\mathbf{r}}$$

$$\nabla \cdot \mathbf{r} = 3$$

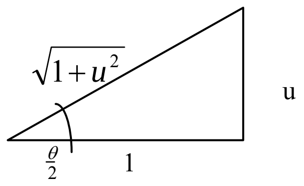
$$\nabla(1/r) = -\frac{\hat{\mathbf{r}}}{r^2}$$

$$\nabla \cdot (\frac{\hat{\mathbf{r}}}{r^2}) = -\nabla^2(1/r) = 4\pi\delta(r)$$

$$\int_0^{2\pi}\frac{4\cos\theta+1}{8\cos\theta+17}d\theta \quad =\int_0^{2\pi}\left(\frac{1}{2}-\frac{15}{2}\frac{1}{8\cos\theta+17}\right)d\theta$$

$$=\int_0^{2\pi}\frac{1}{2}d\theta-\frac{15}{2}\int_0^{2\pi}\frac{1}{8\cos\theta+17}d\theta$$

Let $u=\tan\dfrac{\theta}{2}\quad\Rightarrow\cos\theta=\dfrac{1-u^2}{1+u^2},\ \ d\theta=\dfrac{2}{1+u^2}du$



$\cos\theta=2\cos^2\frac{\theta}{2}-1=\frac{1-u^2}{1+u^2}$

$du=\frac{1}{2}\sec^2\frac{\theta}{2}d\theta$

因爲 $\cos\theta$ 爲週期函數，所以積分路徑可以改爲由-π積到π，來避開 $\tan(\frac{\pi}{2})$ 的不連續

$$=\pi-\tfrac{15}{2}\int_{\tan(-\frac{\pi}{2})}^{\tan(\frac{\pi}{2})}\frac{2du}{8(1-u^2)+17(1+u^2)}$$

$$=\pi-15\int_{\tan(-\frac{\pi}{2})}^{\tan(\frac{\pi}{2})}\frac{du}{25+9u^2}$$

$$=\pi-\left(\tan^{-1}(\frac{3}{5}\tan\tfrac{\pi}{2})-\tan^{-1}(\frac{3}{5}\tan\tfrac{-\pi}{2})\right)$$

$$=0$$

另解:複變理論留數定理亦可解，待工數(四)分曉

Gradient-方向導微

$\phi(x,y)=x^2+y^2$

Gradient

切向

$$\frac{\partial\phi}{\partial t}=\frac{\phi(x+dx,y+dy)-\phi(x,y)}{ds}=\boxed{\nabla\phi}\cdot t$$

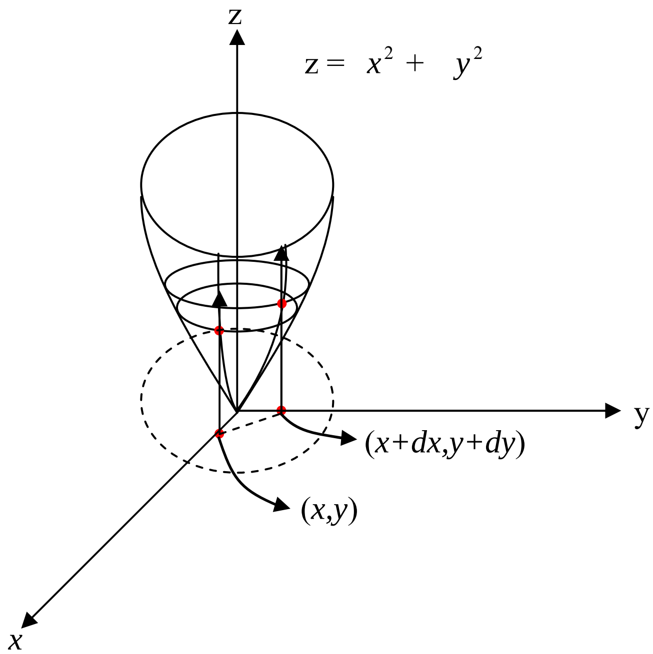
$$(ds)^2=(dx)^2+(dy)^2$$

$$\underset{\sim}{t}=\left(\frac{dx}{ds},\frac{dy}{ds}\right)$$

法向

$$\frac{\partial\phi}{\partial n}=\frac{\phi(x+dy,y-dx)-\phi(x,y)}{ds}=\nabla\phi\cdot\underset{\sim}{n}$$

$$\underset{\sim}{n}=\left(\frac{dy}{ds},\frac{-dx}{ds}\right)$$



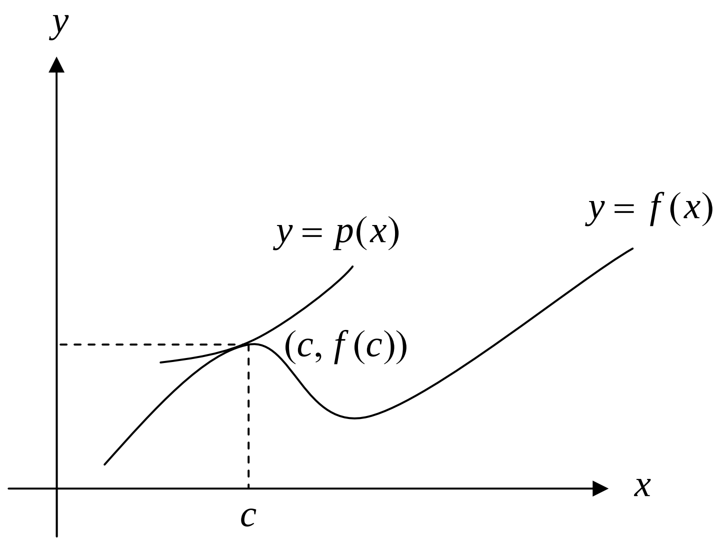
任意向

$$\frac{\partial\phi}{\partial v}=\nabla\phi\cdot\underset{\sim}{v}$$ $\underset{\sim}{v}$ 爲單位向量

Taylor 多項式及逼近

目的：用多項式函數來逼近其他的基本函數，其意思爲對於一個函數 $f(x)$，其在 c 點的值 $f(c)$，希望找到一個多項式函數 $p(x)$，使得 $p(c)=f(c)$。

從圖形來看，即 $p(x)$ 要通過 $(c, f(c))$ 這一點，也稱多項式 $p(x)$ 在 c 點展開。
(expansion at c)



我們更希望 $p(x)$ 在 c 點附近的行爲，也很像 $f(x)$，因此，我們增加對 $p(x)$ 的限制條件，如 $p(x)$ 在 c 點的斜率與 $f(x)$ 在 c 點的斜率一致，即 $p'(c)=f'(c)$。

由 $p(c)=f(c)$ 及 $p'(c)=f'(c)$ 兩個條件，我們可以得到 $f(x)$ 的一個線性逼近。

Ex. 1 $f(x)=e^x$，找一個 $p_1(x)=a_1x+a_0$，使得 $p_1(0)=f(0)$，$p_1'(0)=f'(0)$。

Sol. $p_1(0)=f(0)=1$

$\Rightarrow a_0=1$

$\because p_1'(0)=f'(0)=1$

$\Rightarrow a_1=1$

$\therefore p_1(x)=x+1$

由 Ex.1 得知，當 x 遠離 0 時， $f(x)$ 與 $p_1(x)$ 的差值漸大，如何再度修正呢？

故再多加 $p''(0)=f''(0)$ ，即 $p(x)$ 要滿足 $p_2(0)=f(0)$ 、 $p_2'(0)=f'(0)$ 、

$p_2''(0)=f''(0)$ ，可得 $p_2(x)=1+x+\frac{1}{2}x^2$ 。

同理，欲使 $p(x)$ 更逼近 $f(x)$ ，與 $f(x)$ 越相近，所增加的限制條件越多，所得項數就越多，如此下去，如果需求 $p^n(0)=f^n(0)$ 則可得到 $f(x)=e^x$ 的更好逼近，且 $p_n(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{n!}x^n$

Taylor 及 Maclaurin 級數

上面的例子是找一個多項式 $p(x)$ ，對 $f(x)=e^x$ 在 $x=0$ 展開。而一般情形下，想找 $p(x)$ ，對於函數 $f(x)$ 在 $x=c$ 的展開(expansion)式。

可表示成 $p_n(x)=a_0+a_1(x-c)+a_2(x-c)^2+\cdots+a_n(x-c)^n$ ，

即爲了要滿足 $\forall n$ ， $p_n(c)=f(c)$ ，即 $p_n^{(n)}(c)=f^{(n)}(c)$ 。

經過多次微分

$$
\begin{aligned}
&p_n^1(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+\cdots na_n(x-c)^{n-1}\\
&p_n^2(x)=2a_2+2\cdot 3a_3(x-c)+\cdots n\cdot(n-1)\cdot a_n(x-c)^{n-2}\\
&\quad\vdots\\
&p_n^{(n)}(x)=n\cdot(n-1)\cdot\cdots\cdot 3\cdot 2\cdot 1\cdot a_n=n!a_n
\end{aligned}
\text{。}
$$

將 $x=c$ 帶入上式

$$
\begin{aligned}
&p_n^1(c)=a_1=f^1(c)\\
&p_n^2(c)=2a_2=f^2(c)\Rightarrow a_2=\frac{f^2(c)}{2!}\\
&\quad\vdots\\
&p_n^{(n)}(x)=n!a_n=f^{(n)}(c)\Rightarrow a_n=\frac{f^{(n)}(c)}{n!}
\end{aligned}
$$

則我們可定義：若 $f(x)$ 在 c 點的 n 階導數存在，則 n 次多項式

$$p_n(x) = f(c) + f'(c)\cdot(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n,$$

稱爲 $f(x)$ 在 c 點的 n 階 Taylor 多項式(展開式)。

若 $c=0$ 時，則 $p_n(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \cdots + \frac{f^{(n)}(c)}{n!}\cdot x^n$

稱爲 $f(x)$ 的 Maclaurin 展開式。

Ex. 2 求 $f(x)=e^x$ 的 6 階 Maclaurin 多項式。

Sol.

$$p(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f^{(3)}(0)}{3!}\cdot x^3 + \frac{f^{(4)}(0)}{4!}\cdot x^4 + \frac{f^{(5)}(0)}{5!}\cdot x^5 + \frac{f^{(6)}(0)}{6!}\cdot x^6$$

$$\begin{aligned}
f(x) &= e^x & f(0) &= 1\\
f'(x) &= e^x & f'(0) &= 1\\
f''(x) &= e^x & f''(0) &= 1\\
f^{(3)}(x) &= e^x & f^{(3)}(0) &= 1\\
f^{(4)}(x) &= e^x & f^{(4)}(0) &= 1\\
f^{(5)}(x) &= e^x & f^{(5)}(0) &= 1\\
f^{(6)}(x) &= e^x & f^{(6)}(0) &= 1
\end{aligned}$$

$$\therefore\ \boxed{p(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6}$$

Ex. 3 求 $f(x)=\cos(x)$ 的 6 階 Maclaurin 多項式。

Sol.

$$p(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}\cdot x^2+\frac{f^{(3)}(0)}{3!}\cdot x^3+\frac{f^{(4)}(0)}{4!}\cdot x^4+\frac{f^{(5)}(0)}{5!}\cdot x^5+\frac{f^{(6)}(0)}{6!}\cdot x^6$$

$$\begin{aligned}
f(x)&=\cos(x) & f(0)&=1\\
f'(x)&=-\sin(x) & f'(0)&=0\\
f''(x)&=-\cos(x) & f''(0)&=-1\\
f^{(3)}(x)&=\sin(x) & f^{(3)}(0)&=0\\
f^{(4)}(x)&=\cos(x) & f^{(4)}(0)&=1\\
f^{(5)}(x)&=-\sin(x) & f^{(5)}(0)&=0\\
f^{(6)}(x)&=-\cos(x) & f^{(6)}(0)&=-1
\end{aligned}$$

$$\therefore \boxed{p(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6}$$

Ex. 4 求 $f(x)=\sin(x)$ 的 6 階 Maclaurin 多項式。

Sol.

$$p(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}\cdot x^2+\frac{f^{(3)}(0)}{3!}\cdot x^3+\frac{f^{(4)}(0)}{4!}\cdot x^4+\frac{f^{(5)}(0)}{5!}\cdot x^5+\frac{f^{(6)}(0)}{6!}\cdot x^6$$

$$\begin{aligned}
f(x)&=\sin(x) & f(0)&=0\\
f'(x)&=\cos(x) & f'(0)&=1\\
f''(x)&=-\sin(x) & f''(0)&=0\\
f^{(3)}(x)&=-\cos(x) & f^{(3)}(0)&=-1\\
f^{(4)}(x)&=\sin(x) & f^{(4)}(0)&=0\\
f^{(5)}(x)&=\cos(x) & f^{(5)}(0)&=1\\
f^{(6)}(x)&=-\sin(x) & f^{(6)}(0)&=0
\end{aligned}$$

$$\therefore \boxed{p(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5}$$

若 $f(x)=e^{x}$，我們把 $x=i\theta$ 代入，

則
$$e^{i\theta}=1+i\theta-\frac{1}{2!}\theta^{2}-i\frac{1}{3!}\theta^{3}+\frac{1}{4!}\theta^{4}+i\frac{1}{5!}\theta^{5}-\frac{1}{6!}\theta^{6}+\cdots$$
$$=(1-\frac{1}{2!}\theta^{2}+\frac{1}{4!}\theta^{4}-\frac{1}{6!}\theta^{6}+\cdots)+i(\theta-\frac{1}{3!}\theta^{3}+\frac{1}{5!}\theta^{5}+\cdots)$$

而 $\cos(\theta)=1-\frac{1}{2!}\theta^{2}+\frac{1}{4!}\theta^{4}+-\frac{1}{6!}\theta^{6}+\cdots$，

$\sin(\theta)=\theta-\frac{1}{3!}\theta^{3}+\frac{1}{5!}\theta^{5}+\cdots$，

因此我們得到 $e^{i\theta}=\cos(\theta)+i\sin(\theta)$，此公式即爲尤拉公式(Euler formula)。

Taylor′s expansion

(1) 自變數-單變數

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + H.O.T.$$

(2) 自變數-雙變數

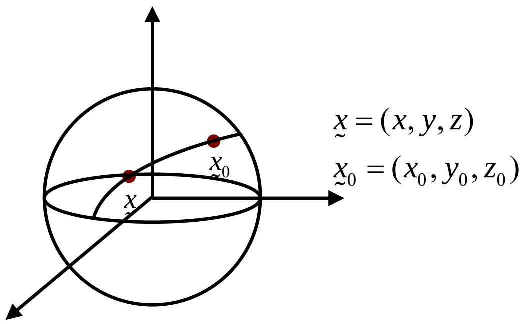
$$f(x,y) = f(x_0,y_0) + \frac{f_x(x_0,y_0)}{1!}(x - x_0) + \frac{f_y(x_0,y_0)}{1!}(y - y_0) + H.O.T.$$

$$\begin{cases} f_x = \dfrac{\partial}{\partial x} \\ f_y = \dfrac{\partial}{\partial y} \end{cases}$$

(3) 自變數-三變數

$$f(x,y) = f(x_0,y_0,z_0) + \frac{f_x(x_0,y_0,z_0)}{1!}(x - x_0) + \frac{f_y(x_0,y_0,z_0)}{1!}(y - y_0) + \frac{f_y(x_0,y_0,z_0)}{1!}(z - z_0) + H.O.T.$$

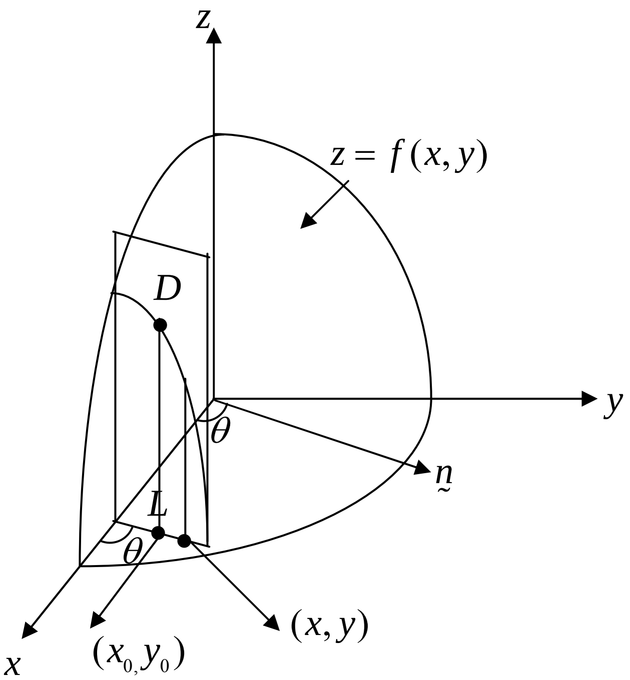
$$\begin{cases} f_x = \dfrac{\partial}{\partial x} \\ f_y = \dfrac{\partial}{\partial y} \\ f_z = \dfrac{\partial}{\partial z} \end{cases}$$



方向導微(directional derivatives)

目的：為了求某點在n方向的導數。

例如我們想求$P(x_0, y_0, f(x_0, y_0))$在n方向的導數



$$\frac{\partial f}{\partial n} = \nabla f \cdot \underset{\sim}{n}$$

首先過P點在n方向取一個垂直於xy平面且和曲面相交的曲線為C，而在xy平面相交的直線為L，而此直線的參考方程式可設為：

$$x = x_0 + S\cos\theta, y = y_0 + S\sin\theta$$

其中θ為L與軸的夾角

因此過$(x_0, y_0, f(x_0, y_0))$及$(x, y, f(x, y))$的割線斜率為：

$$\frac{f(x,y) - f(x_0,y_0)}{S}$$

$$= \frac{f(x_0 + S\cos\theta, y_0 + S\sin\theta) - f(x_0,y_0)}{S}$$

$$= \lim_{s\to 0}\frac{f(x_0 + S\cos\theta, y_0 + S\sin\theta) - f(x_0,y_0)}{S}$$

$$= \frac{\partial f}{\partial n}$$

$$f'(S) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial S} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \cdot (\frac{\partial x}{\partial S}, \frac{\partial y}{\partial S}) = \nabla f \cdot n$$

$$\therefore \frac{\partial f}{\partial n} = \nabla f \cdot n$$

The temperature field, $T(x,y)$, on the surface of a metal plate is

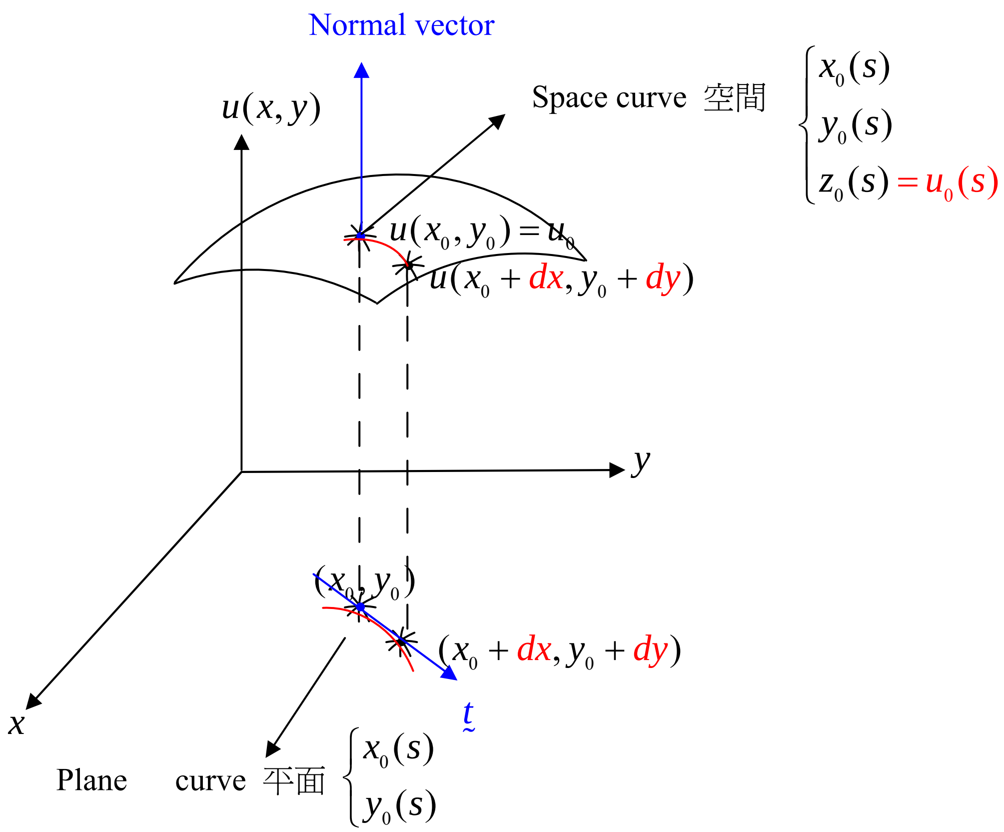
$T(x,y) = 12 - x^2 - 3y^2$.

(a) Please determine the direction derivative at the point $P\ (3,1)$ in the direction $\underset{\sim}{v}(1,-1)$.

(b) Please determine the maximum rate of the temperature field at the point P.

(c) Please determine the tangent plane across the point $(3,1,T(3,1))$.

Direction derivative



$$u(x,y) \Rightarrow u(x_0, y_0) \Rightarrow u_0(s) \qquad u < {}^{x}_{y} > s$$

$$u_0(s) = u(x_0(s), y_0(s))$$

$$\frac{du_0(s)}{ds} = \frac{\partial u_0}{\partial x_0}\frac{dx_0}{ds} + \frac{\partial u_0}{\partial y_0}\frac{dy_0}{ds} \longrightarrow = \left(\frac{\partial u_0}{\partial x_0}, \frac{\partial u_0}{\partial y_0}\right) \cdot \left(\frac{dx_0}{ds}, \frac{dy_0}{ds}\right) = \nabla u \cdot \underset{\sim}{t}$$

$$\left(-\frac{\partial u_0}{\partial x_0}, -\frac{\partial u_0}{\partial y_0}, 1\right) \cdot \left(\frac{dx_0}{ds}, \frac{dy_0}{ds}, \frac{du_0(s)}{ds}\right) = 0$$

曲面上過 (x_0, y_0, u_0) 點之法向量　　在曲面上過 (x_0, y_0, u_0) 點之曲線切向量

Ex.1 The temperature field, $T(x,y)$, on the surface of a metal plate is $T(x,y)=12-x^2-3y^2$.

(a) Please determine the direction derivative at the point $P\,(3,1)$ in the direction $\underset{\sim}{v}(1,-1)$.

(b) Please determine the maximum rate of the temperature field at the point P.

(c) Please determine the tangent plane across the point $(3,1,T(3,1))$.

Ex.2 Given a spatial curve described by time-like parameter as follows：

$$\begin{cases} x(t)=\cos(t) \\ y(t)=\sin(t) \\ z(t)=t \end{cases}$$

Try to find the following solutions:

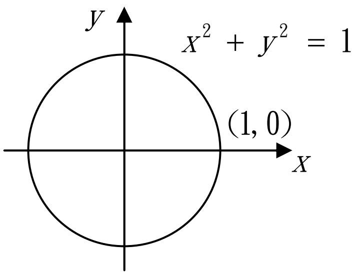
（1）Try to transform the time-like into space-like parameter. ($t\to s$, where s is arc length)

（2）Find the unit tangential vector, normal vector and bi-normal vector of the curve, $\tau(s),\upsilon(s)\ and\ \beta(s)$.

（3）Please determine the radius of curvature for ρ and σ, try to plot the curve.

（4）If $\underset{\sim}{r}(t)=\left(x(t),\,y(t),\,z(t)\right)$, please determine $\left(\frac{d\underset{\sim}{r}}{ds}\times\frac{d^2\underset{\sim}{r}}{ds^2}\right)\cdot\frac{d^3\underset{\sim}{r}}{ds^3}$.

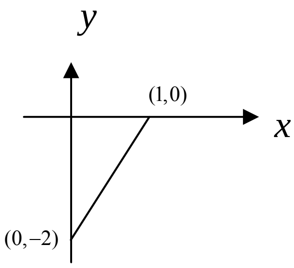
Ex.3



1. Find the parameter form for the curve.
2. Find the total arc length of the curve.
3. Use the Green theorem to calculate the area bounded by the curve.

   (Hint: $2A = \oint xdy - ydx$)
4. Find the radius of curvature for the curve at the point (1, 0).

Ex.4 試求此三角形的面積? 請利用 $\iint_A 1dxdy=\frac{1}{2}\int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\,d\Gamma$ ，面積分跟線積分各算一次。



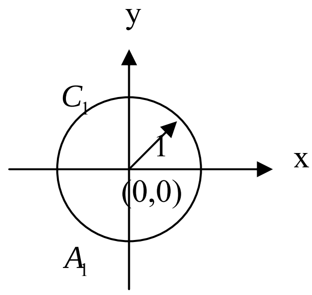
工數(二)大考 1　　魯蛋製 2008/04/02

NTOU/MSV

年級：________ 姓名：________ 學號：________

國立台灣海洋大學河海工程學系２００8 工程數學（二）Green theorem ?

1．$\oint_C (\frac{-y}{x^2+y^2})dx + (\frac{x}{x^2+y^2})dy = ?$



where C_1 and A_1 are shown below.

$x = \cos\theta \quad dx = -\sin\theta d\theta$

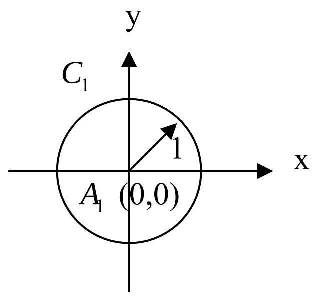
$y = \sin\theta \quad dy = \cos\theta d\theta$

$$\oint_C (\frac{-y}{x^2+y^2})dx + (\frac{x}{x^2+y^2})dy = \oint (\sin^2\theta + \cos^2\theta)d\theta = 2\pi$$

2．$\iint_A [\frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) - \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2})]dA = ?$

wher e C_1 and A_1 are shown below.

$$\iint_A [\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}]dA = 0$$



由於Ｐ與Ｑ在Ａ中之原點（０，０）不可解析(奇異)，所以 Green theorem 不適用。

Green 恆等式(Green's Identity)

B94520143 李家瑋
（2007/07/24）

Green第一恆等式

一維	二、三維
$\int_a^b u\frac{d^2u}{dx^2}dx = u\frac{du}{dx}\Big\vert_a^b - \int_a^b \frac{du}{dx}\frac{du}{dx}dx$	$\int_V u\nabla^2 u dV = \int_B u\frac{\partial u}{\partial n}dB - \int_V \nabla u\cdot\nabla u dV$

Green第二恆等式

一維	二、三維
$\int_a^b u\frac{d^2v}{dx^2}dx = u\frac{dv}{dx}\Big\|_a^b - \int_a^b \frac{du}{dx}\frac{dv}{dx}dx$	$\int_V u\nabla^2 v dV = \int_B u\frac{\partial v}{\partial n}dB - \int_V \nabla u\cdot\nabla v dV$

Green第三恆等式

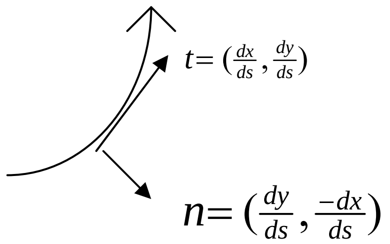
一維	二、三維
$\int_a^b u\frac{d^2v}{dx^2}dx - \int_a^b v\frac{d^2u}{dx^2}dx = u\frac{dv}{dx}\Big\|_a^b - v\frac{du}{dx}\Big\|_a^b$	$\int_V u\nabla^2 v dV - \int_V v\nabla^2 u dV = \int_B u\frac{\partial v}{\partial n}dB - \int_B v\frac{\partial u}{\partial n}dB$

Two views of Green's theorem

$$\int P dx + \int Q dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$ Green's theorem

(1) Divergence theorem

$$\int \underset{\sim}{v} \cdot \underset{\sim}{n}\, ds = \iint \nabla \cdot v dA$$



Choose $v_1 = Q$

$v_2 = -P$

$$\underset{\sim}{n} = \left(\frac{dy}{ds}, \frac{-dx}{ds}\right)$$

$$\int \underset{\sim}{v} \cdot \underset{\sim}{n}\, ds = \oint (Q \underset{\sim}{i} - P \underset{\sim}{j}) \cdot \left(\frac{dy}{ds} \underset{\sim}{i} - \frac{dx}{ds} \underset{\sim}{j}\right) ds = \int (P dx + Q dy)$$

$$\iint \nabla \cdot v dA = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

(2) Stokes' theorem

$$\oint \underset{\sim}{v} \cdot \underset{\sim}{t}\, ds = \iint \nabla \times \underset{\sim}{v}\, dA$$

$$\underset{\sim}{v} = P \underset{\sim}{i} + Q \underset{\sim}{j} + 0 \underset{\sim}{k}$$

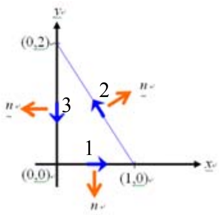
$$\underset{\sim}{t} = \left(\frac{dx}{ds} \underset{\sim}{i} + \frac{dy}{ds} \underset{\sim}{j}\right)$$

$$\iint \nabla \times \underset{\sim}{v}\, dA$$

$$\nabla \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \underset{\sim}{k}$$

$$\iint \nabla \times \underset{\sim}{v}\, dA = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

例題1.計算面積



$$y = 2-2x,\ x = \frac{2-y}{2}$$

$$\left(\frac{d\Gamma}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2 = \left(\frac{d}{dy}\left(\frac{2-y}{2}\right)\right)^2 + \left(\frac{dy}{dy}\right)^2$$

$$\frac{d\Gamma}{dy} = \frac{\sqrt{5}}{2} \Rightarrow d\Gamma = \frac{\sqrt{5}}{2}dy$$

$$A = \frac{1}{2}\int_\Gamma \nabla y^2 \cdot \underset{\sim}{n}\, d\Gamma$$

$y = 0$

$$= \int_{\Gamma_1}(0,y)\cdot(0,-1)d\Gamma + \int_{\Gamma_2}(0,y)\cdot\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)d\Gamma + \int_{\Gamma_3}(0,y)\cdot(-1,0)d\Gamma$$

$$= \int_0^2 \frac{1}{\sqrt{5}}\, y\cdot\frac{\sqrt{5}}{2}dy = \frac{1}{4}y^2\Big|_0^2 = 1$$

2008／03／27　Y.J.Lin 製　1

推導通式 $\iint_A y^n\, dxdy = \int_A y^n\, dA$ 轉線積分＝？

$$\int_A \nabla\phi\cdot\nabla\varphi dA = \int_\Gamma \phi\nabla\varphi\cdot \underset{\sim}{n}\, d\Gamma - \int_A \phi\nabla^2\varphi\, dA$$

$$\int_A y^n\, dA \Rightarrow \int_A \nabla\phi\cdot\nabla\varphi dA$$

$$\text{Let:}\ \phi = y^2 \quad ,\varphi = y^n$$

$$\int_A y^n\, dA = \frac{1}{2n}\int_A \nabla y^2\cdot\nabla y^n dA$$

$$\int_A \nabla y^2\cdot\nabla y^n dA = \int_\Gamma y^2\nabla y^n\cdot \underset{\sim}{n}\, d\Gamma - \int_A y^2\nabla^2 y^n\, dA$$

$$\Rightarrow \int_A y^n\, dA = \frac{1}{2n}\left[\int_\Gamma y^2\nabla y^n\cdot \underset{\sim}{n}\, d\Gamma - \int_A y^2\nabla^2 y^n\, dA\right]$$

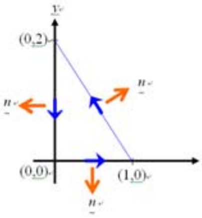
$$2n\int_A y^n\, dA = \int_\Gamma y^2\nabla y^n\cdot \underset{\sim}{n}\, d\Gamma - \left(n(n-1)\right)\int_A y^n\, dA$$

$$\left(n^2 - n + 2n\right)\int_A y^n\, dA = \int_\Gamma y^2\nabla y^n\cdot \underset{\sim}{n}\, d\Gamma$$

$$\Rightarrow \int_A y^n\, dA = \frac{1}{n(n+1)}\int_\Gamma y^2\nabla y^n\cdot \underset{\sim}{n}\, d\Gamma$$

2008／03／27　Y.J.Lin 製　2

例題2.計算形心 n=1



$$y = 2-2x,\ x=\frac{2-y}{2}$$

$$\left(\frac{d\Gamma}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2=\left(\frac{d}{dy}\left(\frac{2-y}{2}\right)\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{d\Gamma}{dy}=\frac{\sqrt{5}}{2}\ \Rightarrow d\Gamma=\frac{\sqrt{5}}{2}dy$$

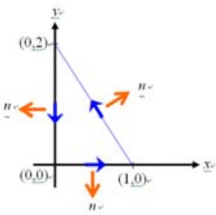
$$A\overline{y}=1\cdot\overline{y}=\int_A y\,dA=\frac{1}{2}\int_\Gamma y^2\nabla y\cdot \underset{\sim}{n}\,d\Gamma$$

$y=0$

$$=\frac{1}{2}\int_{\Gamma_1}(0,y^2)\cdot(0,-1)\,d\Gamma+\frac{1}{2}\int_{\Gamma_2}(0,y^2)\cdot\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)d\Gamma+\frac{1}{2}\int_{\Gamma_3}(0,y^2)\cdot(-1,0)d\Gamma$$

$$=\frac{1}{2}\int_0^2\frac{1}{\sqrt{5}}y^2\cdot\frac{\sqrt{5}}{2}dy=\frac{1}{2}\cdot\left(\frac{y^3}{6}\bigg|_0^2\right)=\frac{2}{3}$$

例題3.計算慣性矩 n=2



$$y = 2-2x,\ x=\frac{2-y}{2}$$

$$\left(\frac{d\Gamma}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2=\left(\frac{d}{dy}\left(\frac{2-y}{2}\right)\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{d\Gamma}{dy}=\frac{\sqrt{5}}{2}\ \Rightarrow d\Gamma=\frac{\sqrt{5}}{2}dy$$

$$I_{xx}=\int_A y^2\,dA=\frac{1}{6}\int_\Gamma y^2\nabla y^2\cdot \underset{\sim}{n}\,d\Gamma$$

$y=0$

$$=\frac{1}{3}\int_{\Gamma_1}(0,y^3)\cdot(0,-1)\,d\Gamma+\frac{1}{3}\int_{\Gamma_2}(0,y^3)\cdot\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)d\Gamma+\frac{1}{3}\int_{\Gamma_3}(0,y^3)\cdot(-1,0)d\Gamma$$

$$=\frac{1}{3}\int_0^2\frac{1}{\sqrt{5}}y^3\cdot\frac{\sqrt{5}}{2}dy=\frac{1}{3}\cdot\left(\frac{y^4}{8}\bigg|_0^2\right)=\frac{2}{3}$$

推導通式 $\iint_A x^n y^m \, dxdy = \int_A x^n y^m \, dA$ 轉線積分＝？

$$\int_A \nabla\phi\cdot\nabla\varphi dA = \int_\Gamma \phi\nabla\varphi\cdot \underset{\sim}{n}\, d\Gamma - \int_A \phi\nabla^2\varphi\, dA$$

$$\int_A x^n y^m \, dA \Rightarrow \int_A \nabla\phi\cdot\nabla\varphi dA$$

Let: $\phi = x^n y^2 \quad , \varphi = y^m$

$$\int_A x^n y^m \, dA = \frac{1}{2m}\int_A \nabla x^n y^2 \cdot \nabla y^m dA$$

$$\int_A \nabla x^n y^2 \cdot \nabla y^m dA = \int_\Gamma x^n y^2 \nabla y^m \cdot \underset{\sim}{n}\, d\Gamma - \int_A x^n y^2 \nabla^2 y^m \, dA$$

$$\Rightarrow \int_A x^n y^m \, dA = \frac{1}{2m}\left[\int_\Gamma x^n y^2 \nabla y^m \cdot \underset{\sim}{n}\, d\Gamma - \int_A x^n y^2 \nabla^2 y^m \, dA\right]$$

$$2m\int_A x^n y^m \, dA = \int_\Gamma x^n y^2 \nabla y^m \cdot \underset{\sim}{n}\, d\Gamma - \left(m\left(m-1\right)\right)\int_A \left(x^n y^2\right) y^{m-2} \, dA$$

$$(m^2+m)\int_A x^n y^m \, dA = \int_\Gamma x^n y^2 \nabla y^m \cdot \underset{\sim}{n}\, d\Gamma$$

$$\int_A x^n y^m \, dA = \frac{1}{m(m+1)}\int_\Gamma x^n y^2 \nabla y^m \cdot \underset{\sim}{n}\, d\Gamma$$

２００８／０３／２７ Y.J.Lin 製 5

雙重積分變成單重積分

$$\int_a^x \int_a^{x_1} f(\tau)d\tau dx_1 \Leftrightarrow \int_a^x (x-\xi)f(\xi)d\xi$$

Method.1 Proof using Leibniz rule

將左式微分，可得 $\frac{d}{dx}\int_a^x \int_a^{x_1} f(\tau)d\tau dx_1 = \int_a^x f(\tau)d\tau$

透過萊布尼茲法則

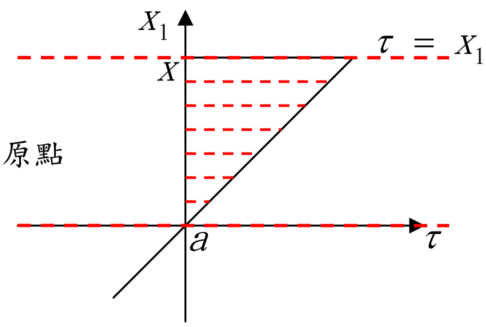
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt + f(x,b(x))b'(x) - f(x,a(x))a'(x)$$

可將右式微分 $\frac{d}{dx}\int_a^x (x-\xi)f(\xi)d\xi = \int_a^x f(\xi)d\xi$

左式等於右式，而且當 $x = a$ 時 $\int_a^a \int_a^{x_1} f(\tau)d\tau dx_1 = \int_a^a (a-\xi)f(\xi)d\xi$ 其值相等。

故可得證 $\int_a^x \int_a^{x_1} f(\tau)d\tau dx_1 = \int_a^x (x-\xi)f(\xi)d\xi$

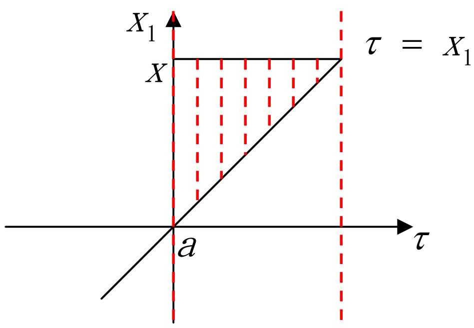
Method.2 變數變換



透過圖形平移可以假設 a 點為原點

$$\int_0^x \int_0^{x_1} f(\tau)d\tau dx_1 \Leftrightarrow$$

$\Updownarrow$ $\Updownarrow$



$$\int_0^x \int_\tau^x f(\tau)dx_1 d\tau \Leftrightarrow$$

$$= \int_0^x (x-\tau)f(\tau)d\tau$$

$$\int_0^X \int_0^{X_1} f(\tau)d\tau dx_1 = \int_0^X (X-\tau)f(\tau)d\tau \Leftrightarrow \int_a^X \int_a^{X_1} f(\tau)d\tau dx_1 = \int_a^X (X-\tau)f(\tau)d\tau$$

Method.3 幾何方法，見班主任(重積分轉線積分.ppt)

雙重積分變成單重積分.yu 製 2008/04/01

重積分轉線積分

$$\int_a^x \int_a^{x_1} f(\xi)d\xi dx_1 = \int_a^x (x-\xi) f(\xi)d\xi$$

指導教授：陳正宗終身特聘教授

報告者：高聖凱

日期：2008/04/01

關係說明

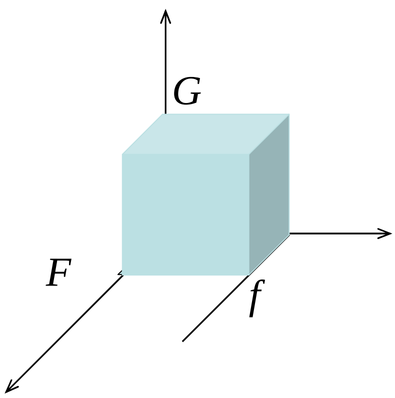
f：線

積分 ↓ ↑ 微分

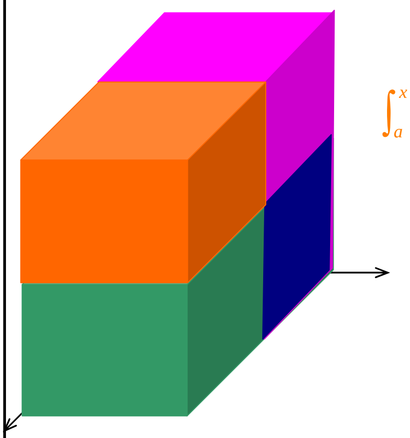
F：面

積分 ↓ ↑ 微分

G：體積

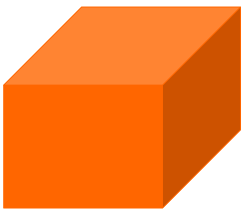


$$\int_a^x \int_a^{x_1} f(\xi)d\xi dx_1$$



$$\int_a^x \int_a^{x_1} f(\xi)d\xi dx_1 = \int_a^x \left[F(x_1) - F(a)\right]dx_1$$

$$= G(x) - G(a) - xF(a) + aF(a)$$



$$\int_a^x \int_a^{x_1} f(\xi)d\xi dx_1$$

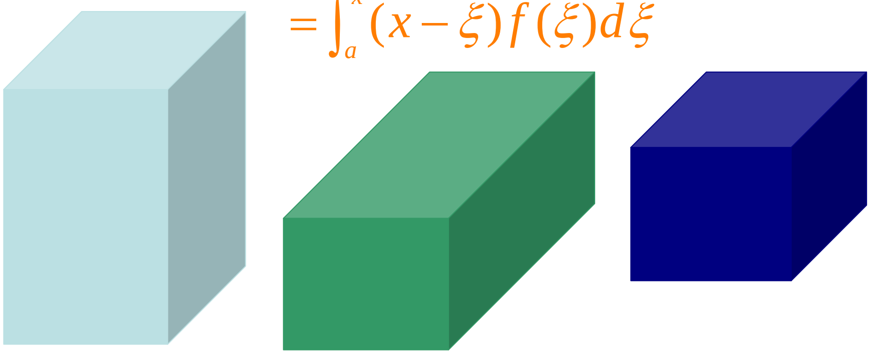
$$= \left[G(x) - G(a)\right] - \left[(x-a)F(a)\right]$$

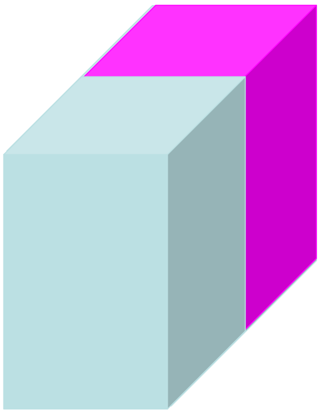
$$= \int_a^x F(\xi)d\xi - \int_a^x F(a)d\xi \quad ?$$

$$= \int_a^x F(\xi)d\xi + (x-\xi)F(\xi)\Big|_a^x$$

$$= (x-\xi)F(\xi)\Big|_a^x - \int_a^x -F(\xi)d\xi$$

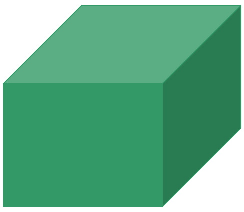
$$= \int_a^x (x-\xi)f(\xi)d\xi$$





$$x\int_a^x f(\xi)d\xi$$

$$= xF(x) - xF(a)$$



$$-[G(x)-G(a)]+[xF(x)-aF(a)]$$

$$= -\int_a^x F(\xi)d\xi + \xi F(\xi)\Big|_a^x$$

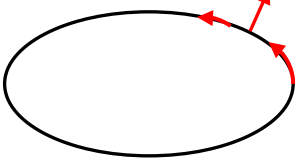
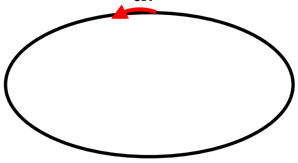
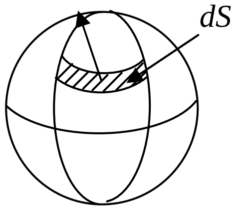
$$= \xi F(\xi)\Big|_a^x - \int_a^x F(\xi)d\xi = \int_a^x \xi f(\xi)d\xi$$

$xF(x)$ $\quad -[G(x)-G(a)]$ $\quad -aF(a)$

$$\int_a^x \int_a^{x_1} f(\xi)\, d\xi\, dx_1$$

$$= \int_a^x x f(\xi)\, d\xi - \int_a^x \xi f(\xi)\, d\xi$$

$$= \int_a^x (x-\xi) f(\xi)\, d\xi$$

	Gauss(散度)定理	Stokes 定理
	法向	切向
1-2D	$\oint \underset{\sim}{v}\cdot\underset{\sim}{n}\,ds$ $=\iint \nabla\cdot\underset{\sim}{v}\,dA$  	$\oint \underset{\sim}{F}\cdot dr$ $=\iint \nabla\times F\cdot\underset{\sim}{n}\,dS$ 
2-3D	$\iint_S \underset{\sim}{v}\cdot\underset{\sim}{n}\,dS$ $=\iiint_V \nabla\cdot\underset{\sim}{v}\,dV$  	沒了 因爲面上切向量 怎麼定？
	$\underset{\sim}{v}$:velocity 流量跑出	$\underset{\sim}{F}$:force 作功
	縱波	橫波
	P 波	S 波

Evaluate $\oint_c zdx + xdy + ydz$**, where** C **is the boundary of the plane** $z = 1 - y$**. See the Figure 1.**

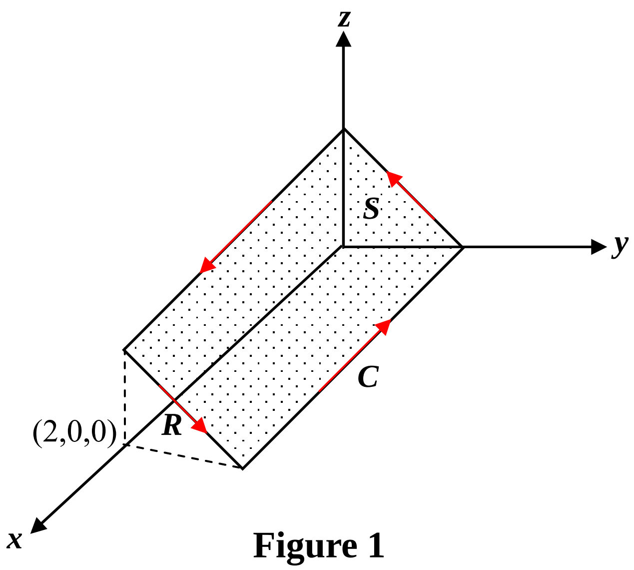


Figure 1

Sol:

$$\underset{\sim}{F} = zi + xj + yk$$

$$\nabla \times \underset{\sim}{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & k \end{vmatrix} = (i, j, k)$$

$$\underset{\sim}{n} = (\frac{0}{\sqrt{2}} i, \frac{1}{\sqrt{2}} j, \frac{1}{\sqrt{2}} k)$$

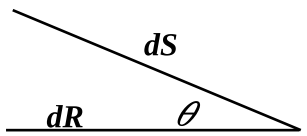
$$\oint_c zdx + xdy + ydz = \iint_S \nabla \times \underset{\sim}{F} \cdot \underset{\sim}{n} dS$$

$$= \iint_S (i, j, k) \cdot (\frac{0}{\sqrt{2}} i, \frac{1}{\sqrt{2}} j, \frac{1}{\sqrt{2}} k) dS$$

$$= \iint_S (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) dS = \sqrt{2} \iint_S dS$$

$$= \sqrt{2} \iint_S dS = \sqrt{2} \iint_R \sqrt{2} dR$$

$$= 4$$



$$\frac{dR}{dS} = \cos\theta \Rightarrow dS = \frac{dR}{\cos\theta} = \sqrt{2} dR$$

Fig.1

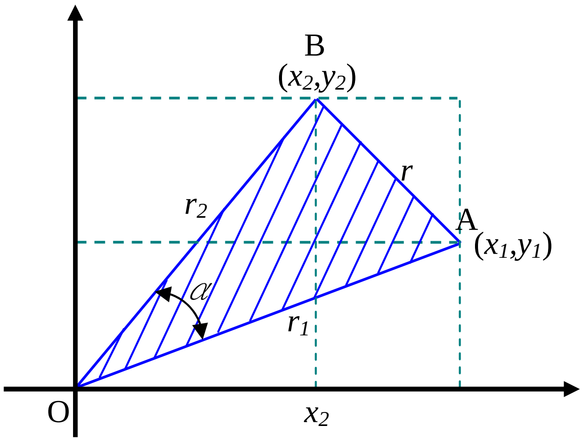
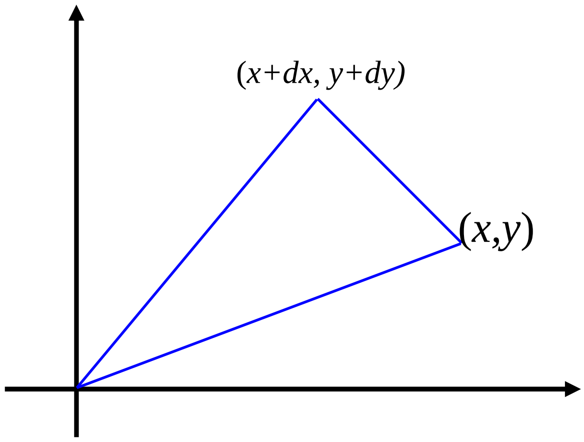


Fig.2



(a). Please determine the area of triangle OAB composed by the three points O:(0,0), A:(2,1), B:(1,2)as shown in Fig.1, where the origin is (0,0), (x1,y1)=(2,1), (x2,y2)=(1,2).

(b). Please determine the area of triangle composed by the three points (0,0), (x,y), (x+dx, y+dy)as shown in Fig.2, where (x+dx,y+dy) is the neighborhood of (x,y).

(c). What is the Green theorem ?

(d). Please determine $\oint_C r \cdot n ds$, where r is the position vector of (x,y) and n is the normal vector of contour C, ds is the path integration, where C is the closed contour along boundaries of triangle OAB: (0,0), (2,1), (1,2) as shown in Fig.1.

(e). Please determine the divergence of position vector r i.e., $\nabla \cdot r = ?$ where r=x i +y j ?

Small test 08-01.doc 魯蛋製

1. Vector calculus:

Fig.1

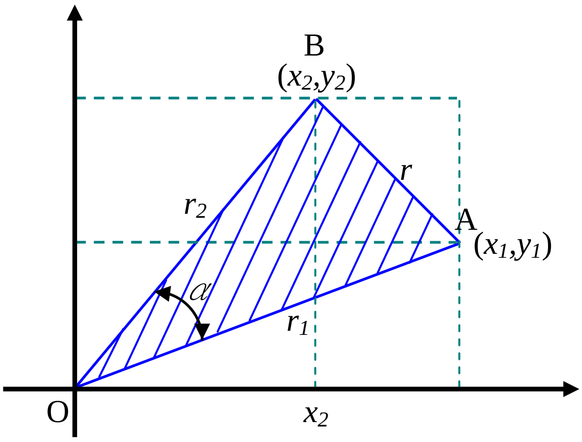
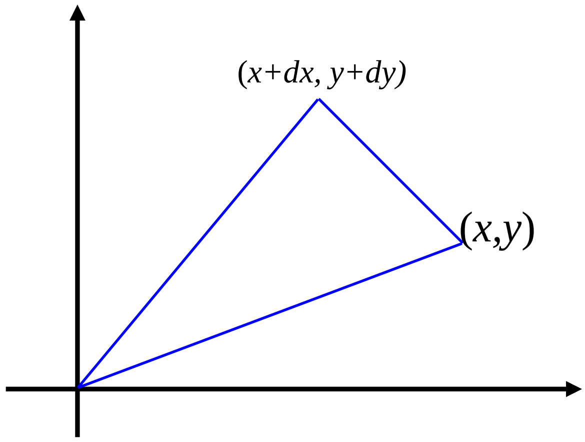


Fig.2



(a). Please determine the area of triangle OAB composed by the three points O:(0,0), A:(2,1), B:(1,2) as shown in Fig.1, where the origin is (0,0), (x1,y1)=(2,1), (x2,y2)=(1,2). (5 %)

$$\frac{1}{2}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = \frac{3}{2}$$

(b). Please determine the area of triangle composed by the three points (0,0), (x,y), (x+dx, y+dy) as shown in Fig.2, where (x+dx,y+dy) is the neighborhood of (x,y). (5 %)

$$\frac{1}{2}\begin{vmatrix} x+dx & y+dy \\ x & y \end{vmatrix} = \frac{1}{2}\oint xdy - ydx$$

(c). What is the Green theorem ? (5 %)

$$A=\oint Pdx + Qdy$$

$$= \iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA\right)$$

$$P = y$$
$$Q = -x$$

(d). Please determine $\oint_C r \cdot n ds$, where r is the position vector of (x,y) and n is the normal vector of contour C, ds is the path integration, where C is the closed contour along boundaries of triangle OAB: (0,0), (2,1), (1,2) as shown in Fig.1. (5 %)

$$\oint r \cdot n ds$$

$$= \int (x, 3-x) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) ds$$

$$+ \int \left(x, \frac{1}{2}x\right) \cdot \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) ds$$

$$+ \int (x, 2x) \cdot \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) ds$$

$$= \int_0^{\sqrt{2}} \left(\frac{1}{\sqrt{2}}x + \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}}x\right) ds$$

$$= \frac{3}{\sqrt{2}} \int_0^{\sqrt{2}} ds = 3$$

$$r = (x, 3-x) \qquad n = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$r_1 = \left(x, \frac{1}{2}x\right) \qquad n_1 = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$$

$$r_2 = (x, 2x) \qquad n_2 = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

(e). Please determine the divergence of position vector r i.e., $\nabla \cdot r = ?$ where r=x i +y j (5%).

$$\iint_A \nabla \cdot r dA \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \quad r = (x, y)$$

$$\nabla \cdot r = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial y}\right) = 1 + 1 = 2$$

海洋大學河海工程學系　工程數學（二）－補救教學練習考２　姓名：

1.求函數 $y = x^2$ ，在點 $(2,4)$ 的曲率及曲率半徑。

2. $\begin{cases} x(t) = t\cos(t) \\ y(t) = t\sin(t) \\ z(t) = \sqrt{3}t \end{cases}$ ，時域參數表示法轉空域參數表示法，求 t 與 s 的關係?(s : 路徑長)

3. $\begin{cases} x(t) = \dfrac{1}{2}e^{2t} \\ y(t) = \sqrt{2}e^{t} \\ z(t) = t \end{cases}$ ，時域參數表示法轉空域參數表示法，求 t 與 s 的關係? (s : 路徑長)

Small test 08-02.2.doc 魯蛋製

1. 求函數 $y=x^2$，在點 $(2,4)$ 的曲率及曲率半徑。

$$y'=2x$$
$$y''=2$$

$$\frac{1}{\rho}=\frac{y''}{\left[1+\left(y'\right)^2\right]^{3/2}}$$

$$\frac{1}{\rho}=\frac{2}{\left[1+\left(2x\right)^{3/2}\right]^{3/2}}$$

$$\frac{1}{\rho}=\frac{2}{\left[1+16\right]^{3/2}}=0.029$$

$$\rho=35.046$$

2. $\begin{cases} x(t)=t\cos(t) \\ y(t)=t\sin(t) \\ z(t)=\sqrt{3}t \end{cases}$，時域參數表示法轉空域參數表示法，求 t 與 s 的關係?(s :路徑長)

$$x'=\cos\left(t\right)-t\sin\left(t\right)$$
$$y'=\sin\left(t\right)+t\cos\left(t\right)$$
$$z'=\sqrt{3}$$
$$\left(\frac{dx}{dt}\right)^2=t^2\sin t^2-2t\cos t\sin t+\cos t^2$$
$$\left(\frac{dy}{dt}\right)^2=t^2\cos t^2+2t\cos t\sin t+\sin t^2$$
$$\left(\frac{dz}{dt}\right)^2=3$$

$$(ds)^2=(dx)^2+(dy)^2+(dz)^2$$
$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$
$$=\sqrt{t^2+1+3}$$
$$=\sqrt{4+t^2}$$

$$\therefore s=\frac{1}{2}(t\sqrt{t^2+4}+4\ln\left|t+\sqrt{t^2+4}\right|)+c$$
$$\because t=0\rightarrow s=0$$
$$\therefore c=-2.77$$
$$\therefore s=\frac{1}{2}(t\sqrt{t^2+4}+4\ln\left|t+\sqrt{t^2+4}\right|)-2.77$$

3. $\begin{cases} x(t)=\dfrac{1}{2}e^{2t} \\ y(t)=\sqrt{2}e^{t} \\ z(t)=t \end{cases}$ ，時域參數表示法轉空域參數表示法，求t與s的關係? (s:路徑長)

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$

$$=\sqrt{e^{4t}+2e^{2t}+1}$$

$$\frac{ds}{dt}=\left(e^{4t}+2e^{2t}+1\right)^{\frac{1}{2}}$$

$$ds=\left(e^{2t}+1\right)dt$$

$$s=\frac{1}{2}e^{2t}+t+c$$

$\because t=0\rightarrow s=0$

$\therefore c=-\dfrac{1}{2}$

$\therefore s=\dfrac{1}{2}e^{2t}+t-\dfrac{1}{2}$

1.在一個平面盤上溫度的分布函數爲$T(x,y)=20-4x^2-y^2$ 。

求(1) 在$(2,3)$位置，哪個方向溫度增加最快?

(2) 變化率的最大值爲何?

2.一球體之溫度分部爲$T(x,y,z)=x^2+xy+y^2+2yz$ ，求在點$(1,1,1)$延著$P(\frac{1}{2},1,2)$到$Q(\frac{3}{2},0,3)$的方向導微?

Small test 08-03.doc 魯蛋製

1. 在一個平面盤上溫度的分布函數為$T(x,y)=20-4x^2-y^2$。

求(1) 在$(2,3)$位置，哪個方向溫度增加最快?

(2) 變化率的最大值為何?

(1)

$$\frac{\partial T}{\partial a}=\nabla T\cdot \underset{\sim}{a}=(-8x,2y)\cdot \underset{\sim}{a}=(-16,-6)\cdot a$$
$$|\nabla T|\cdot|a|\cos\theta=\sqrt{292}\times 1\times\cos\theta$$

$$when\quad \cos\theta=0\Rightarrow \underset{\sim}{a}=(\frac{-16}{\sqrt{292}},\frac{-6}{\sqrt{292}})$$

(2)

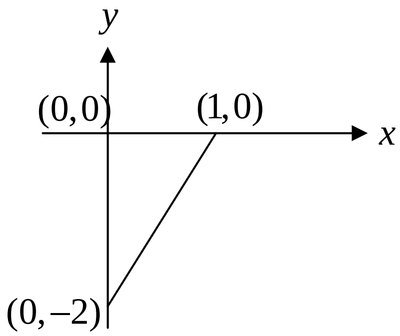
$$|\nabla T|\cdot\underbrace{\left|\frac{\nabla T}{|\nabla T|}\right|}_{=1}=|\nabla T|=\sqrt{292}$$

2.一球體之溫度分部為$T(x,y,z)=x^2+xy+y^2+2yz$，求在點$(1,1,1)$延著$P(\frac{1}{2},1,2)$到$Q(\frac{3}{2},0,3)$的方向導微?

$$\overrightarrow{PQ}=(1,-1,1)=\underset{\sim}{a}$$
$$\nabla T=(2x+y,x+2y+2z,2y)$$
$$\nabla T\cdot\underset{\sim}{a}=(3,5,2)\cdot(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}})$$
$$=\frac{3}{\sqrt{3}}-\frac{5}{\sqrt{3}}+\frac{2}{\sqrt{3}}$$
$$=0$$

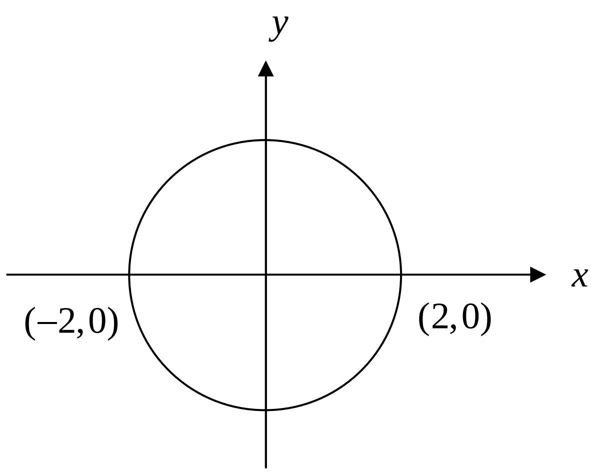
海洋大學河海工程學系　　工程數學（二）－補救教學練習考 4　姓名：
學號：

1.



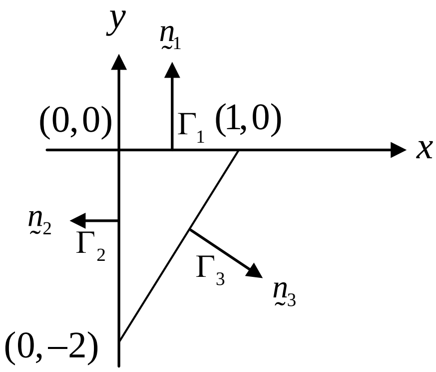
試求此三角形的面積?　請利用 $\iint_A 1dxdy = \frac{1}{2}\int_\Gamma \nabla y^2 \cdot \underset{\sim}{n}\, d\Gamma$，面積分跟線積分各算一次。

2.



試算出點 $(-2,0)$ 的曲率半徑?(請用今天上課所交的方法)

1.



試求此三角形的面積?　請利用$\iint_A 1dxdy=\frac{1}{2}\int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\,d\Gamma$，面積分跟線積分各算一次。

Sol:

面積分:

$$A=\iint 1\,dxdy=\int_{-2}^{0}\int_{0}^{\frac{y+2}{2}}dxdy=\int_{-2}^{0}\frac{y+2}{2}\,dy=\frac{y^2}{4}+y\Big|_{-2}^{1}=1$$

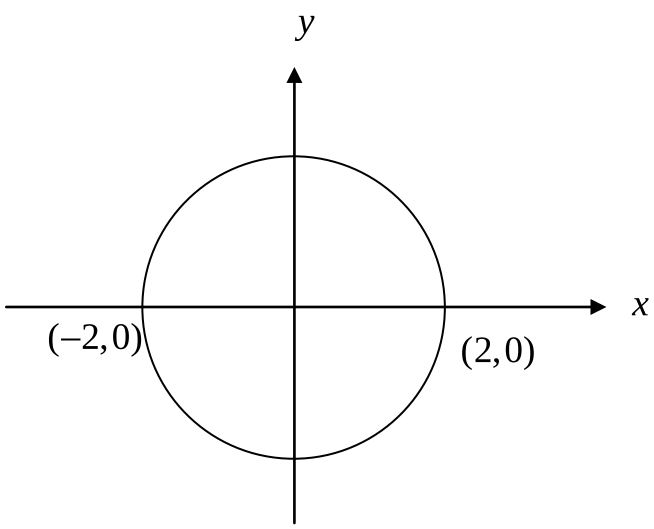
線積分:

$$A=\frac{1}{2}\int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\,d\Gamma$$

$$=\int_{\Gamma_1}(0,y)\cdot(0,1)d\Gamma+\int_{\Gamma_2}(0,y)\cdot(-1,0)d\Gamma+\int_{\Gamma_3}(0,y)\cdot(\frac{2}{\sqrt{5}},\frac{-1}{\sqrt{5}})d\Gamma$$

$$=\int_{-2}^{0}\frac{-1}{\sqrt{5}}y\cdot\frac{\sqrt{5}}{2}\,dy=-\frac{1}{4}y^2\Big|_{-2}^{0}=1$$

2.



試算出點$(-2,0)$的曲率半徑?(請用今天上課所交的方法)

Set:

參數表示式:

$$let:\begin{cases}x=2\cos\theta\\ y=2\sin\theta\end{cases}\Rightarrow\begin{cases}\dot{x}=-2\sin\theta\\ \dot{y}=\ \ 2\cos\theta\end{cases}\Rightarrow\begin{cases}\ddot{x}=-2\cos\theta\\ \ddot{y}=-2\sin\theta\end{cases}$$

$$\rho=\frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=\frac{\left[(-2\sin\theta)^2+(2\cos\theta)^2\right]^{\frac{3}{2}}}{(-2\sin\theta)(-2\sin\theta)-(2\cos\theta)(-2\cos\theta)}=\frac{(4)^{\frac{3}{2}}}{4}=2$$

變成弧長表示式:

$$let:\begin{cases}x=2\cos\theta\\ y=2\sin\theta\end{cases}\Rightarrow\begin{cases}\dfrac{dx}{d\theta}=-2\sin\theta\\ \dfrac{dy}{d\theta}=\ \ 2\cos\theta\end{cases}$$

$$ds=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=\sqrt{(-2\sin\theta)^2+(2\cos\theta)^2}$$

$$s\Rightarrow 2\theta$$

$$\begin{cases}x=2\cos\dfrac{s}{2}\\ y=2\sin\dfrac{s}{2}\end{cases}\Rightarrow\begin{cases}\dot{x}=-\sin\dfrac{s}{2}\\ \dot{y}=\ \ \cos\dfrac{s}{2}\end{cases}\Rightarrow\begin{cases}\ddot{x}=-\dfrac{1}{2}\cos\dfrac{s}{2}\\ \ddot{y}=-\dfrac{1}{2}\sin\dfrac{s}{2}\end{cases}$$

(a) $$\rho=\frac{\dot{X}^2+\dot{Y}^2}{\dot{X}\ddot{Y}-\ddot{X}\dot{Y}}=\frac{\left(-\sin\frac{s}{2}\right)^2+\left(\cos\frac{s}{2}\right)^2}{\left(-\sin\frac{s}{2}\right)\left(-\frac{1}{2}\sin\frac{s}{2}\right)-\left(-\frac{1}{2}\cos\frac{s}{2}\right)\left(\cos\frac{s}{2}\right)}=2$$

(b) $$\rho=\frac{1}{\dot{X}\ddot{Y}-\ddot{X}\dot{Y}}=\frac{1}{\left(-\sin\frac{s}{2}\right)\left(-\frac{1}{2}\sin\frac{s}{2}\right)-\left(-\frac{1}{2}\cos\frac{s}{2}\right)\left(\cos\frac{s}{2}\right)}=2$$

(c) $$\rho=\frac{1}{\sqrt{\ddot{X}^2+\ddot{Y}^2}}=\frac{1}{\sqrt{\left(-\frac{1}{2}\cos\frac{s}{2}\right)^2+\left(-\frac{1}{2}\sin\frac{s}{2}\right)^2}}=2$$

工數第一次大考(Vector calculus) 18:00-20:00 April 10, 2008

日期：2008 年 4 月 10 日　姓名：＿＿＿＿＿＿＿　學號：＿＿＿＿＿＿＿

1 Evaluate $\oint_c \underset{\sim}{F}\cdot d\underset{\sim}{r} = \oint_c zdx + xdy + ydz$, where the force field $\underset{\sim}{F} = (z, x, y)$, and C is the trace of the cylinder $x^2 + y^2 = 1$ in the plane $y + z = 2$. (See Figure 1.) (a) Use Stokes' theorem to determine the $\oint_c \underset{\sim}{F}\cdot d\underset{\sim}{r}$. (6%) (b) Use line integral determine the $\oint_c \underset{\sim}{F}\cdot d\underset{\sim}{r}$. (6%)

(c) Determine $\nabla\times\underset{\sim}{F}, \nabla\cdot\underset{\sim}{F}$. (6%) (d) Is the force field $\underset{\sim}{F}$ conservative? (2%) (e) Why? (5%)

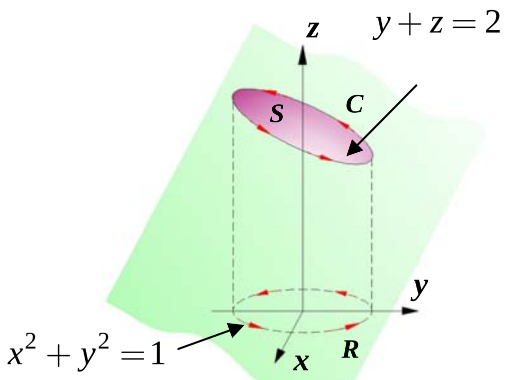


Figure 1

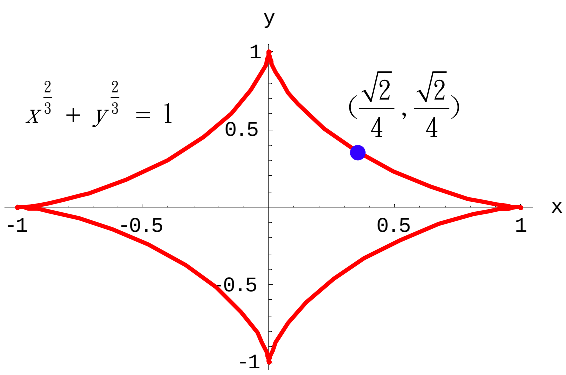


Figure 2

2 See Figure 2.

a. Find the parameter form for the curve.(Hint: $\begin{cases} x(\theta) = \cos^3\theta \\ y(\theta) = ? \end{cases}$) (2%)

b. Find the total arc length of the curve. (3%)

c. Use the Green theorem to calculate the area bounded by the curve. (Hint: $2A = \oint xdy - ydx$) (5%)

d. Find the radius of curvature for the curve at the point $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$.(10%)

3 Given a spatial curve described by a time-like parameter as follows：

$$\begin{cases} x(t) = \dfrac{t}{\sqrt{3}}\cos(\ln(t)) \\ y(t) = \dfrac{t}{\sqrt{3}}\sin(\ln(t)), t > 0 \\ z(t) = \dfrac{t}{\sqrt{3}} \end{cases} \text{ and } (x(0), y(0), z(0)) = (0, 0, 0) \Rightarrow \begin{cases} X(s) \\ Y(s)\ ? \\ Z(s) \end{cases}$$

Try to find the following solutions:

a. Transform the time-like into space-like parameter. ($t \to s$, where s is arc length) (7%)

b. Find the unit tangential vector, normal vector and bi-normal vector of the curve, $\tau(s), \upsilon(s)$ and $\beta(s)$. (10%)

c. Please determine the radius of curvature for ρ and σ, try to plot the curve. (6%)

d. If $\underset{\sim}{r}(t) = (x(t), y(t), z(t)) \Rightarrow \underset{\sim}{R}(s) = (X(s), Y(s), Z(s))$, please determine $\left(\frac{d\underset{\sim}{R}}{ds}\times\frac{d^2\underset{\sim}{R}}{ds^2}\right)\cdot\frac{d^3\underset{\sim}{R}}{ds^3}$ at $t = \sqrt{3}$. (7%)

4 The temperature field, $T(x,y)$, on the surface of a metal plate is

$$T(x,y)=\frac{\pi}{6}+4\sqrt{3}-6\cos(x)-4\cos(x)\sin(y).$$

a. Please determine the directional derivative at the point $P\,(\frac{\pi}{6},\frac{\pi}{6})$ in the direction from $a(-1,2)$ to $b(1,3)$. (3%)

b. In which directions the temperature increase and decrease most rapidly at the point P? (Please express your answer in a unit vector). (4%) Why? (4%)

c. Please determine the minimum rate of the temperature field at the point P. (3%)

d. In which directions the temperature has no change at the point P? (Please express your answer in a unit vector). (4%) Why? (4%)

e. Please determine the tangent plane across the point $(\frac{\pi}{6},\frac{\pi}{6},T(\frac{\pi}{6},\frac{\pi}{6}))$.(See Figures 3 and 4.) (3%)

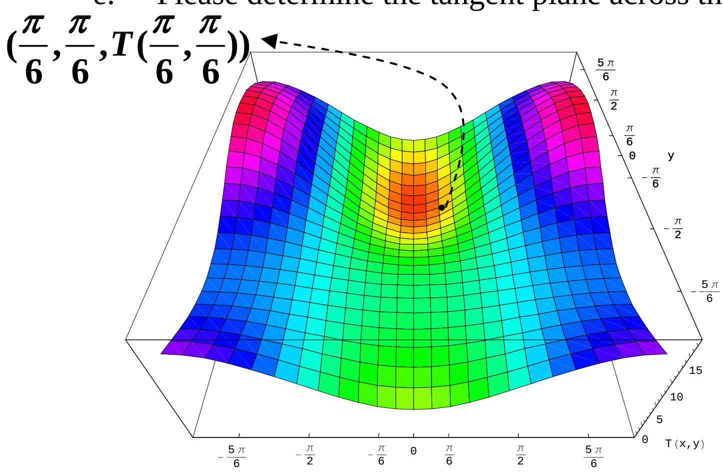


Figure 3

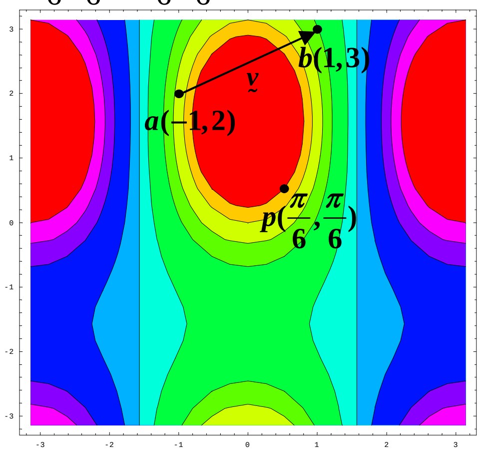
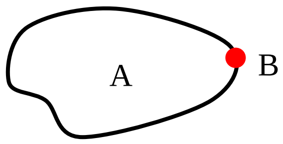


Figure 4

5 Domain integral $\leftrightarrow$ boundary integral (See Figure 5.)



a. Please derive the formula $\iint_A 1\,dxdy=\frac{1}{2}\oint_\Gamma \frac{\partial y^2}{\partial n}\,d\Gamma$ (5%)

b. Please calculate the area (Figure 5) by using the following formula (5%)

(Please use both integrals and compare with each other.) $\iint_A 1\,dxdy=\frac{1}{2}\oint_\Gamma \frac{\partial y^2}{\partial n}\,d\Gamma$

c. Use the following formulas to find the centroid $(\bar{x},\bar{y})$ (10%)

$$A\bar{x}=\iint_A x\,dxdy=\int_A x\,dA=\frac{1}{n(n+1)}\int_\Gamma x^2\nabla x\cdot \underset{\sim}{n}\,d\Gamma$$

$$A\bar{y}=\iint_A y\,dxdy=\int_A y\,dA=\frac{1}{n(n+1)}\int_\Gamma y^2\nabla y\cdot \underset{\sim}{n}\,d\Gamma$$

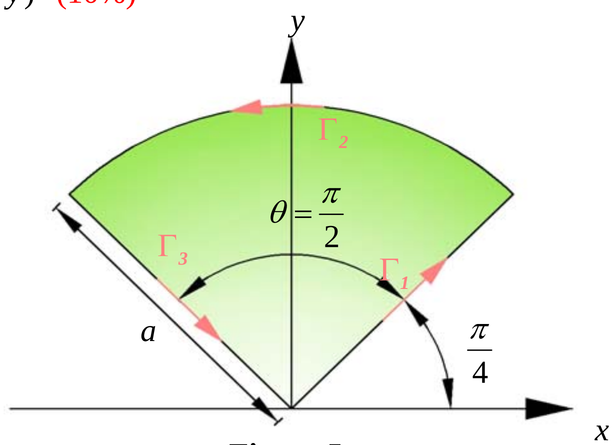


Figure 5

工數第一次大考(Vector calculus) 18:00-20:00 April 10, 2008

日期：2008 年 4 月 10 日　姓名：__________　學號：__________

1 Evaluate $\oint_c \underset{\sim}{F} \cdot d\underset{\sim}{r} = \oint_c zdx + xdy + ydz$, where the force field $\underset{\sim}{F} = (z, x, y)$, and C is the trace of the cylinder $x^2 + y^2 = 1$ in the plane $y + z = 2$. (See Figure 1.) (a) Use Stokes' theorem to determine the $\oint_c \underset{\sim}{F} \cdot d\underset{\sim}{r}$. (6%) (b) Use line integral determine the $\oint_c \underset{\sim}{F} \cdot d\underset{\sim}{r}$. (6%)

(c) Determine $\nabla \times \underset{\sim}{F}, \nabla \cdot \underset{\sim}{F}$. (6%) (d) Is the force field $\underset{\sim}{F}$ conservative? (2%) (e) Why? (5%)

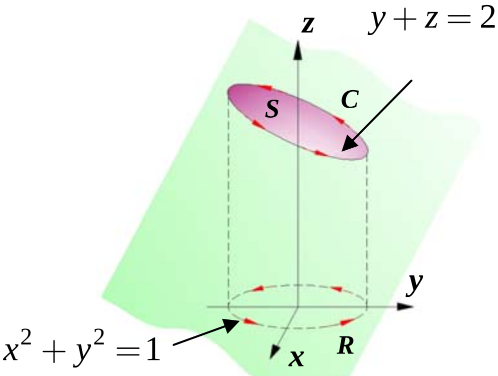


Figure 1

Sol:

(a)Using Stokes' Theorem $\oint_C F \cdot \underset{\sim}{t} ds = \iint_S (\nabla \times F) \cdot \underset{\sim}{n} dS$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = i + j + k$$

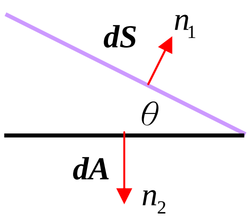
$$\underset{\sim}{n} = \left(\frac{0}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k\right)$$

$$\oint_C F \cdot \underset{\sim}{t} ds = \iint_S (\nabla \times F) \cdot \underset{\sim}{n} dS$$

$$= \iint_S (i + j + k) \cdot \left(\frac{0}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k\right) dS$$

$$= \sqrt{2} \iint_S dS$$

$$= \sqrt{2} \iint_R \sqrt{2} dA = 2\pi$$



$$\frac{dA}{dS} = \cos\theta \Rightarrow dS = \frac{dA}{\cos\theta}$$

$$\therefore dS = \sqrt{2} dA$$

$$n_1 = (0,1,1)$$

$$n_2 = (0,0,1)$$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{1}{\sqrt{2}}$$

(b) Use line integral

$$\oint_c zdx + xdy + ydz = \oint_C zdx + \oint_C xdy + \oint_C ydz$$

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 2 - \sin\theta \end{cases} \Rightarrow \begin{cases} dx = -\sin\theta d\theta \\ dy = \cos\theta d\theta \\ dz = -\cos\theta d\theta \end{cases}$$

$$\oint_C zdx = \int_0^{2\pi} (2 - \sin\theta) - \sin\theta d\theta$$

$$= \int_0^{2\pi} \sin^2\theta - 2\sin\theta d\theta = \pi$$

$$\oint_C xdy = \int_0^{2\pi} \cos\theta \cos\theta d\theta$$

$$= \int_0^{2\pi} \cos^2\theta d\theta = \pi$$

$$\oint_C ydz = \int_0^{2\pi} \sin\theta(-\cos\theta) d\theta$$

$$= -\int_0^{2\pi} \sin\theta \cos\theta d\theta = 0$$

$$\therefore \oint_C zdx + \oint_C xdy + \oint_C ydz = \pi + \pi + 0 = 2\pi$$

(c) Determine $\nabla \times \underset{\sim}{F}, \nabla \cdot \underset{\sim}{F}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = i + j + k$$

$$\nabla \cdot F = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \cdot (z + x + y) = (\frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z}) = 0$$

(d) Is the force field $\underset{\sim}{F}$ **conservative?**

No

(e) Why?

因爲 $\nabla \times \underset{\sim}{F} \neq 0$

2 See Figure 2.

a. Find the parameter form for the curve.(Hint: $\begin{cases} x(\theta) = \cos^3\theta \\ y(\theta) = ? \end{cases}$) (2%)

b. Find the total arc length of the curve. (3%)

c. Use the Green theorem to calculate the area bounded by the curve.
(Hint: $2A = \oint xdy - ydx$) (5%)

d. Find the radius of curvature for the curve at the point $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$.(10%)

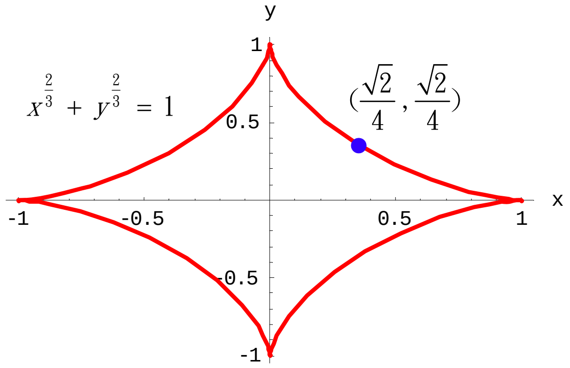


Figure 2

Sol.

(a) $\because \cos^2\theta + \sin^2\theta = 1, \therefore$ set $\begin{cases} x = \cos^3\theta \\ y = \sin^3\theta \end{cases}$

(b) Set $\begin{cases} x = \cos^3\theta \\ y = \sin^3\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{d\theta} = -3\cos^2\theta\sin\theta \\ \dfrac{dy}{d\theta} = 3\sin^2\theta\cos\theta \end{cases}$

$$ds = \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}d\theta = |3\sin\theta\cos\theta|\,d\theta$$

$$s = \int ds = 4\int_0^{\frac{\pi}{2}} 3\sin\theta\cos\theta d\theta = 6\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = 6$$

(c) $A = \dfrac{1}{2}\oint (xdy - ydx)$

$$A = \frac{1}{2}\oint (3\cos^4\theta\sin^2\theta d\theta + 3\sin^4\theta\cos^2\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} 3\cos^2\theta\sin^2\theta d\theta = \frac{3}{8}\int_0^{2\pi}\sin^2 2\theta d\theta = \frac{3}{8}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{3\pi}{8}$$

(d) <u>Method1</u>

$\because$ *the point*$(\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4})=(\cos^3\frac{\pi}{4},\sin^3\frac{\pi}{4})$ $\therefore$ *we konw* $\theta=\frac{\pi}{4}$ *and* $\frac{ds}{d\theta}=3\sin\theta\cos\theta$

$$\begin{cases}\bar{x}(s)=x(\theta)=\cos^3\theta\\ \bar{y}(s)=y(\theta)=\sin^3\theta\end{cases}$$

$$\begin{cases}\frac{d\bar{x}(s)}{ds}=\frac{dx(\theta)}{d\theta}\frac{d\theta}{ds}=-\cos\theta\\ \frac{d\bar{y}(s)}{ds}=\frac{dy(\theta)}{d\theta}\frac{d\theta}{ds}=\sin\theta\end{cases}$$

$$\begin{cases}\frac{d^2\bar{x}(s)}{ds^2}=\frac{1}{d\theta}(\frac{d\bar{x}(s)}{ds})\frac{d\theta}{ds}=\frac{d(-\cos\theta)}{d\theta}\frac{d\theta}{ds}=\frac{\sin\theta}{3\sin\theta\cos\theta}\\ \frac{d^2\bar{y}(s)}{ds^2}=\frac{1}{d\theta}(\frac{d\bar{y}(s)}{ds})\frac{d\theta}{ds}=\frac{d(\sin\theta)}{d\theta}\frac{d\theta}{ds}=\frac{\cos\theta}{3\sin\theta\cos\theta}\end{cases}$$

$$\rho=\frac{(\frac{dx}{ds})^2+(\frac{dy}{ds})^2}{\left|\frac{dx}{ds}\frac{d^2y}{ds^2}-\frac{d^2x}{ds^2}\frac{dy}{ds}\right|}$$

$$=\frac{(-\cos\theta)^2+(\sin\theta)^2}{\left|(-\cos\theta)\frac{\cos\theta}{3\sin\theta\cos\theta}-\sin\theta\frac{\sin\theta}{3\sin\theta\cos\theta}\right|}=\left|3\sin\theta\cos\theta\right|$$

So *we konw* $\theta=\frac{\pi}{4},\rho=\left|3\sin\theta\cos\theta\right|=\left|3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right|=\frac{3}{2}$

<u>Method2</u>

$$\rho=\frac{1}{\sqrt{(\frac{d^2x}{ds^2})^2+(\frac{d^2y}{ds^2})^2}}=\frac{1}{\sqrt{(\frac{\sin\theta}{3\sin\theta\cos\theta})^2+(\frac{\cos\theta}{3\sin\theta\cos\theta})^2}}=\left|3\sin\theta\cos\theta\right|\text{So}$$

we konw $\theta=\frac{\pi}{4},\rho=\left|3\sin\theta\cos\theta\right|=\left|3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right|=\frac{3}{2}$

<u>Method3</u>

$$\begin{cases}x(\theta)=\cos^3\theta\\ y(\theta)=\sin^3\theta\end{cases}\Rightarrow\begin{cases}\frac{dx(\theta)}{d\theta}=-3\cos^2\theta\sin\theta\\ \frac{dy(\theta)}{d\theta}=3\sin^2\theta\cos\theta\end{cases}$$

$$\begin{cases}\frac{d^2x(\theta)}{d\theta^2}=-3\cos^3\theta+6\sin^2\theta\cos\theta\\ \frac{d^2y(\theta)}{d\theta^2}=-3\sin^3\theta+6\cos^2\theta\sin\theta\end{cases}$$

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\frac{dy}{d\theta}\right|} = \frac{\left[(9\cos^2\theta\sin^2\theta\right]^{\frac{3}{2}}}{\left|-9\cos^2\theta\sin^2\theta\right|} = \left|3\sin\theta\cos\theta\right|$$

So *we konw* $\theta = \frac{\pi}{4}, \rho = \left|3\sin\theta\cos\theta\right| = \left|3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right| = \frac{3}{2}$

3 Given a spatial curve described by a time-like parameter as follows：

$$\begin{cases} x(t) = \frac{t}{\sqrt{3}}\cos\left(\ln(t)\right) \\ y(t) = \frac{t}{\sqrt{3}}\sin\left(\ln(t)\right), t > 0 \quad \text{and} \quad \left(x(0), y(0), z(0)\right) = \left(0, 0, 0\right) \Rightarrow \begin{cases} X(s) \\ Y(s) \; ? \\ Z(s) \end{cases} \\ z(t) = \frac{t}{\sqrt{3}} \end{cases}$$

Try to find the following solutions:

a. Transform the time-like into space-like parameter. ($t \to s$, where s is arc length) (7%)

b. Find the unit tangential vector, normal vector and bi-normal vector of the curve, $\tau(s), \upsilon(s)$ *and* $\beta(s)$. (10%)

c. Please determine the radius of curvature for ρ and σ, try to plot the curve. (6%)

d. If $\underset{\sim}{r}(t) = \left(x(t), y(t), z(t)\right) \Rightarrow \underset{\sim}{R}(s) = \left(X(s), Y(s), Z(s)\right)$, please determine $\left(\frac{d\underset{\sim}{R}}{ds} \times \frac{d^2\underset{\sim}{R}}{ds^2}\right) \cdot \frac{d^3\underset{\sim}{R}}{ds^3}$ at $t = \sqrt{3}$. (7%)

Sol.

(a)

$$\begin{cases} \dot{x}(t) = \frac{1}{\sqrt{3}}\cos\left(\ln(t)\right) - \frac{1}{\sqrt{3}}\sin\left(\ln(t)\right) \\ \dot{y}(t) = \frac{1}{\sqrt{3}}\sin\left(\ln(t)\right) + \frac{1}{\sqrt{3}}\cos\left(\ln(t)\right), t > 0 \\ \dot{z}(t) = \frac{1}{\sqrt{3}} \end{cases}$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= \left[\frac{1}{\sqrt{3}}\cos\left(\ln(t)\right) - \frac{1}{\sqrt{3}}\sin\left(\ln(t)\right)\right]^2 + \left[\frac{1}{\sqrt{3}}\sin\left(\ln(t)\right) + \frac{1}{\sqrt{3}}\cos\left(\ln(t)\right)\right]^2 + \left[\frac{1}{\sqrt{3}}\right]^2 = 1$$

$ds = dt \to s = t$

$$\begin{cases} x(s) = \dfrac{s}{\sqrt{3}}\cos(\ln(s)) \\ y(s) = \dfrac{s}{\sqrt{3}}\sin(\ln(s)),\ s>0 \\ z(s) = \dfrac{s}{\sqrt{3}} \end{cases}$$

(b)

$$\underset{\sim}{\tau}(s) = \left(\frac{dx(s)}{ds}, \frac{dy(s)}{ds}, \frac{dz(s)}{ds}\right)$$

$$= \left[\frac{1}{\sqrt{3}}\cos(\ln(s)) - \frac{1}{\sqrt{3}}\sin(\ln(s))\right]\underset{\sim}{i} + \left[\frac{1}{\sqrt{3}}\sin(\ln(s)) + \frac{1}{\sqrt{3}}\cos(\ln(s))\right]\underset{\sim}{j} + \left[\frac{1}{\sqrt{3}}\right]\underset{\sim}{k}$$

$$\underset{\sim}{\upsilon}(s) // \frac{d}{ds}\underset{\sim}{\tau}(s)$$

$$\frac{d}{ds}\underset{\sim}{\tau}(s) = \left[-\frac{1}{\sqrt{3}s}\sin(\ln(s)) - \frac{1}{\sqrt{3}s}\cos(\ln(s))\right]\underset{\sim}{i} + \left[\frac{1}{\sqrt{3}s}\cos(\ln(s)) - \frac{1}{\sqrt{3}s}\sin(\ln(s))\right]\underset{\sim}{j} + [0]\underset{\sim}{k}$$

$$\left\|\frac{d}{ds}\underset{\sim}{\tau}(s)\right\| = \sqrt{\frac{2}{3s^2}} = \frac{\sqrt{2}}{\sqrt{3}s}$$

$$\underset{\sim}{\upsilon}(s) = -\left[\frac{1}{\sqrt{2}}\sin(\ln(s)) + \frac{1}{\sqrt{2}}\cos(\ln(s))\right]\underset{\sim}{i} + \left[\frac{1}{\sqrt{2}}\cos(\ln(s)) - \frac{1}{\sqrt{2}}\sin(\ln(s))\right]\underset{\sim}{j} + [0]\underset{\sim}{k}$$

$$\underset{\sim}{\beta}(s) = \underset{\sim}{\tau}(s) \times \underset{\sim}{\upsilon}(s)$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{1}{\sqrt{3}}\cos(\ln(s)) - \frac{1}{\sqrt{3}}\sin(\ln(s)) & \frac{1}{\sqrt{3}}\sin(\ln(s)) + \frac{1}{\sqrt{3}}\cos(\ln(s)) & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}}\sin(\ln(s)) + \frac{-1}{\sqrt{2}}\cos(\ln(s)) & \frac{1}{\sqrt{2}}\cos(\ln(s)) - \frac{1}{\sqrt{2}}\sin(\ln(s)) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\sqrt{3}}\sin(\ln(s)) + \frac{1}{\sqrt{3}}\cos(\ln(s)) & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}}\cos(\ln(s)) - \frac{1}{\sqrt{2}}\sin(\ln(s)) & 0 \end{vmatrix}\underset{\sim}{i} - \begin{vmatrix} \frac{1}{\sqrt{3}}\cos(\ln(s)) - \frac{1}{\sqrt{3}}\sin(\ln(s)) & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}}\sin(\ln(s)) + \frac{-1}{\sqrt{2}}\cos(\ln(s)) & 0 \end{vmatrix}\underset{\sim}{j}$$

$$+ \begin{vmatrix} \frac{1}{\sqrt{3}}\cos(\ln(s)) - \frac{1}{\sqrt{3}}\sin(\ln(s)) & \frac{1}{\sqrt{3}}\sin(\ln(s)) + \frac{1}{\sqrt{3}}\cos(\ln(s)) \\ \frac{-1}{\sqrt{2}}\sin(\ln(s)) + \frac{-1}{\sqrt{2}}\cos(\ln(s)) & \frac{1}{\sqrt{2}}\cos(\ln(s)) - \frac{1}{\sqrt{2}}\sin(\ln(s)) \end{vmatrix}\underset{\sim}{k}$$

$$= -\left[\frac{1}{\sqrt{6}}\cos(\ln(s)) - \frac{1}{\sqrt{6}}\sin(\ln(s))\right]\underset{\sim}{i} - \left[\frac{1}{\sqrt{6}}\sin(\ln(s)) + \frac{1}{\sqrt{6}}\cos(\ln(s))\right]\underset{\sim}{j} + \left[\frac{2}{\sqrt{6}}\right]\underset{\sim}{k}$$

(c)

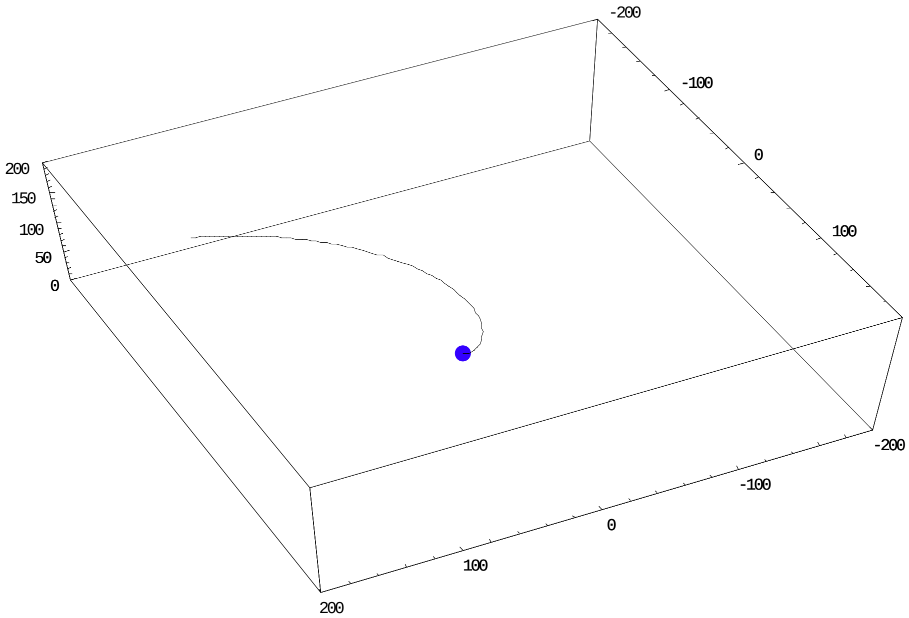
$$\frac{1}{\rho}=\left\|\frac{d}{ds}\underset{\sim}{\tau}(s)\right\|=\sqrt{\frac{2}{3s^2}}=\frac{\sqrt{2}}{\sqrt{3}s}$$

$$\rho=\sqrt{\frac{3}{2}}s$$

$$\frac{d}{ds}\underset{\sim}{\beta}=\left[\frac{1}{\sqrt{6}s}\sin(\ln(s))+\frac{1}{\sqrt{6}s}\cos(\ln(s))\right]\underset{\sim}{i}-\left[\frac{1}{\sqrt{6}s}\cos(\ln(s))-\frac{1}{\sqrt{6}s}\sin(\ln(s))\right]\underset{\sim}{j}+[0]\underset{\sim}{k}$$

$$\frac{1}{\sigma}=\left\|-\frac{d\underset{\sim}{\beta}}{ds}\right\|=\frac{1}{\sqrt{3}s}$$

$$\sigma=\sqrt{3}s$$



(d)

$$\left(\frac{d\underset{\sim}{r}}{ds}\times\frac{d^2\underset{\sim}{r}}{ds^2}\right)\cdot\frac{d^3\underset{\sim}{r}}{ds^3}=\left(\underset{\sim}{\tau}\times\frac{1}{\rho}\underset{\sim}{\upsilon}\right)\cdot\left[\frac{1}{\rho}\left(\frac{d}{ds}\underset{\sim}{\upsilon}\right)\right]$$

$$=\frac{1}{\rho}\underset{\sim}{\beta}\cdot\left[\frac{1}{\rho}\left(-\frac{1}{\rho}\underset{\sim}{\tau}+\frac{1}{\sigma}\underset{\sim}{\beta}\right)\right]$$

$$=\frac{1}{\rho^2}\frac{1}{\sigma}$$

$$=\frac{2}{3\sqrt{3}s^3}$$

$$=\frac{2}{27}$$

4 The temperature field, $T(x,y)$, on the surface of a metal plate is

$$T(x,y)=\frac{\pi}{6}+4\sqrt{3}-6\cos(x)-4\cos(x)\sin(y).$$

a. Please determine the directional derivative at the point $P\,(\frac{\pi}{6},\frac{\pi}{6})$ in the direction from $a(-1,2)$ to $b(1,3)$. (3%)

b. In which directions the temperature increase and decrease most rapidly at the point P? (Please express your answer in a unit vector). (4%) Why? (4%)

c. Please determine the minimum rate of the temperature field at the point P. (3%)

d. In which directions the temperature has no change at the point P? (Please express your answer in a unit vector). (4%) Why? (4%)

e. Please determine the tangent plane across the point $(\frac{\pi}{6},\frac{\pi}{6},T(\frac{\pi}{6},\frac{\pi}{6}))$.(See Figures 3 and 4.) (3%)

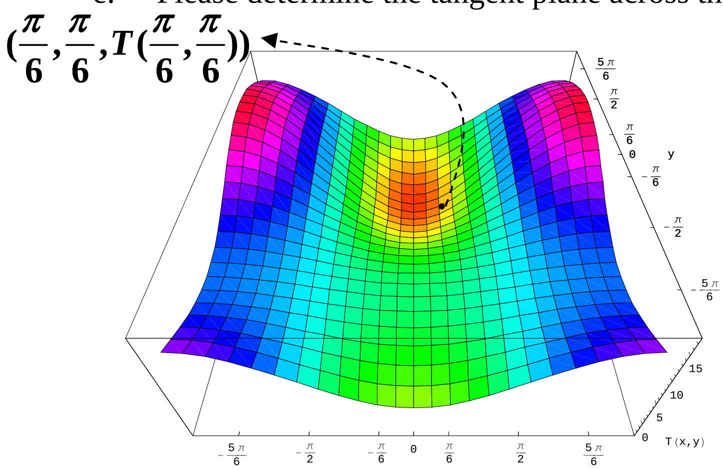


Figure 3

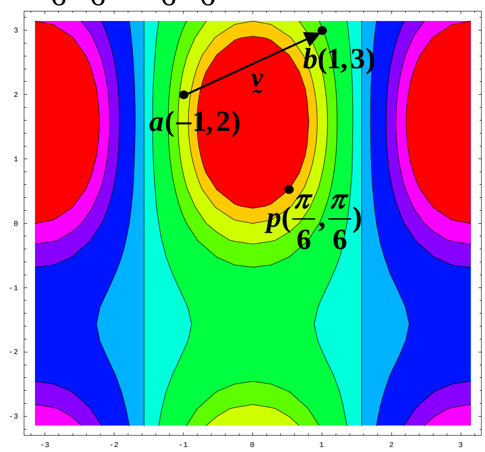


Figure 4

Sol.

(a) $\underset{\sim}{v}=\frac{(2,1)}{\sqrt{2^2+1^2}}=(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})$

$$\nabla T(x,y)=(6\sin(x)+4sin(x)\sin(y),-4\cos(x)\cos(y))$$

$$\Rightarrow \nabla T(\frac{\pi}{6},\frac{\pi}{6})=(4,-3)$$

$$\therefore \frac{\partial T(\frac{\pi}{6},\frac{\pi}{6})}{\partial v}=\nabla T(\frac{\pi}{6},\frac{\pi}{6})\cdot\underset{\sim}{v}=(4,-3)\cdot(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})=\sqrt{5}$$

(b) $\max \Rightarrow (\frac{4}{5},-\frac{3}{5})$

$\min \Rightarrow (-\frac{4}{5},\frac{3}{5})$

因爲 $\frac{\partial T}{\partial v}=\nabla T\cdot\underset{\sim}{n}=|\nabla T||\underset{\sim}{n}|\cos(\theta)$

$$-1\le\cos(\theta)\le 1$$

所以 $\dfrac{\partial T}{\partial n} \Rightarrow \begin{cases} \max \Rightarrow \theta = 0^\circ \\ \min \Rightarrow \theta = 180^\circ \end{cases}$

因此溫度增加最快的方向即為梯度方向 $(\frac{4}{5}, -\frac{3}{5})$ 。

溫度降低最快的方向即為梯度方向的反向 $(-\frac{4}{5}, \frac{3}{5})$ 。

(c) $\dfrac{\partial T}{\partial v} = \nabla T \cdot \underset{\sim}{n} = |\nabla T||\underset{\sim}{n}|\cos(\theta)$

$\because -1 \le \cos(\theta) \le 1$

$\therefore \left(\dfrac{\partial T}{\partial n}\right)_{\min} = \nabla T \cdot \underset{\sim}{n} = |\nabla T||\underset{\sim}{n}|\cos(180^\circ) = -|\nabla T| = -5$

(d) $(\frac{3}{5}, \frac{4}{5})$ *and* $(-\frac{3}{5}, -\frac{4}{5})$

因為 $\dfrac{\partial T}{\partial t} = \nabla T \cdot \underset{\sim}{t} = |\nabla T||\underset{\sim}{t}|\cos(\theta)$

$-1 \le \cos(\theta) \le 1$

若溫度不改變的話，也就是變化率為 0，即方向導微為 0，也就是

$\theta = 90^\circ$ *and* $\theta = 270^\circ$ 。

因此溫度不改變的方向即跟梯度方向垂直 $\Rightarrow (\frac{3}{5}, \frac{4}{5})$ *and* $(-\frac{3}{5}, -\frac{4}{5})$ 。

(e) 切於此點 $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ 平面的法向量，為 $\underset{\sim}{n} = (4, -3, -1)$ ，

令平面方程式為 $4x - 3y - z = k$ ， $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ 代入，得 $k = 0$ 。

所以切於此點 $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ 的平面方程式為 $4x - 3y - z = 0$ 。

5 Domain integral ↔ boundary integral (See Figure 5.)

a. Please derive the formula $\iint_A 1\, dxdy = \frac{1}{2}\oint_\Gamma \frac{\partial y^2}{\partial n}\, d\Gamma$ (5%)

b. Please calculate the area (Figure 5) by using the following formula (5%)

(Please use both integrals and compare with each other.) $\iint_A 1\, dxdy = \frac{1}{2}\oint_\Gamma \frac{\partial y^2}{\partial n}\, d\Gamma$

c. Use the following formulas to find the centroid $(\bar{x}, \bar{y})$ (10%)

$$A\bar{x} = \iint_A x\, dxdy = \int_A x\, dA = \frac{1}{n(n+1)}\int_\Gamma x^2 \nabla x \cdot \underset{\sim}{n}\, d\Gamma$$

$$A\bar{y} = \iint_A y\, dxdy = \int_A y\, dA = \frac{1}{n(n+1)}\int_\Gamma y^2 \nabla y \cdot \underset{\sim}{n}\, d\Gamma$$

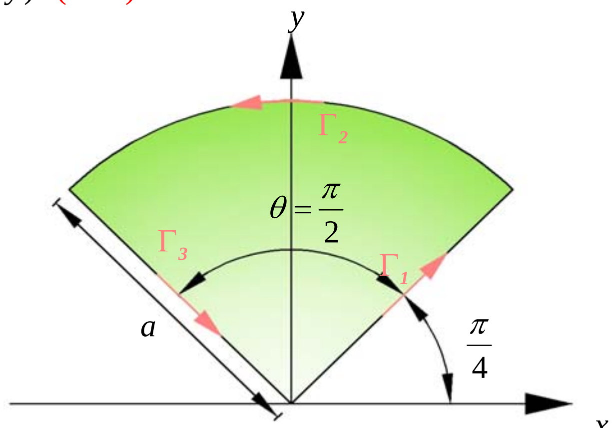


Figure 5

Sol:

(a).試推導由面積分轉線積分 $\iint_A 1\, dxdy = \frac{1}{2}\oint_\Gamma \frac{\partial y^2}{\partial n}\, d\Gamma$

利用格林第二恆等式：

$$\int_A \phi \nabla^2 \varphi\, dA = \int_\Gamma \phi \nabla\varphi \cdot \underset{\sim}{n}\, d\Gamma - \int_A \nabla\phi \cdot \nabla\varphi dA$$

Let: $\phi = 1$, $\varphi = y^2$ or x^2

$$\int_A \nabla^2 y^2\, dA = \int_\Gamma \nabla y^2 \cdot \underset{\sim}{n}\, d\Gamma$$

$$\Rightarrow \int_A \nabla^2 y^2\, dA = \int_\Gamma \nabla y^2 \cdot \underset{\sim}{n}\, d\Gamma$$

$$\Rightarrow \frac{1}{2}\int_A \nabla^2 y^2\, dA = \int_A 1\, dA = \frac{1}{2}\int_\Gamma \nabla y^2 \cdot \underset{\sim}{n}\, d\Gamma$$

(b).以上列公式面積分及線積分計算面積

面積分:

$x^2 + y^2 = a$; $x = a\cos\theta$, $y = a\sin\theta$

$$\iint_A 1\, dxdy = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^a a\, da d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\frac{a^2}{2}\right]_0^a d\theta$$

$$= \left[\frac{a^2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{a^2}{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi a^2}{4}$$

線積分:

$$A=\frac{1}{2}\int_{\Gamma}\nabla y^2\cdot \underset{\sim}{n}\, d\Gamma$$

$$=\int_{\Gamma_1}(0,y)\cdot(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_1+\int_{\Gamma_2}(0,y)\cdot(\cos\theta,\sin\theta)d\Gamma_2+\int_{\Gamma_3}(0,y)\cdot(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_3$$

第一線段：

$$\int_{\Gamma_1}(0,y)\cdot(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_1=\int_{\Gamma_1}\frac{-y}{\sqrt{2}}d\Gamma_1$$

$$\left(\frac{d\Gamma_1}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{dx}{dy}=1\ ;\frac{dy}{dy}=1$$

$$d\Gamma_1=\sqrt{2}\ dy$$

$$\int_{\Gamma_1}\frac{-y}{\sqrt{2}}d\Gamma_1=\int_0^{\frac{a}{\sqrt{2}}}\left(\frac{-y}{\sqrt{2}}\right)\cdot\left(\sqrt{2}\right)dy=-\frac{y^2}{2}\bigg|_0^{\frac{a}{\sqrt{2}}}=-\frac{a^2}{4}$$

第二線段：

$$\int_{\Gamma_2}(0,y)\cdot(\cos\theta,\sin\theta)d\Gamma_2=\int_{\Gamma_2}y\sin\theta d\Gamma_2$$

$$\left(\frac{d\Gamma}{d\theta}\right)^2=\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2$$

$$x=a\cos\theta\,;y=a\sin\theta$$

$$\frac{dx}{d\theta}=-a\sin\theta\ ;\frac{dy}{d\theta}=a\cos\theta$$

$$d\Gamma_2=a\ d\theta$$

$$\int_{\Gamma_2}-y\sin\theta d\Gamma_2=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(-y\sin\theta\right)ad\theta=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left[\left(a\sin\theta\right)\cdot\sin\theta\right]a\ d\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}a^2\sin^2\theta\ d\theta=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}a^2\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)\bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}=\frac{\pi a^2}{4}+\frac{a^2}{2}$$

第三線段：

$$A=\int_{\Gamma_3}(0,y)\cdot(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_3=\int_{\Gamma_1}\frac{-y}{\sqrt{2}}d\Gamma_3$$

$$\left(\frac{d\Gamma_3}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{dx}{dy}=1\ ;\frac{dy}{dy}=1$$

$$d\Gamma_3=\sqrt{2}\,dy$$

$$\int_{\Gamma_3}\frac{-y}{\sqrt{2}}d\Gamma_3=\int_0^{\frac{a}{\sqrt{2}}}\left(\frac{-y}{\sqrt{2}}\right)\cdot\left(\sqrt{2}\right)dy=-\frac{y^2}{2}\Bigg|_0^{\frac{a}{\sqrt{2}}}=-\frac{a^2}{4}$$

$$A=-\frac{a^2}{4}+\frac{\pi a^2}{4}+\frac{a^2}{2}-\frac{a^2}{4}=\frac{\pi a^2}{4}$$

(c).以下列公式計算形心 $\overline{y}$

$$A\overline{y}=\int_A y\,dA=\frac{1}{2}\int_\Gamma y^2\nabla y\cdot \underset{\sim}{n}\,d\Gamma$$

$$=\int_{\Gamma_1}(0,y^2)\cdot(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_1+\int_{\Gamma_2}(0,y^2)\cdot(\cos\theta,\sin\theta)d\Gamma_2+\int_{\Gamma_3}(0,y^2)\cdot(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_3$$

第一線段：

$$\int_{\Gamma_1}(0,y^2)\cdot(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_1=\int_{\Gamma_1}\frac{-y^2}{\sqrt{2}}d\Gamma_1$$

$$\left(\frac{d\Gamma_2}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{dx}{dy}=1\ ;\frac{dy}{dy}=1$$

$$d\Gamma_2=\sqrt{2}\,dy$$

$$\int_{\Gamma_1}\frac{-y^2}{\sqrt{2}}d\Gamma_1=\int_0^{\frac{a}{\sqrt{2}}}\left(\frac{-y^2}{\sqrt{2}}\right)\cdot\left(\sqrt{2}\right)dy=-\frac{y^3}{3}\Bigg|_0^{\frac{a}{\sqrt{2}}}=-\frac{a^3}{6\sqrt{2}}$$

第二線段：

$$\int_{\Gamma_2}(0,y^2)\cdot(\cos\theta,\sin\theta)d\Gamma_2=\int_{\Gamma_2}y^2\sin\theta d\Gamma_2$$

$$\left(\frac{d\Gamma_2}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$x=a\cos\theta\,;y=a\sin\theta$$

$$\frac{dx}{dy}=-a\sin\theta\ ;\frac{dy}{dy}=a\cos\theta$$

$$d\Gamma_2=a\,d\theta$$

$$\int_{\Gamma_2}-y^2\sin\theta d\Gamma_2=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(-y^2\sin\theta\right)ad\theta=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left[\left(a\sin\theta\right)^2\cdot\sin\theta\right]a\,d\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}a^3\sin^3\theta\,d\theta=\frac{5a^3}{3\sqrt{2}}$$

第三線段：

$$\int_{\Gamma_3}(0,y^2)\cdot(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})d\Gamma_3=\int_{\Gamma_1}\frac{-y^2}{\sqrt{2}}d\Gamma_3$$

$$\left(\frac{d\Gamma_3}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{dx}{dy}=1\ ;\frac{dy}{dy}=1$$

$$d\Gamma_3=\sqrt{2}\,dy$$

$$\int_{\Gamma_3}\frac{-y^2}{\sqrt{2}}d\Gamma_3=\int_0^{\frac{a}{\sqrt{2}}}\left(\frac{-y^2}{\sqrt{2}}\right)\cdot\left(\sqrt{2}\right)dy=-\frac{y^3}{3}\Bigg|_0^{\frac{a}{\sqrt{2}}}=-\frac{a^3}{6\sqrt{2}}$$

$$A\overline{y}=-\frac{a^3}{6\sqrt{2}}+\frac{5a^3}{3\sqrt{2}}-\frac{a^3}{6\sqrt{2}}=\frac{4a^3}{3\sqrt{2}}$$

$$\overline{y}=\frac{4a^3}{3\sqrt{2}}\Bigg/\frac{\pi a^2}{4}=\frac{4\sqrt{2}a}{3\pi}$$

$$\overline{x}=0$$

第二章　Fourier 級數

2.1 Fourier 預備知識

粒徑篩分析與 Fourier series 分析------------------------------------99
傳立葉級數在圓邊表示法--100
傳立葉級數(相關內容)--101
傳立葉級數(預備知識) --102
奇偶函數對稱與反對稱--116
向量空間到函數空間--117
向量內外積談三角和公式--118
振幅與相差與週期--119
振幅與相差以及週期(補救教學) ------------------------------------121
時空三角函數--126
cos(kx-wt)k and w 的意義--127

2.2 Fourier series

Why Fourier series--128
Fourier series expansion(可看到 Gibbs phenomenon) --------------------129
傳立葉級數--131
Fourier coefficient--132
實數與複數 Fourier--133
Decomposition theorem--134
Fourier series 與轉換--135
Fourier series for function with period 2L 與 2π---------------------------136
和差化積與正交關係--137

2.3 Fourier 算例

series-sum(低階含解答) --138
series-sum(高階) --140
Fourier series examples (John Appleby) ---------------------------------141

2.3 Gibbs 現象

Gibbs phenomenon--146
Gibbs 實例--148

2.4 複數 Fourier series

Fourier series--149
複數空間講義--150
通式與特例 Fourier--151
複數與工程(工程師的觀點) --152

2.5 級數微分

Stokes'轉換--154
Alternative series and Stokes' transformation------------------------------156

Alternative series and Stokes' transformation------------------------------158
Cesaro sum for Fourier series--159

2.6 能量

Energy conservation--161
Parseval's 定理與 Fourier coefficient 的新觀點-----------------------------162

2.7 小考

小考與解答--164

Fourier series (重點整理)

● Fourier 預備知識

粒徑篩分析與 Fourier series 分析(傅立葉級數在圓邊表示法)
傅立葉級數(相關內容)
傅立葉級數(預備知識 1)感謝徐文信博士提供
奇偶函數對稱與反對稱
向量空間到函數空間(預備知識 3) ppt 檔(林羿州提供)
向量內外積談三角和公式
振幅與相差與週期(預備知識 2) ppt 檔(謝祥志提供)
時空三角函數
cos(kx-wt)k and w 的意義

● Fourier series

Why Fourier series(範例-可看到 Gibbs phenomenon) 傅立葉級數	
Fourier coefficient ，實數與複數 Fourier	
Decomposition theorem	
Fourier series-2	
Fourier series for function with period 2L 與 2π	
和差化積與正交關係	
算例	series-sum(低階含解答)(高階) Euler 的作法 May 15 2008 上課計畫
	Fourier series examples (John Appleby)

● Gibbs 現象

Gibbs phenomenon (實例)

● 複數 Fourier series

Fourier series-4
複數空間講義
通式與特例 Fourier
複數與工程(工程師的觀點)

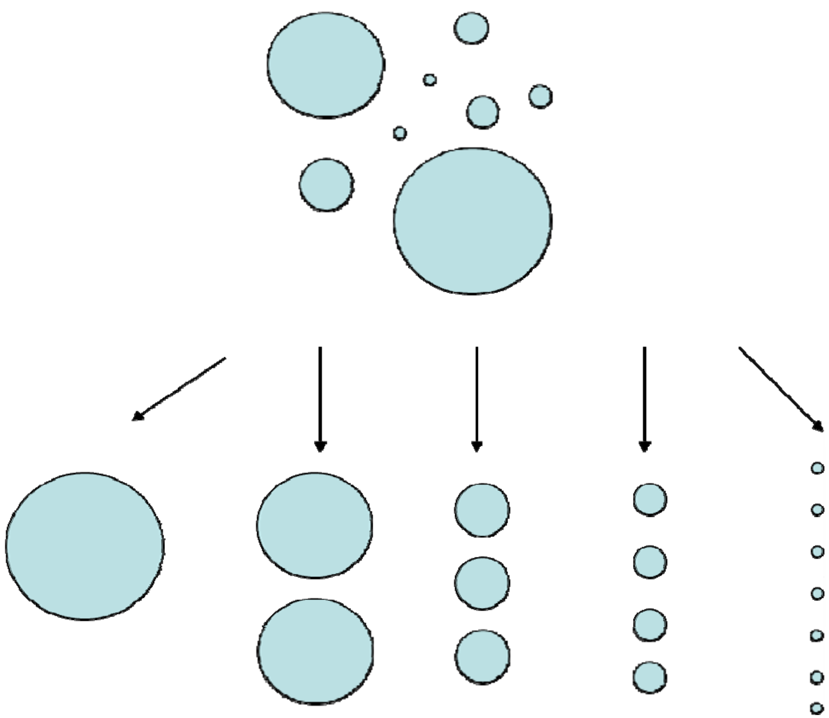
級數微分

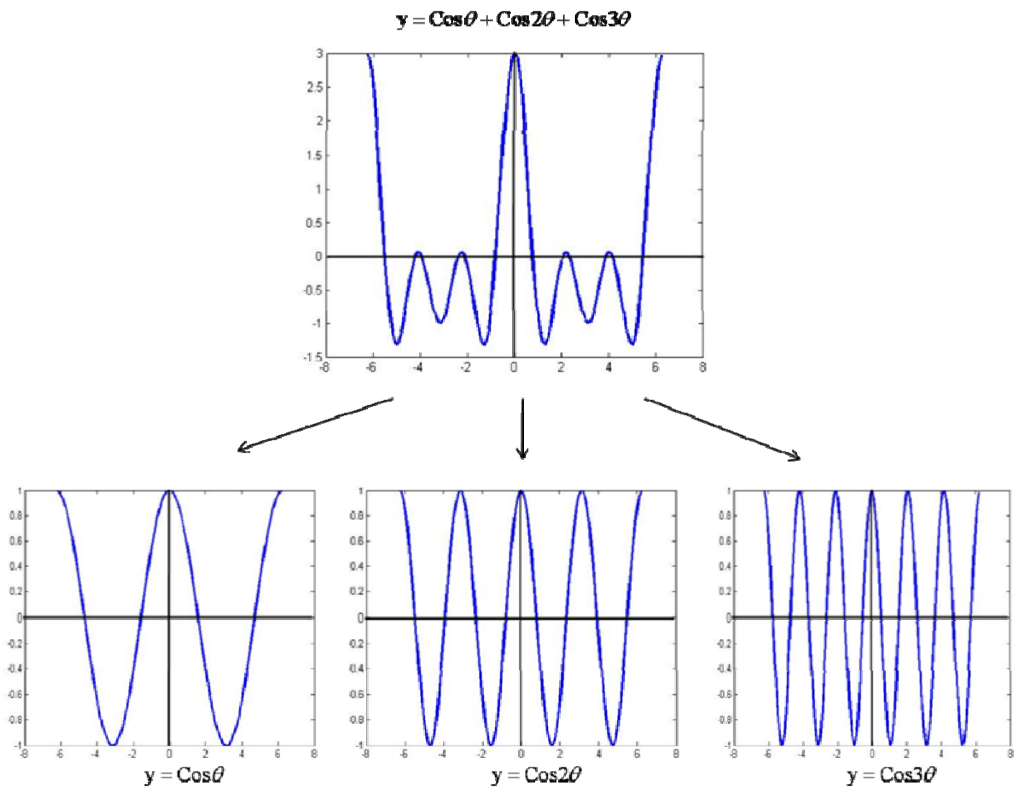
Stokes’ 轉換	Alternative series and Stokes' transformation
	Alternative series and Stokes' transformation
	Cesaro sum for Fourier series

能量

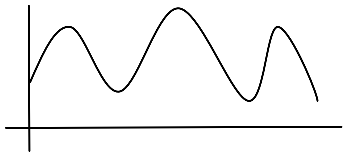
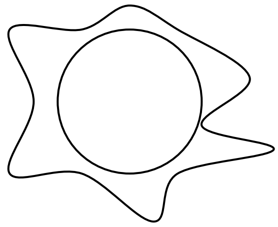
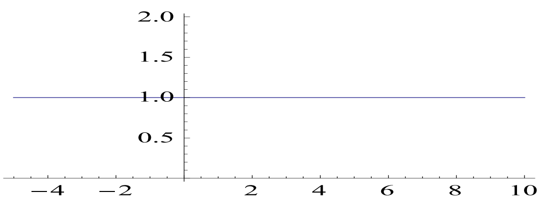
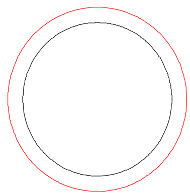
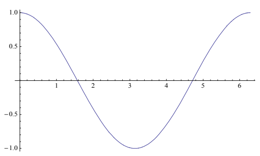
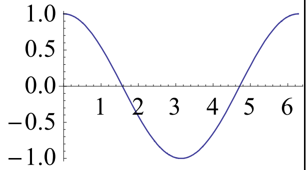
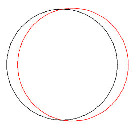
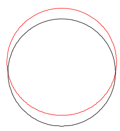
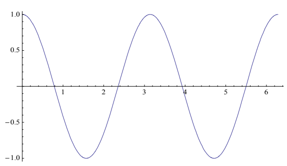
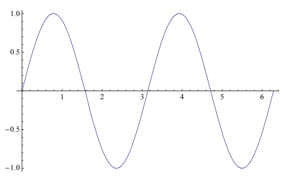
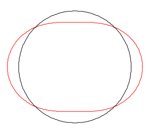
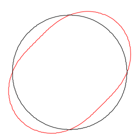
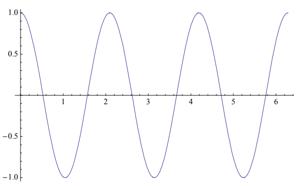
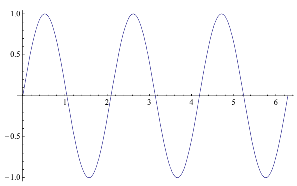
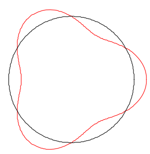
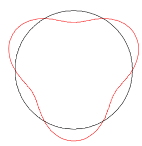
Energy conservation
Parseval's 定理與 Fourier coefficient 的新觀點

粒徑篩分析與Fourier series分析





ntou/msv 劉家如製表 2012/1/20

<table>
<tr><td colspan="5">傅立葉級數在圓邊表示法</td></tr>
<tr><td></td><td colspan="2">$t(\theta)=p_0+\sum_{n=1}^{\infty} p_n\cos\theta+q_n\sin\theta$</td><td colspan="2">$t(\theta)=p_0+\sum_{n=1}^{\infty} p_n\cos\theta+q_n\sin\theta$</td></tr>
<tr><td>f(θ)</td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>f(θ)=1</td><td colspan="2">p_0
</td><td colspan="2">p_0
</td></tr>
<tr><td></td><td>cos(nθ)</td><td>sin(nθ)</td><td>cos(nθ)</td><td>sin(nθ)</td></tr>
<tr><td>n=1</td><td>p_1
</td><td>q_1
</td><td>p_1
</td><td>q_1
</td></tr>
<tr><td>n=2</td><td>p_2
</td><td>q_2
</td><td>p_2
</td><td>q_2
</td></tr>
<tr><td>n=3</td><td>p_3
</td><td>q_3
</td><td>p_3
</td><td>q_3
</td></tr>
</table>

傅立葉級數在圓邊表示法　2012/2/13　許哲崙 製

Fourier series

海大河海系　　陳正宗

periodic function

basis

Euler formula

alternative formula

half expansion formula

Even function

Odd function

half-range cosine expansion

half-range sine expansion

another represenattion

applications

frequency content

wave number content

wave

signal

termwise differentiation

Gibbs phenomenon

Stokes' transformation

Approximation

Distance

Norm

Least square

Dirichelet condition

Jump

complex exponetial Fourier series

loading simulation by harmonic function

感謝國立屏東科技大學土木系徐文信老師提供

0. 預備知識

1. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

2. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

3. $\sin mx\cos nx=\frac{1}{2}\left[\sin(m+n)x+\sin(m-n)x\right]$

4. $\sin mx\sin nx=\frac{-1}{2}\left[\cos(m+n)x-\cos(m-n)x\right]$

5. $\cos mx\cos nx=\frac{1}{2}\left[\cos(m+n)x+\cos(m-n)x\right]$

6. $ce^{i\theta} = \ os\theta+i\sin\theta$ (尤拉公式)

7. $\int x\cos bxdx=\frac{x}{b}\sin bx+\frac{1}{b^2}\cos bx+C$

8. $\int x\sin bxdx=-\frac{x}{b}\cos bx+\frac{1}{b^2}\sin bx+C$

9. $\int x^2\cos bxdx=\frac{x^2}{b}\sin bx+\frac{2x}{b^2}\cos bx-\frac{2}{b^3}\sin bx+C$

10. $\int x^2\sin bxdx=-\frac{x^2}{b}\cos bx+\frac{2x}{b^2}\sin bx+\frac{2}{b^3}\cos bx+C$

11. $\int e^{ax}\cos bxdx=\frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]+C$

12. $\int e^{ax}\sin bxdx=\frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]+C$

13. $\int_{-\pi}^{\pi}\sin mx\cos nxdx=0 \quad (m\neq\pm n)$

14. $\int_{-\pi}^{\pi}\sin mx\sin nxdx=0 \quad (m\neq\pm n)$

15. $\int_{-\pi}^{\pi}\cos mx\cos nxdx=0 \quad (m\neq\pm n)$

16. $\int_{-\pi}^{\pi}\sin mx\cos mxdx=0$

17. $\int_{-\pi}^{\pi} \sin mx \sin mx dx = \pi$

18. $\int_{-\pi}^{\pi} \cos mx \cos mx dx = \pi$

19. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad x \in R$

20. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x^1}{1!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad x \in R$

21. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad x \in R$

22. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + \cdots + x^n + \cdots \qquad -1 < x < 1$

23. $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \frac{x}{1} - \frac{x^2}{2} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots \qquad -1 < x \le 1$

什麼是級數?

大學時期學習到的級數有泰勒級數(Taylor series)、傅立葉級數(Fourier series)以及洛倫級數(Laurent series)，但像是泰勒級數和洛倫級數這種冪級數型式的展開式是體現不出週期性的。

1. 週期函數

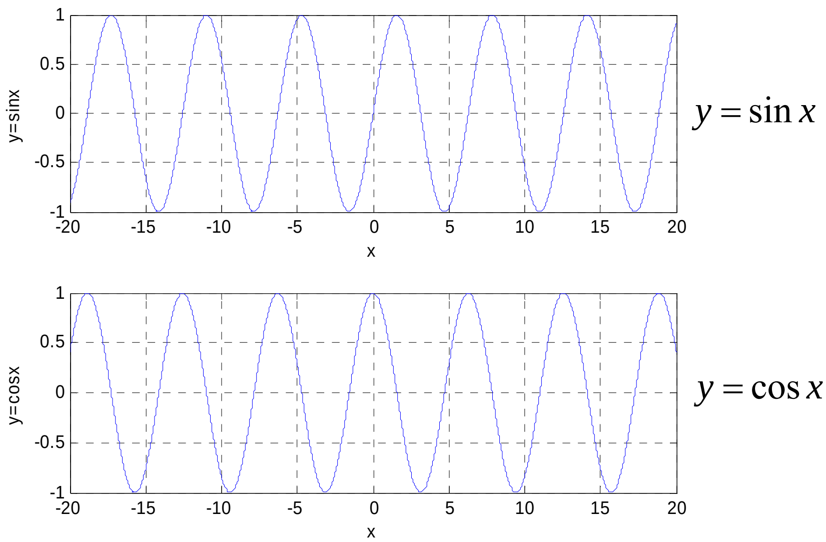
$$f(x+T) = f(x) \quad (T \text{ 為常數}) \ (1)$$

週期的定義:

(i) 滿足(1)式的T值中的最小正數，即為該函數的週期。

(ii) 一個常數以任何正數為週期

例: $\sin\ x$ 和 $\cos x$ 為週期 2π 之函數。$(0\sim 2\pi$ 或 $-\pi\sim\pi)$

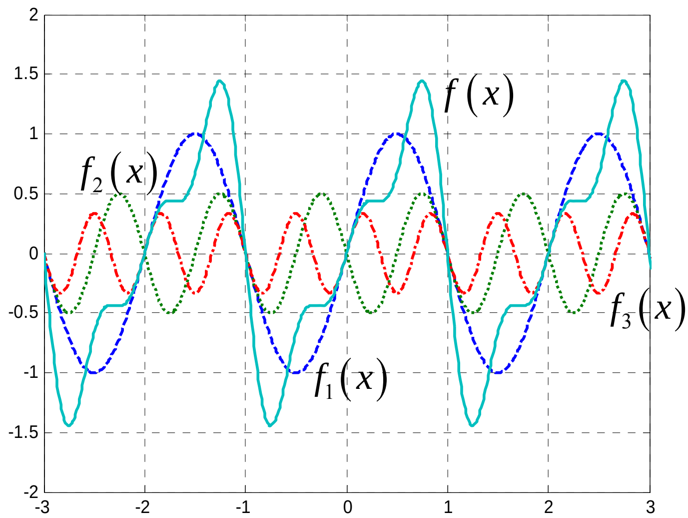


2. 基本三角函數系

$$1\,,\cos\frac{\pi x}{\ell}\,,\sin\frac{\pi x}{\ell}\,,\cos\frac{2\pi x}{\ell}\,,\sin\frac{2\pi x}{\ell}\,,\cdots,\cos\frac{n\pi x}{\ell}\,,\sin\frac{n\pi x}{\ell}\,,\cdots \qquad (2)$$

例如 $\cos\frac{n\pi x}{\ell}$ 和 $\sin\frac{n\pi x}{\ell}$ 的週期為 $\frac{2\ell}{n}$，但它們的共同週期為 2ℓ，即所有週期的最小公倍數，通常這個週期命名為函數系的週期。

例: $f(x)=f_1(x)+f_2(x)+f_3(x)=\sin\frac{\pi}{1}x-\frac{1}{2}\sin\frac{2\pi}{1}x+\frac{1}{3}\sin\frac{3\pi}{1}x$



3. 傳立葉級數(Fourier series)

逆問題:

如果給定一個週期為2ℓ的任意週期函數$f(x)$，

$$f(x+2\ell)=f(x) \tag{3}$$

我們能否將它表示成簡單的三角函數(有限個或無限個)之合呢? 即:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{\ell}x+b_n\sin\frac{n\pi}{\ell}x\right) \tag{4}$$

如果能實現這種分解，那麼對許多複雜的函數就可以通過簡單的三角函數來研究其性質。

上述問題的答案是肯定的，且稱(4)式為函數$f(x)$的**傳立葉級數**(狹義傳立葉級數)。

若函數$f(x)$按非三角函數系$\{\phi_n(x)\}(n=1,2,3...)$進行展開所得的級數稱為廣義傳立葉展開。

但首先需要解決兩個問題:

(i) 在什麼條件下$f(x)$才能按基本三角函數系展開?

(ii) 如何確定展開式中的係數? 即$a_0,a_1,...b_0,b_1...$。

4. 完備正交函數系

討論一週期型 Sturm-Liouville 邊界值問題

$$y'' + \lambda y = 0 \text{ ， } x \in [d, d+2\ell] \text{ ， } y(d) = y(d+2\ell) \text{ ， } y'(d) = y'(d+2\ell)$$

經由完備正交系的討論可得:

在區間 $[-\ell, \ell]$ 的函數系 $\left\{1, \cos\frac{n\pi}{\ell}, \sin\frac{n\pi}{\ell}\right\}_{n=1}^{\infty}$ 為一組正交函數序列，於是，$f(x)$ 按基本三角函數系的展開式為第(4)式，其中係數

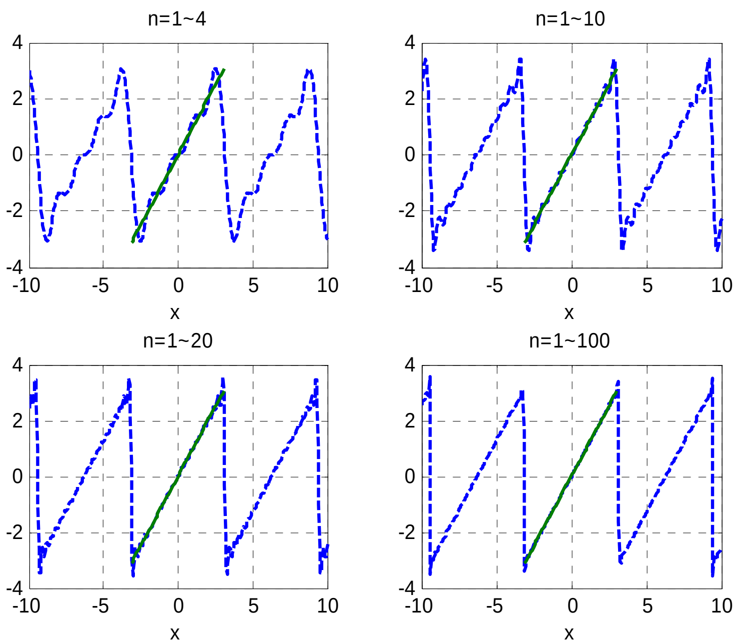
$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi}{\ell}x\,dx \quad n = 0,1,2,3,\cdots \tag{5}$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi}{\ell}x\,dx \quad n = 1,2,3,\cdots \tag{6}$$

Example 13.1

$f(x) = x$ ， $-\pi \le x \le \pi$ ，試展成傅立葉級數。

Ans: $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx \quad x \text{ on } [-\pi, \pi]$

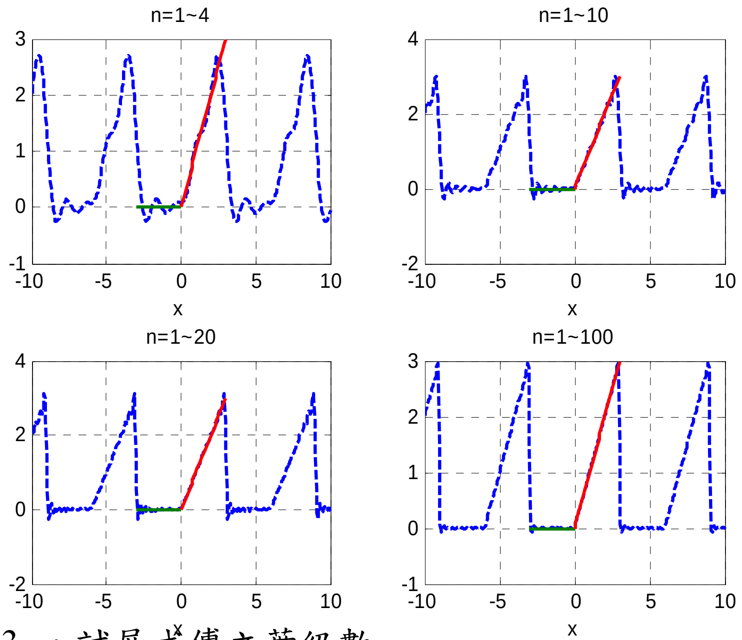


Example 13.2

$$f(x)=\begin{cases}0 & -3\le x\le 0\\ x & 0\le x\le 3\end{cases}$$，試展成傅立葉級數。

Ans:

$$f(x)=\frac{3}{4}+\sum_{n=1}^{\infty}\left(\frac{3}{n^2\pi^2}\left[(-1)^n-1\right]\cos\frac{n\pi x}{3}+\frac{3}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{3}\right)\quad x \text{ on } [-3,3]$$

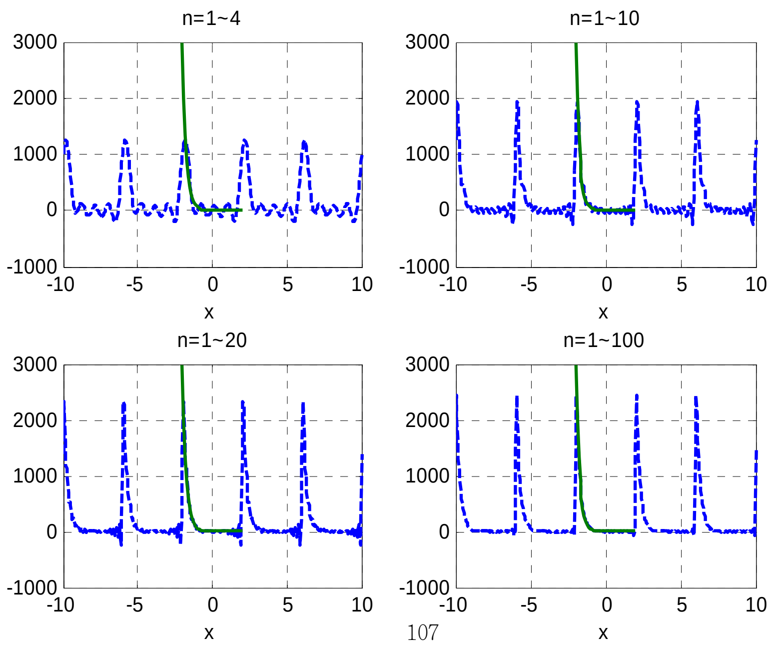


Example 13.3 ，試展成傅立葉級數。

$$f(x)=e^{-4x}\quad , \quad -2\le x\le 2$$

Ans:

$$f(x)=\frac{1}{16}\left(e^{8}-e^{-8}\right)+\left(e^{8}-e^{-8}\right)\sum_{n=1}^{\infty}\left(\frac{8(-1)^n}{64+n^2\pi^2}\cos\frac{n\pi x}{2}+\frac{n\pi(-1)^n}{64+n^2\pi^2}\sin\frac{n\pi x}{2}\right)\quad x \text{ on } [-2,2]$$



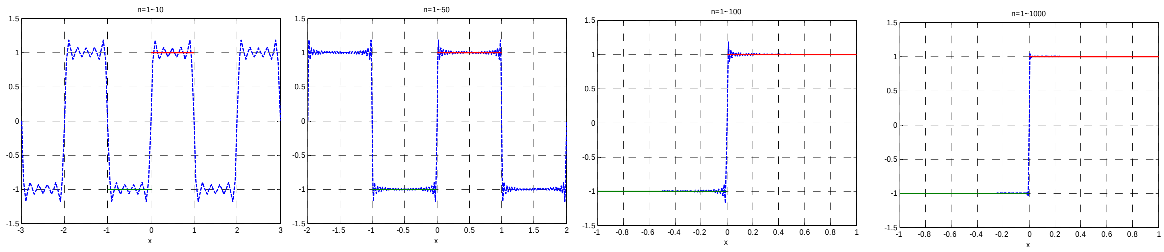
5. 吉布斯現象(The Gibbs Phenomenon)

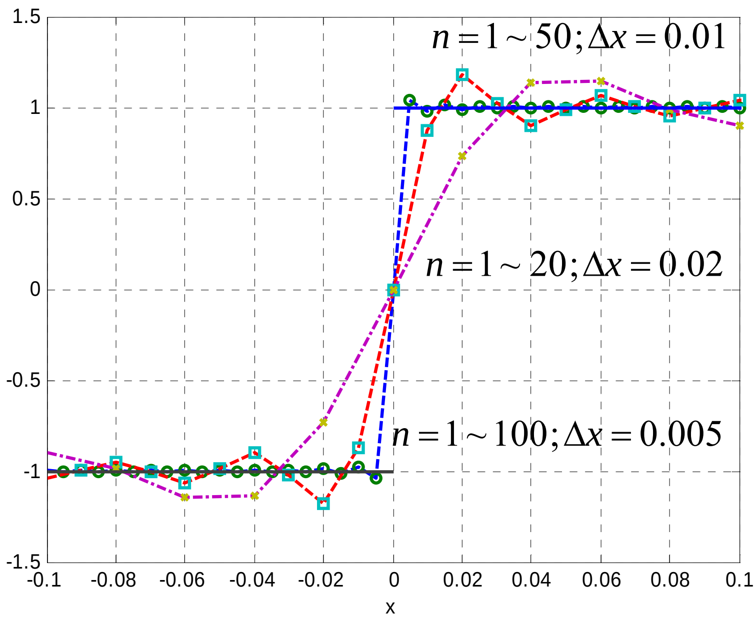
考慮方波 $f(x)=\begin{cases}1 & 0\le x\le 1\\ -1 & -1\le x\le 0\end{cases}$，

可得 $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x \quad x \text{ on } [-1,1]$，即使 n 趨近無窮大，

誤差項在不連續處總是存在，且超出幅度約 18%，稱為吉布斯現象。

後來就部分和在間斷點附近的異常行為稱為吉布斯現象。





6. 傅立葉級數的性質

收斂性: 狄利克雷定理(Dirichlet's theorem)

(i) 若 $f(x)$ 在 $[-\ell,\ell]$ 上連續或者只有有限個間斷點，且在間斷處函數的左右極限都存在。

(ii) $f(x)$在$[-\ell,\ell]$上只有有限個極大值點與及小值點。

((i)(ii)表 $f(x)$為片段連續，可進行積分。)

(iii) 積分$\int_{-\ell}^{\ell}|f(x)|dx$為有限值。(保證$a_n$與$b_n$存在。)

(iv) $f(x)$在$[-\ell,\ell]$外是週期函數，其週期為2ℓ，則級數

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{\ell}x+b_n\sin\frac{n\pi}{\ell}x\right)=\begin{cases} f(x) & \text{,在連續點處} \\ \frac{1}{2}\left[f(x^+)+f(x^-)\right] & \text{,在間斷點處}\end{cases}$$

Example 13.6

$$f(x)=\begin{cases} 5 & x=\pi \\ x & -\pi<x<1 \\ 1-x^2 & 1\le x<2 \\ 4 & 2\le x\le\pi \end{cases}$$

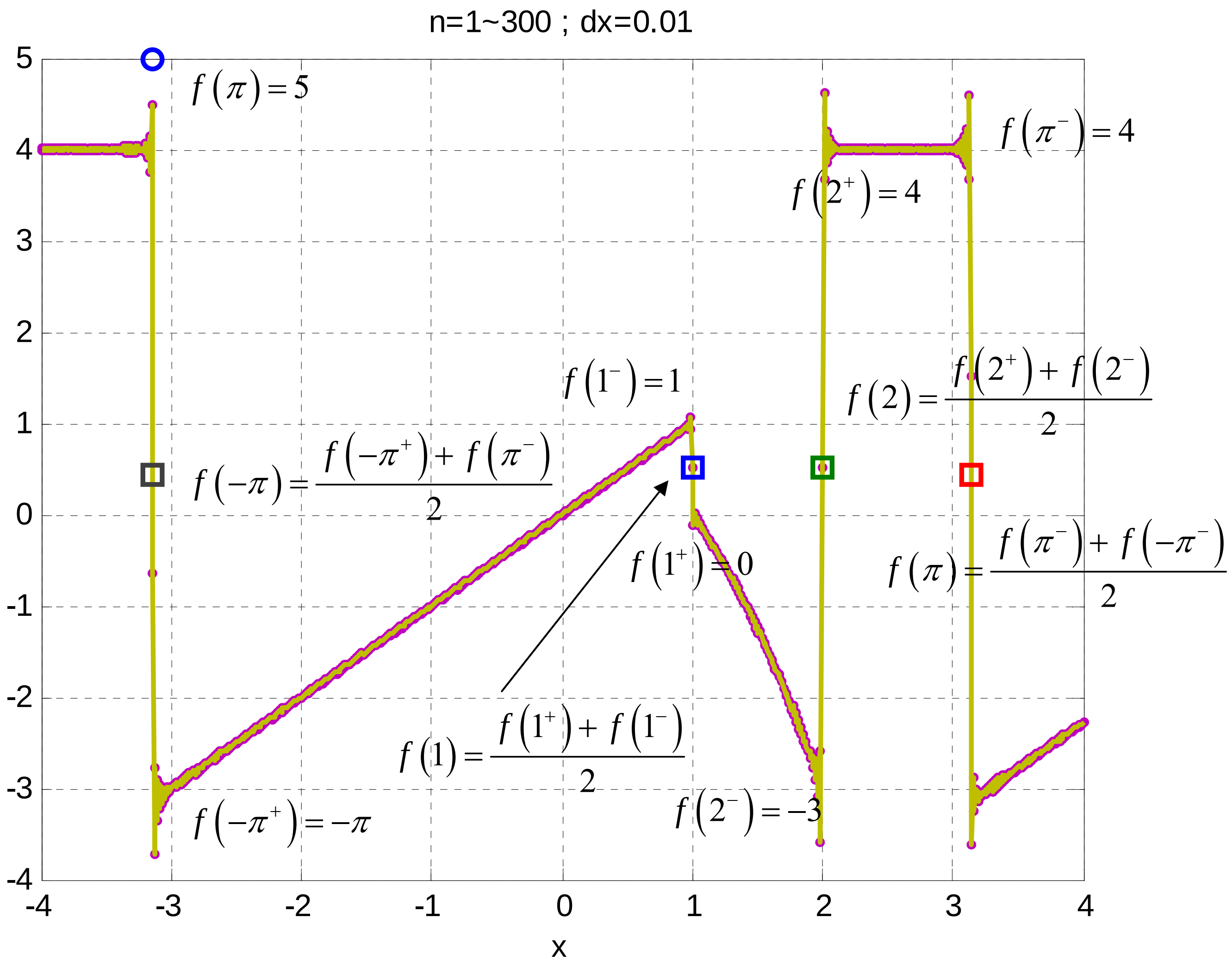
展成傅立葉級數:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

$$a_0=\frac{1}{\pi}\left(4\pi-\frac{\pi^2}{2}-\frac{53}{6}\right),$$

$$a_n=\frac{1}{\pi}\left[\frac{1}{n^2}(-1)^{n+1}+\left(\frac{1}{n}-\frac{2}{n^3}\right)\sin n+\frac{3}{n^2}\cos n+\left(\frac{2}{n^3}-\frac{7}{n}\right)\sin 2n-\frac{4}{n^2}\cos 2n\right]$$

$$b_n=\frac{1}{\pi}\left[\left(\frac{\pi}{n}+\frac{4}{n}\right)(-1)^{n+1}+\frac{3}{n^2}\sin n+\left(\frac{2}{n^2}-\frac{1}{n}\right)\cos n-\frac{4}{n^2}\sin 2n+\left(\frac{7}{n}-\frac{2}{n^3}\right)\cos 2n\right]$$



Example 13.7

Example 13.8

Example 13.11

7. 奇函數(odd function)與偶函數(even function)

(i) 奇函數

特徵:$G(x)=-G(-x)$ (7)

故可知在區間$[-\ell,\ell]$上的奇函數可寫為

$$G(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

其中 $a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}G(x)\cos\frac{n\pi x}{\ell}dx=0 \quad (n=0,1,2,3\cdots)$

$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}G(x)\sin\frac{n\pi x}{\ell}dx=\frac{2}{\ell}\int_{0}^{\ell}G(x)\sin\frac{n\pi x}{\ell}dx \quad (n=1,2,3\cdots)$

且$G(0)=G(\ell)=0$

奇函數的傅立葉級數中只含正弦項，且$x=0$及$x=\ell$兩端級數收斂為 0。

(ii) 偶函數

特徵:$F(x)=F(-x)$ (8)

故可知在區間$[-\ell,\ell]$上的偶函數可寫為

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}$$

其中

$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)\cos\frac{n\pi x}{\ell}dx=\frac{2}{\ell}\int_{0}^{\ell}F(x)\cos\frac{n\pi x}{\ell}dx \quad (n=0,1,2,3\cdots)$

$b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}F(x)\sin\frac{n\pi x}{\ell}dx=0 \quad (n=1,2,3\cdots)$

且$F'(0)=F'(\ell)=0$

偶函數的傅立葉級數中只含餘弦項，且 $x=0$ 及 $x=\ell$ 兩端級數的導數收斂為 0。

(iii) 四則運算

$E \pm E \Rightarrow E$ ， ，$O \pm O \Rightarrow O$ ， $E \pm O \Rightarrow \times$

$E \div E \Rightarrow E$ ， ，$O \div O \Rightarrow E$ ， $E \div O \Rightarrow O$

$E \times E \Rightarrow E$ ， ，$O \times O \Rightarrow E$ ， $O \times E \Rightarrow O$

Example 13.4
Example 13.5

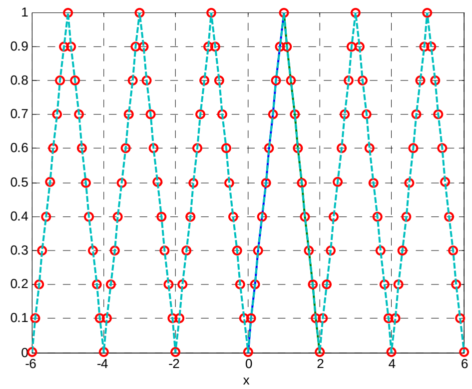
8. 有限區間上的函數的傅立葉級數

現在函數 $f(x)$ 定義在 $[0,\ell]$ 區間上，而在端點和端點之外沒有定義，

考慮 $f(x)=\begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$

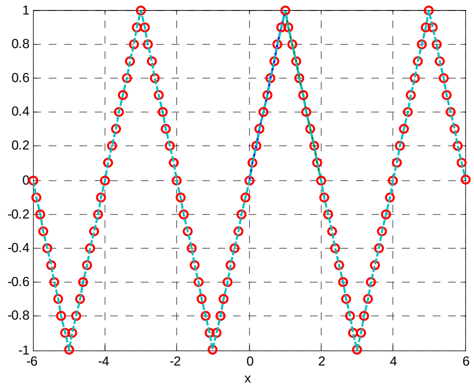
(i) 傅立葉餘弦展開(Fourier cosine series)

$$f(x)=\frac{1}{2}-\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{\left[2(2n-1)\right]^2}\cos(2n-1)\pi x$$



(ii) 傅立葉正弦展開(Fourier sine series)

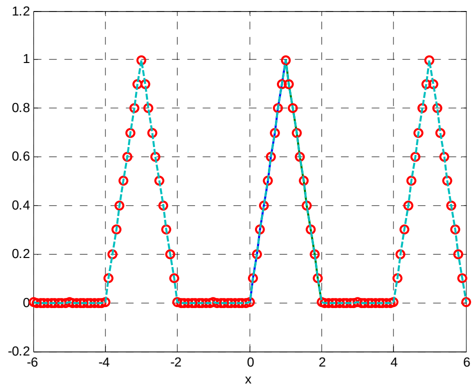
$$f(x)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}$$



(iii) 其它方式

$$f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}\frac{-8}{\left[2\pi(2n-1)\right]^2}\cos(2n-1)\pi x+\frac{4}{\left[\pi(2n-1)\right]^2}\sin\frac{(2n-1)\pi x}{2}$$

可見給 $f(x)$ 在 $[0,\ell]$ 區間之外賦予不同的定義，可以得到不同的傅立葉級數，也就是說，這種問題不能得到唯一解。

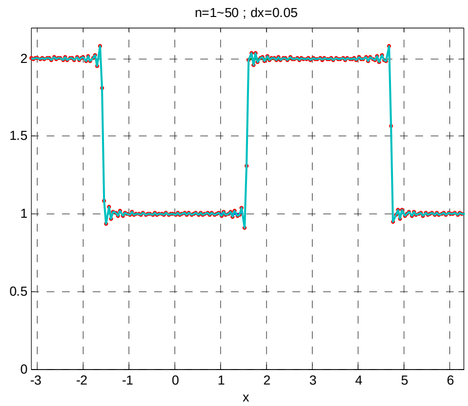


Example 13.20 ，試展成傅立葉級數。

$$f(x)=\begin{cases}1 & 0\le x\le \pi/2\\ 2 & \pi/2\le x\le \pi\end{cases}$$

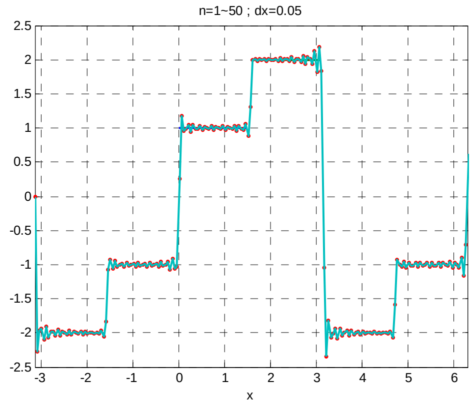
Fourier cosine series

$$f(x)=\frac{3}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos nx$$



Fourier sine series

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\cos\left(\frac{n\pi}{2}\right)+1-2(-1)^n\right)\sin nx$$



9. 傳立葉級數的微分與積分

(i) 微分

考慮三角波 $f(x)=|x| \quad -\ell<x<\ell$

其傳立葉級數為 $f(x)=\frac{\ell}{2}-\frac{4\ell}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{\ell}$

收斂相當快速且在區間$[-\ell,\ell]$上一致收斂，對上式逐項微分

得 $f'(x)=h(x)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin\frac{(2n+1)\pi x}{\ell}$

正是方波$h(x)=\begin{cases}1 & 0<x<\ell \\ -1 & -\ell<x<0\end{cases}$的傳立葉級數，事實上，三角波的導數正是方波，但是微分之後每一個係數前面增添了一個增長因子，因此降低了收斂程度。也有可能微分之後成為一個發散級數，一般而言，微分使級數的收斂程度降低。

(ii) 積分

函數 $f(x)$在$[0,\ell]$區間上分段連續，則

$$\int_{-\ell}^{x}\left[f(t)-\frac{a_0}{2}\right]dt=\frac{-1}{2\ell}\int_{-\ell}^{\ell}\xi f(\xi)d\xi+\sum_{n=1}^{\infty}\frac{\ell}{n\pi}\left(a_n\sin\frac{n\pi x}{\ell}-b_n\cos\frac{n\pi x}{\ell}\right)$$

(9)

(iii) Bessel inequality:

$$\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)^2dx\geq\frac{1}{2}a_0^2+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right) \qquad (10)$$

(iv) Parserval's theorem

$$\frac{1}{\ell}\int_{-\ell}^{\ell} f\left(x\right)^{2} dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{11}$$

10.複指數型式的傅立葉級數

利用尤拉公式可得

$$f\left(x\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{\ell}x} \tag{12}$$

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f\left(x\right) e^{-i\frac{n\pi}{\ell}x} dx \tag{13}$$

11.傅立葉級數的應用

(i) 頻譜分析: 不僅可應用於電路,還可應用於無線電波、光波以及聲波等。

(ii) 計算無窮級數:

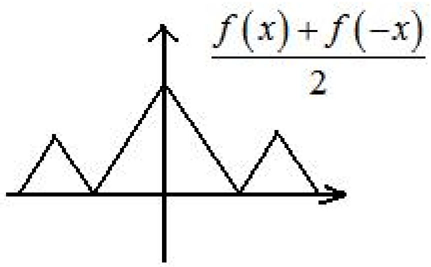
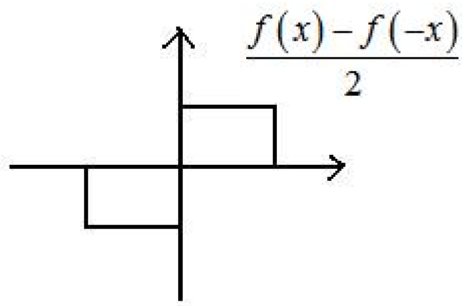
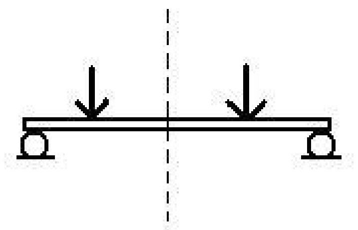
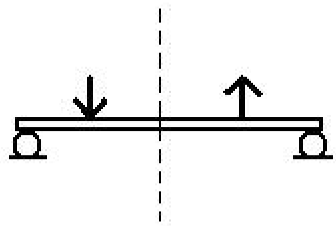
例:由 $h\left(x\right) = \begin{cases} 1 & \left|x\right| < \pi/2 \\ 0 & \pi/2 < \left|x\right| < \pi \end{cases}$,可得 $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$

(iii) 能表示不連續函數: 如矩形波。

(iv) 能表示週期函數: 但注意可能不是唯一展開型式。

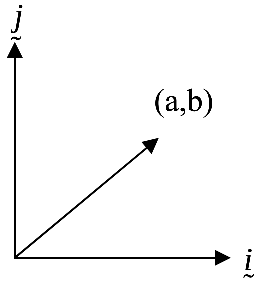
(v) 能對任意函數做調和分析: 如將週期性強迫力的振動系統分析,則很容易求得各分量(三角函數)的反應,再疊加即可。

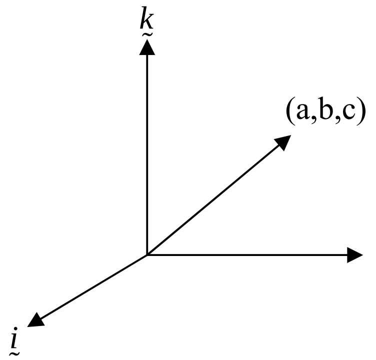
奇偶對稱與反對稱

對稱	反對稱
偶函數	奇函數
平均數$(\frac{a+b}{2})$ 大數$a=\frac{a+b}{2}+\frac{a-b}{2}$	均差值$(\frac{a-b}{2})$ 小數$b=\frac{a+b}{2}-\frac{a-b}{2}$
 $f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}$ $f(x)=f(-x):even$	 $f(x)=\frac{f(x)+f(-x)}{2}-\frac{f(x)-f(-x)}{2}$ $-f(x)=f(-x):odd$
	
$\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ $(\frac{A+A^T}{2})$ $A=\frac{A+A^T}{2}+\frac{A-A^T}{2}$ $A=A^T:symmetric$	$\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$ $(\frac{A-A^T}{2})$ $A=\frac{A+A^T}{2}-\frac{A-A^T}{2}$ $A=-A^T:anti-symmetric$
$\cos kx$	$\sin kx$

奇偶對稱與反對稱.doc 賴芝亭製 2008/04/21

向量空間到函數空間



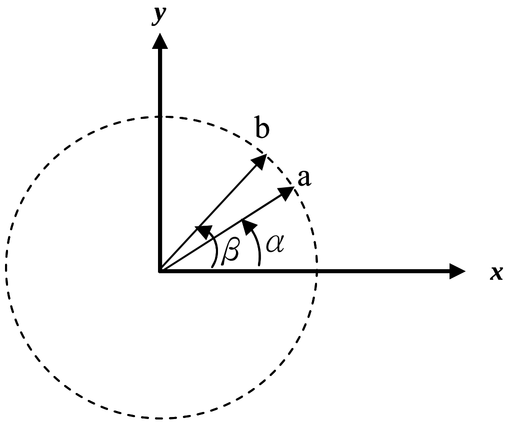


$3\cos t + 4\sin t = 5\cos(t-\phi)$

其中 $\phi = \tan^{-1}(\frac{4}{3})$

	向量空間	函數空間
基底	$\underset{\sim}{i}, \underset{\sim}{j}$ (2D) $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ (3D)	$\cos nt$; $\sin nt$ $\phi_n(t)$
與表達物理量	任意向量	週期 2π 的函數
例子	$3\underset{\sim}{i} + 4\underset{\sim}{j}$ $3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$	$5\cos(t-\phi) = 3\cos t + 4\sin t$
正交性	$\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$; $\underset{\sim}{j} \cdot \underset{\sim}{k} = 0$; $\underset{\sim}{k} \cdot \underset{\sim}{i} = 0$;	$\int_0^{2\pi} \sin mx \sin nx \, dx = 0 \;, m \neq n$ $\int_0^{2\pi} \cos mx \sin nx \, dx = 0 \;, m \neq n$ $\int_0^{2\pi} \cos mx \cos nx \, dx = 0 \;, m \neq n$
投影	$\underset{\sim}{a} = (\underset{\sim}{a} \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{b} \cdot \underset{\sim}{j})\underset{\sim}{j} + (\underset{\sim}{c} \cdot \underset{\sim}{k})\underset{\sim}{k}$	$f(t) = \sum_{n=0}^{\infty} \frac{\langle f(t), \phi_n(t) \rangle}{\langle \phi_n(t), \phi_n(t) \rangle} \phi_n(t)$ $where \; \langle f(t), \phi_n(t) \rangle = \int_0^{2\pi} f(t)\, \phi_n(t) \, dt$
特徵物理量	$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$	$y''(x) = \lambda y(x)$

1. 內積



$\underset{\sim}{a}\cdot\underset{\sim}{b}=|a|\cdot|b|\cos(\beta-\alpha)=|a|\cdot|b|\cos(\alpha-\beta)$

$\Rightarrow\underset{\sim}{a}\cdot\underset{\sim}{b}=(x_1,y_1)\cdot(x_2,y_2)$

$=x_1x_2+y_1y_2$

令 $\underset{\sim}{a}=(\cos\alpha,\sin\alpha)$, $\underset{\sim}{b}=(\cos\beta,\sin\beta)$

$\Rightarrow(\cos\alpha,\sin\alpha)\cdot(\cos\beta,\sin\beta)$

$=1\cdot1\cos(\beta-\alpha)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.......(1)

$\Rightarrow(\cos\beta,\sin\beta)\cdot(\cos\alpha,\sin\alpha)$

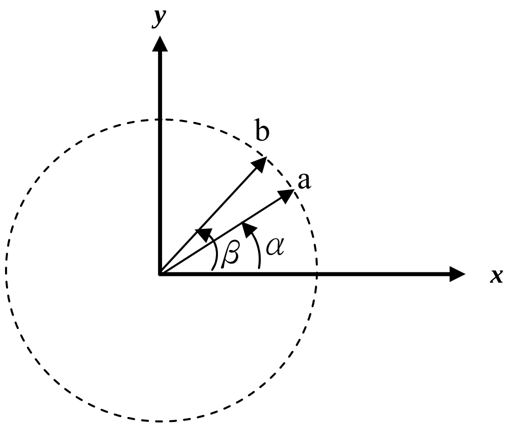
$=1\cdot1\cos(\alpha-\beta)=\sin\alpha\sin\beta+\cos\alpha\cos\beta$.......(2)

由**(1)(2)**故可得

$\cos(\beta-\alpha)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\sin\alpha\sin\beta+\cos\alpha\cos\beta$

2. 外積



$\underset{\sim}{a}\times\underset{\sim}{b}=|a||b|\sin(\beta-\alpha)=-|a||b|\sin(\alpha-\beta)$

$$\underset{\sim}{a}\times\underset{\sim}{b}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ x_1 & y_1 & 0\\ x_2 & y_2 & 0\end{vmatrix}$$

$=(x_1y_2-x_2y_1)\underset{\sim}{k}$

令 $\underset{\sim}{a}=(\cos\alpha,\sin\alpha)$, $\underset{\sim}{b}=(\cos\beta,\sin\beta)$

$(\cos\alpha,\sin\alpha)\times(\cos\beta,\sin\beta)$

$$=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ \cos\alpha & \sin\alpha & 0\\ \cos\beta & \sin\beta & 0\end{vmatrix}$$

$=(\cos\alpha\sin\beta-\sin\alpha\cos\beta)\underset{\sim}{k}$

$=1\cdot1\sin(\beta-\alpha)=-1\cdot1\sin(\alpha-\beta)$

由上式可得

$\sin(\beta-\alpha)=(\cos\alpha\sin\beta-\sin\alpha\cos\beta)$

$\sin(\alpha-\beta)=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

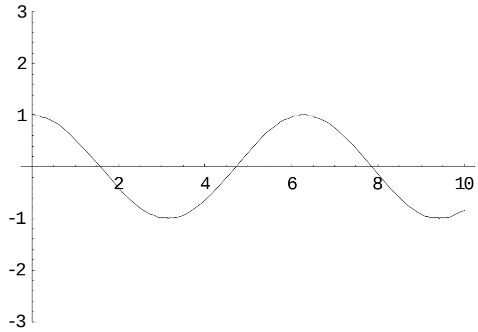
其中 $\underset{\sim}{a}=(x_1,y_1)$

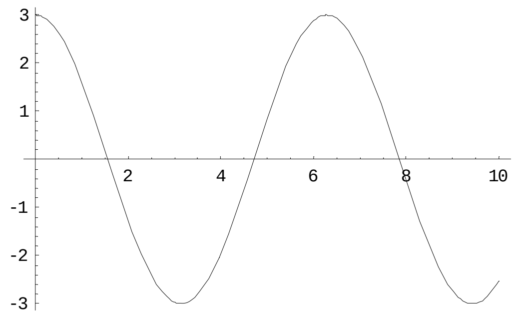
$\underset{\sim}{b}=(x_2,y_2)$

1. 振幅

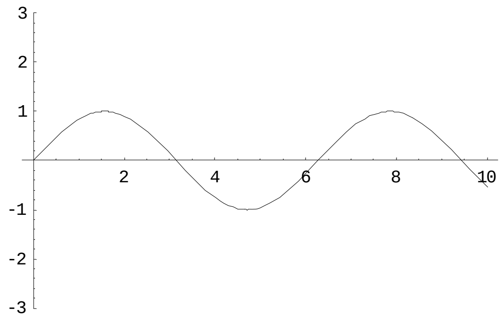
3cos(***t***)+4sin(***t***)=5cos(***t***- φ)

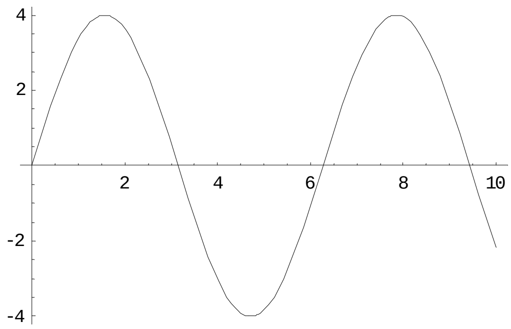
① cos(***t***) 3cos(***t***)





② sin(***t***) 4sin(***t***)

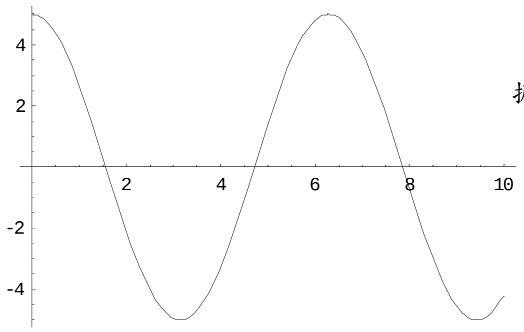


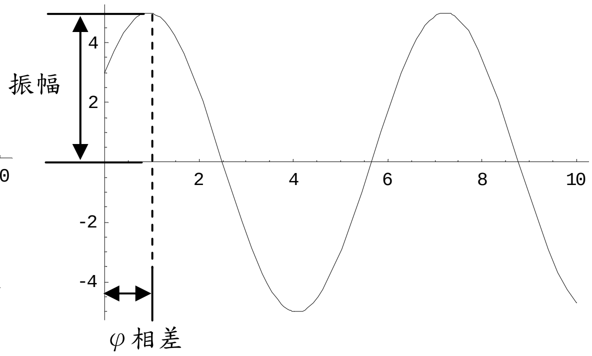


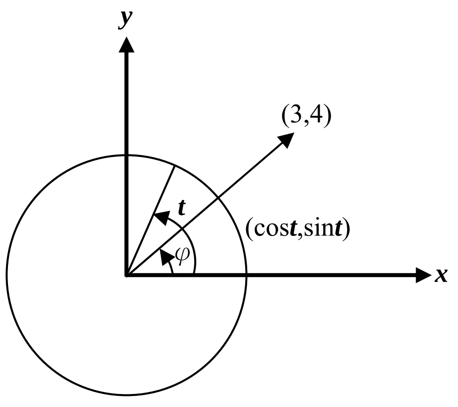
2. 相差

3cos(***t***)+4sin(***t***)=5cos(***t***- φ), φ =0.927295

① 5cos(***t***) 5cos(***t***- φ)







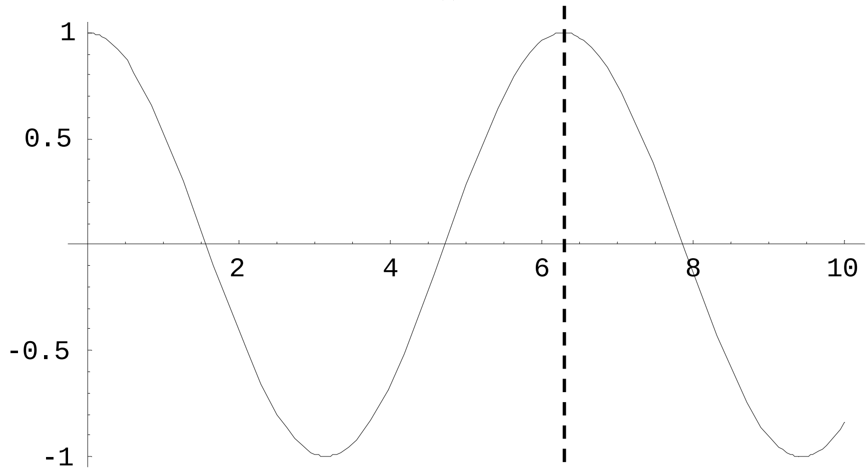
3cos(***t***)+4sin(***t***)=(3,4)·(cos***t***,sin***t***)

=5cos(***t***- φ)

3. 週期

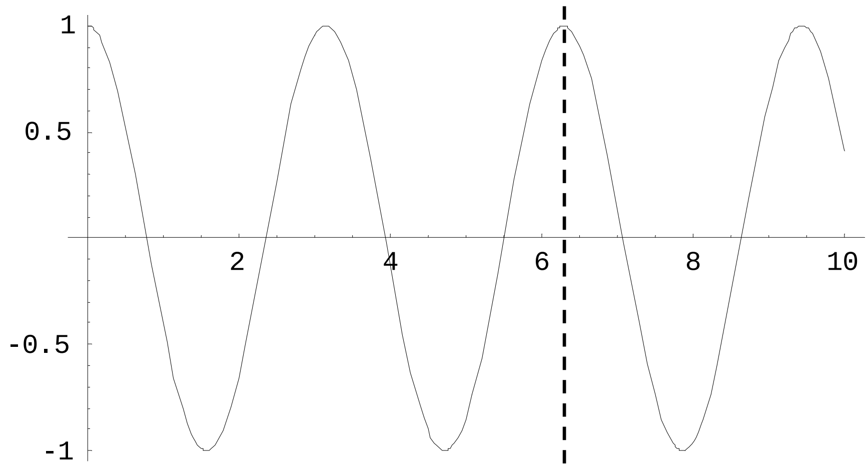
① 2π

cos(*t*)



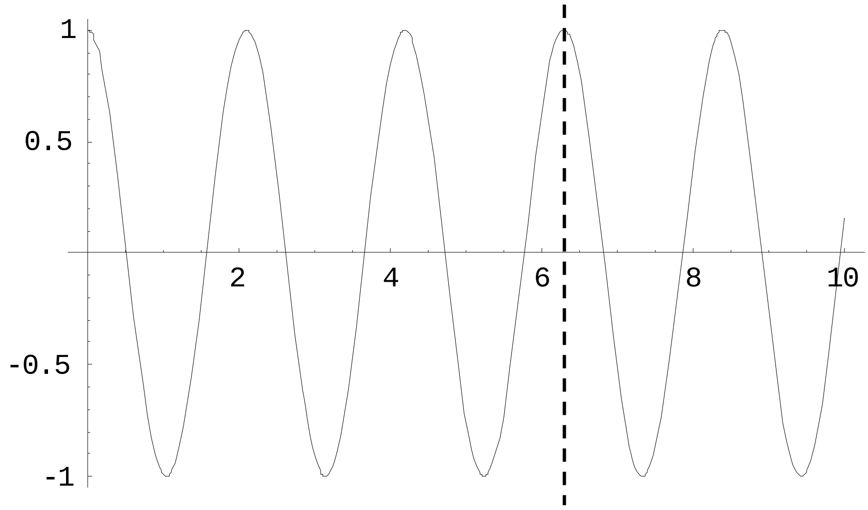
② π

cos(2*t*)



③ 2π/3

cos(3*t*)



2π

補救教學講義

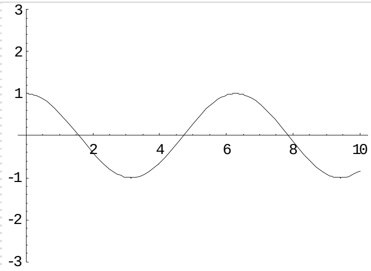
代課助教:謝祥志

目錄

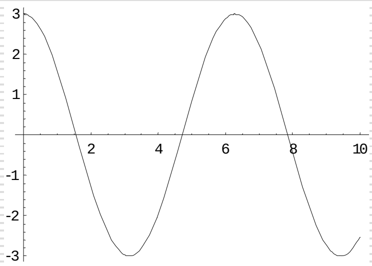
- 振幅
- 相差
- 週期
- end

振幅

cos(***t***)

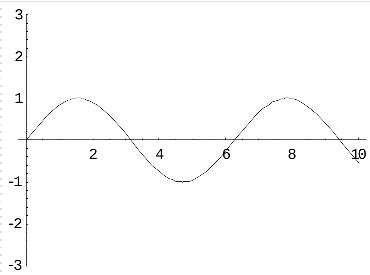


3cos(***t***)

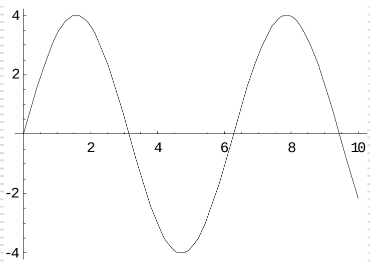


振幅

sin(***t***)

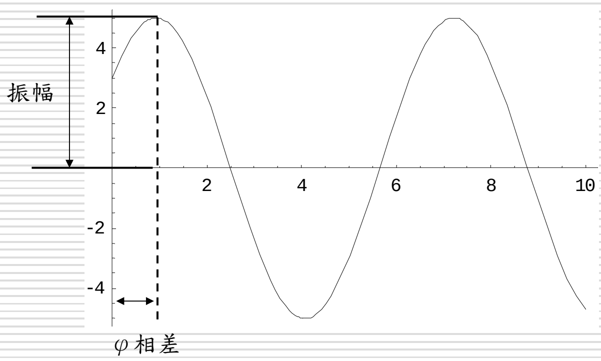


4sin(***t***)



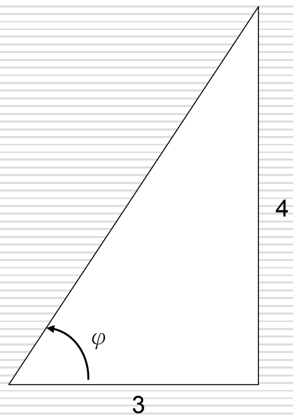
相差(φ)　以 3cos(***t***)+4sin(***t***)=5cos(***t***-φ) 為例

3cos(***t***)+4sin(***t***) 的圖形



工數(二) 補救教學　April.14.2008　謝祥志 製

相差(φ)　以 3cos(***t***)+4sin(***t***)=5cos(***t***-φ) 為例

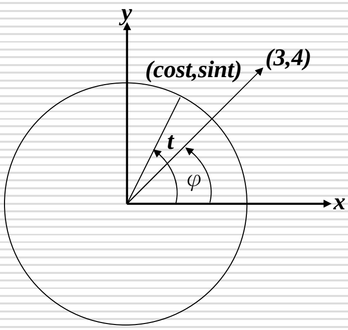


$$\tan\phi = \frac{4}{3}$$

$$\therefore \phi = \tan^{-1}\frac{4}{3}$$

工數(二) 補救教學　April.14.2008　謝祥志 製

物理與數學的結合



數學推導 $\mathbf{3}\cos t + 4\sin t = 5\cos(t-\phi)$

$$\frac{\mathbf{3}}{\mathbf{5}}\cos t + \frac{4}{5}\sin t = \cos\phi\cos t + \sin\phi\sin t$$

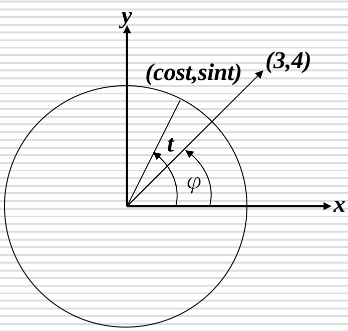
利用和角公式

$$\cos\phi\cos t + \sin\phi\sin t = \cos(t-\phi)$$

$$\Rightarrow \frac{\mathbf{3}}{\mathbf{5}}\cos t + \frac{4}{5}\sin t = \cos(t-\phi)$$

$$\therefore \mathbf{3}\cos t + 4\sin t = 5\cos(t-\phi)$$

物理與數學的結合

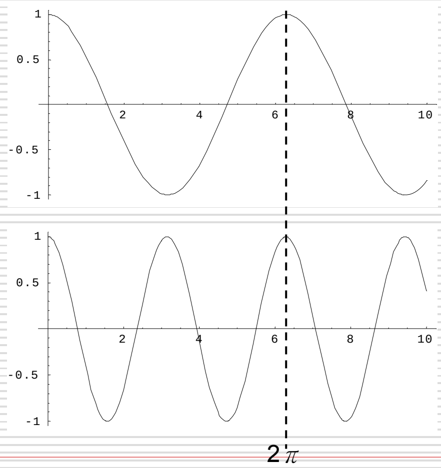


$$\begin{aligned} 3\cos t + 4\sin t &= (3,4)\cdot(\cos t, \sin t) \\ &= 5\cdot 1\cos(t-\phi) \end{aligned}$$

週期

1.週期=2 π

2.週期= π



end

講得不好請多多包容~

下台一鞠躬!!

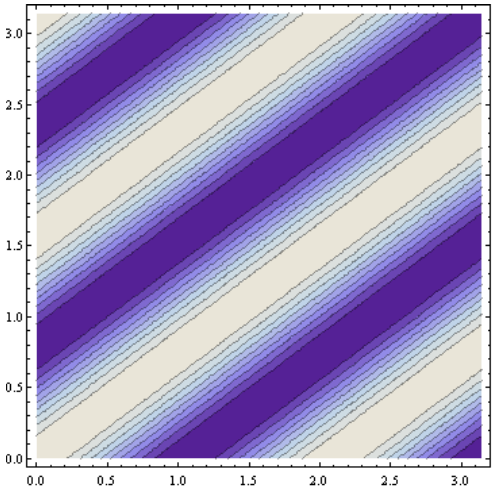
時空三角函數

1.

$\cos(kx - wt)$

$\sin(kx - wt)$

$e^{i(kx-wt)}$



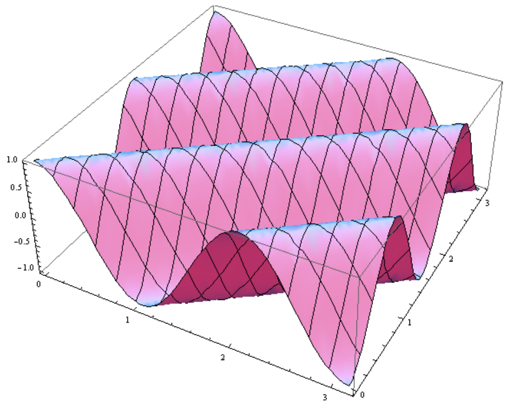
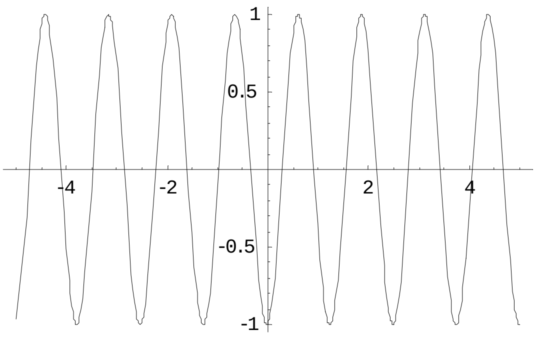


Fig1. $\cos(3x-4t)$ 之 Contour 圖 Fig2. $\cos(3x-4t)$ 之時空圖

2. 定點看時變

$\cos(wt-\phi)$

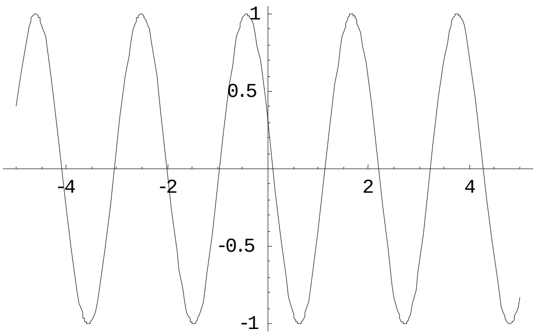
$\sin(wt-\phi)$



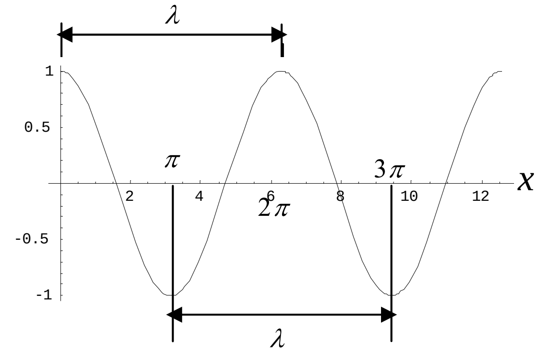
3. 定時看空間變異性

$\cos(kx-\psi)$

$\sin(kx-\psi)$

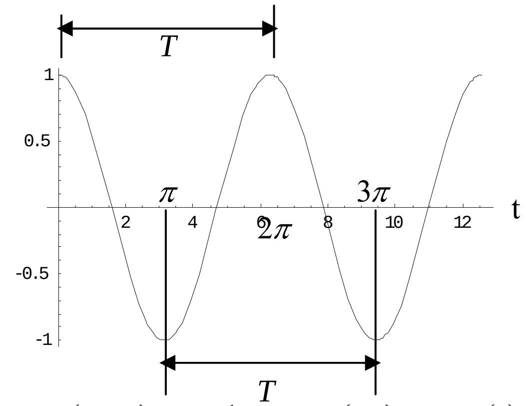


$\cos(kx-\omega t)$



在$t=0$
看空間上的變化
(固定時間)

$\cos(kx), k=1 \Rightarrow \cos(x)$



在$x=0$
看時間上的變化
(固定位置)

$\cos(-\omega t), \omega=1 \Rightarrow \cos(-t)=\cos(t)$

波長$L(\lambda)$:連續兩波峰或波谷間的距離

週期 T:在一定點處，兩波峰或波谷所間隔的時間。(通常取最小值)

由圖得知$\lambda=2\pi$

若$k=2 \Rightarrow \cos(2x)$ 則$\lambda_2=\pi$

若$k=3 \Rightarrow \cos(3x)$ 則$\lambda_3=\frac{2}{3}\pi$

$\Rightarrow \lambda=\frac{2\pi}{k} \therefore k=\frac{2\pi}{\lambda}$

定義k爲波數，即在2π間有幾個波。

同理$T=\frac{2\pi}{\omega} \therefore \omega=\frac{2\pi}{T}$

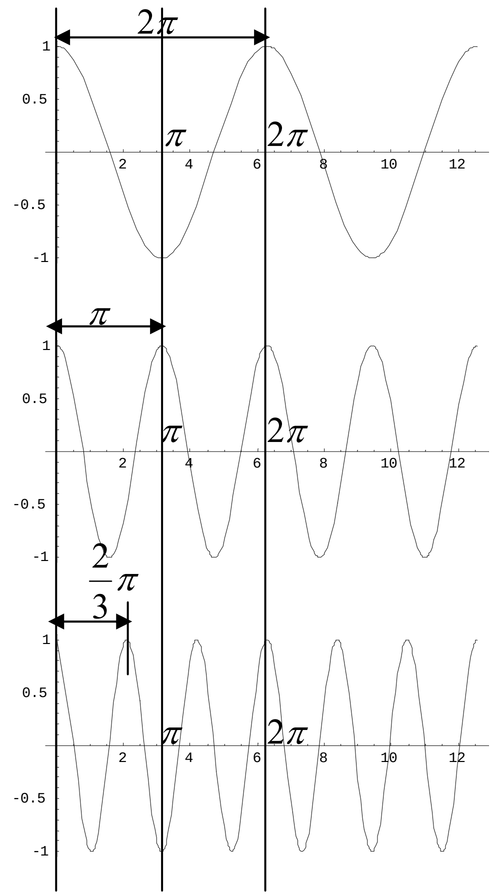
定義ω爲週波率，即在2π的時間裡有幾個波通過。

$\cos(kx-\omega t)$

$=\cos[k(x-\frac{\omega}{k}t)]$

$\Rightarrow \frac{\omega}{k}=\frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}}=\frac{\lambda}{T}=c$ (波速)

$\Rightarrow \cos[k(x-ct)]$



2008 $\cos(kx-\omega t), k$ *and* ω的意義.doc (2008/04/21) 小江製

Why Fourier series

海大河海系　　陳正宗

Loading expansion:

any external force, $f_e(t)$, with a period $2p$, we have

$$f_e(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}$$

$$\ddot{x}(t) + \omega^2 x(t) = f_e(t)$$

Understand the physical phenomenon(dominant freuency of Taipei basin)

Use FFT analyzer

Save storage for memory

To understand the frequency content and wave-number content.

Wave spectrum

Frequency spectrum

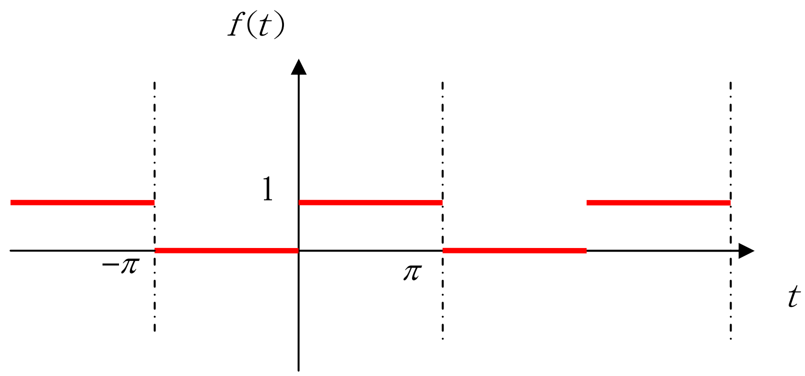
地震能譜、反應譜

食譜、菜譜、音譜

有譜! 無譜!

Fourier series expansion

Find the Fourier series of $f(t) = \begin{cases} 1, & -\pi < t < 0 \\ 0, & 0 < t < \pi \end{cases}$, $f(t) = f(t+2\pi)$



$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi t}{p}) + b_n \sin(\frac{n\pi t}{p})], p = \pi$$

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}(\int_{-\pi}^{0} 0dt + \int_{0}^{\pi} 1dt) = \frac{1}{2}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos(\frac{n\pi t}{p})dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} 0\cos(nt)dt + \int_{0}^{\pi} 1\cos(nt)dt]$$

$$= \frac{\sin(nt)}{n\pi}\bigg|_{0}^{\pi} = 0$$

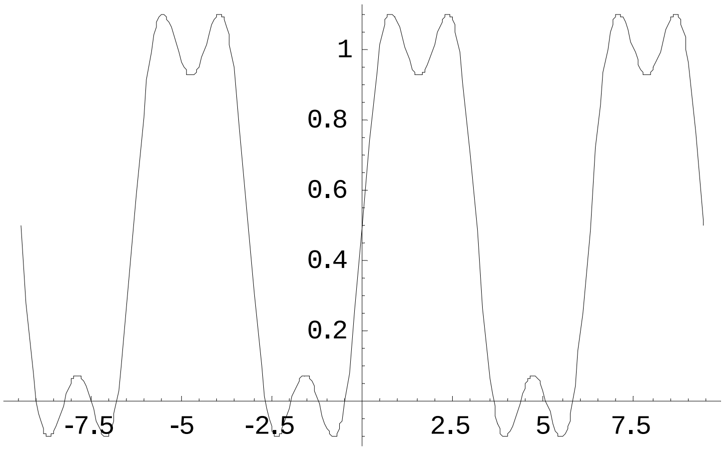
$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin(\frac{n\pi t}{p})dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

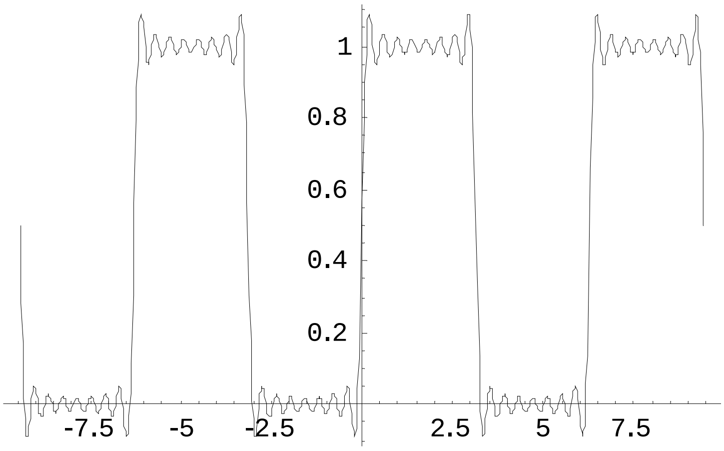
$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\sin(nt)dt + \int_{0}^{\pi} 1\sin(nt)dt$$

$$= \frac{-cos(nt)}{n\pi}\bigg|_{0}^{\pi} = -\frac{(-1)^n - 1}{n\pi} = \frac{1-(-1)^n}{n\pi}$$

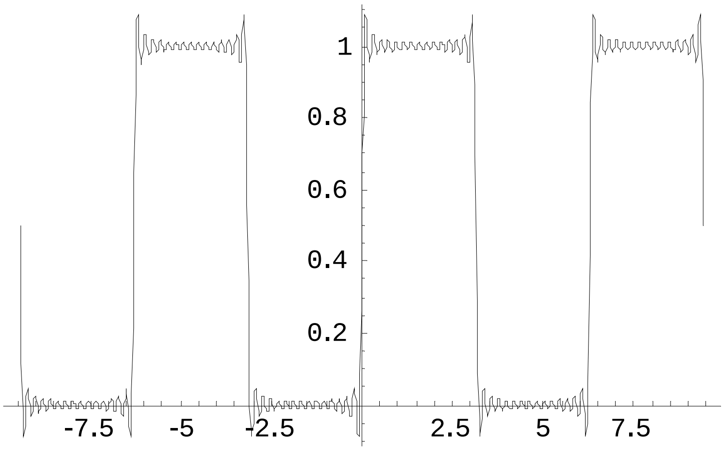
$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}[\frac{1-(-1)^n}{n\pi}]\sin(nt)$$


1
0.8
0.6
0.4
0.2
-7.5
-5
-2.5
2.5
5
7.5

n=3


1
0.8
0.6
0.4
0.2
-7.5
-5
-2.5
2.5
5
7.5

n=15


1
0.8
0.6
0.4
0.2
-7.5
-5
-2.5
2.5
5
7.5

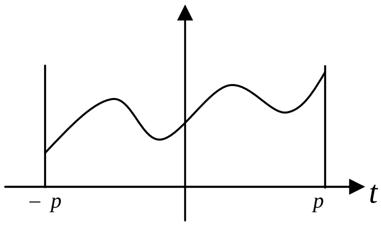
n=30

傅立葉級數

若有一個週期爲T的函數，我們希望把他表示成$\cos(\omega_n t)$與$\sin(\omega_n t)$之和，

其中$\omega_n = \dfrac{2n\pi}{T}, n = 0,1,2,3......$即$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right]$

若今天週期爲$2p$



$$\Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(\frac{n\pi}{p}t) + b_n \sin(\frac{n\pi}{p}t)\right]$$

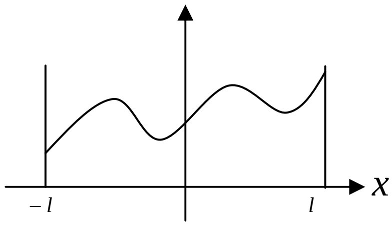
剩下的問題即a_0, a_n, b_n如何決定

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(t)dt$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos(\frac{n\pi}{p}t)dt$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin(\frac{n\pi}{p}t)dt$$

同理，今天波長爲$2l$的函數



$$g(x) = p_0 + \sum_{n=1}^{\infty}[p_n \cos(\frac{n\pi}{l}x) + q_n \sin(\frac{n\pi}{l}x)]$$

$$p_0 = \frac{1}{2l}\int_{-l}^{l} g(x)dx$$

$$p_n = \frac{1}{l}\int_{-l}^{l} g(x)\cos(\frac{n\pi}{l}x)dx$$

$$q_n = \frac{1}{l}\int_{-l}^{l} g(x)\sin(\frac{n\pi}{l}x)dx$$

Fourier coefficients

海大河海系　　陳正宗

Function decomposition: discrete form:

any time function, $f(t)$, with a period $2p$, we have

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}$$

Fourier coefficients:

$$a_n = \frac{1}{p}\int_0^{2p} f(t)cos(\frac{n\pi t}{p})dt$$

$$b_n = \frac{1}{p}\int_0^{2p} f(t)sin(\frac{n\pi t}{p})dt$$

Orthogonal relation for the bases:

$$p\delta_{ij} = \int_0^{2p} cos(\frac{i\pi t}{p})cos(\frac{j\pi t}{p})dt$$

$$p\delta_{ij} = \int_0^{2p} sin(\frac{i\pi t}{p})sin(\frac{j\pi t}{p})dt$$

$$0 = \int_0^{2p} sin(\frac{i\pi t}{p})cos(\frac{j\pi t}{p})dt$$

Minimize the distance, D, between $f(t)$ and the Fourier series:

$$D = \int_0^{2p} \mid \{f(t) - [\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}]\} \mid^2 dt$$

optimal a_n and b_n:

$$\frac{\partial D}{\partial a_n} = 0 \ \rightarrow \ a_n = \frac{1}{p}\int_0^{2p} f(t)cos(\frac{n\pi t}{p})dt$$

$$\frac{\partial D}{\partial b_n} = 0 \ \rightarrow \ b_n = \frac{1}{p}\int_0^{2p} f(t)sin(\frac{n\pi t}{p})dt$$

Fourier base	
real com	plex
$\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{\pi}}\cos(x),\frac{1}{\sqrt{\pi}}\sin(x),...$	$\frac{1}{\sqrt{2\pi}},\frac{e^{ix}}{\sqrt{2\pi}},\frac{e^{-ix}}{\sqrt{2\pi}},\frac{e^{2ix}}{\sqrt{2\pi}},\frac{e^{-2ix}}{\sqrt{2\pi}},...$
$\langle f,g\rangle$	$\langle f,\overline{g}\rangle$
$\lVert f\rVert=1$ $\langle f_m,f_n\rangle=\delta_{mn}$	$\lVert f\rVert=1$ $\langle f_m,\overline{f_n}\rangle=\delta_{mn}$
$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(nx)+q_n\sin(x)$	$f(x)=\sum_{-\infty}^{\infty}c_n e^{-inx}$

Decomposition theorem

海大河海系　　陳正宗

Vector decomposition: discrete form:

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

Function decomposition: discrete form:

any time function, $f(t)$, with a period $2p$, we have

$$f(t) = \sum\{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}$$

where $\omega_n = \frac{n\pi}{p}$.

any space function, $f(x)$, with a wave length 2λ, we have

$$f(x) = \sum\{a_n cos(\frac{n\pi x}{\lambda}) + b_n sin(\frac{n\pi x}{\lambda})\}$$

where $k_n = \frac{n\pi}{\lambda}$.

Function decomposition: continuous form:

any time function, $f(t)$, we have

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} d\omega$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

Fourier series

海大河海系　　　陳正宗

Any time function, $f(t)$, with a period $2p$, we have

$$f(t) = \sum \{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}$$

where $\omega_n = \frac{n\pi}{p}$.

Any space function, $f(x)$, with a wave length 2λ, we have

$$f(x) = \sum \{a_n cos(\frac{n\pi x}{\lambda}) + b_n sin(\frac{n\pi x}{\lambda})\}$$

where $k_n = \frac{n\pi}{\lambda}$.

Any time function, $f(t)$, we have

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} d\omega$$

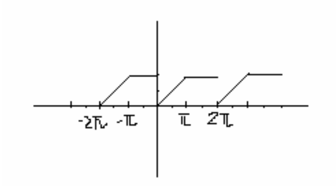
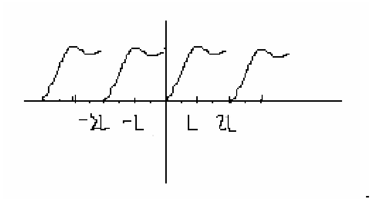
Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

海大河工系陳正宗　工數 (二)

【存檔：c:/ctex/course/math2/fouri2.te】【建檔:Mar./3/'97】

Fourier series for function with period 2L

	Standard form	General form
週期	0～2π	0～2L
f(x)的圖		
基底	1、cos(x)、sin(x)、cos(2x)、sin(2x)、、、cos(nx)、sin(nx)	1、$\cos(\frac{\pi x}{L})$、$\sin(\frac{\pi x}{L})$、$\cos(\frac{2\pi x}{L})$、$\sin(\frac{2\pi x}{L})$、、、、$\cos(\frac{n\pi x}{L})$、$\sin(\frac{n\pi x}{L})$
f(x)	$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$	$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$
長度	$1 \Rightarrow \int_0^{2\pi} 1dx = 2\pi$ $\cos(nx) \Rightarrow \int_0^{2\pi} \cos^2(nx)dx = \pi$ $\sin(nx) \Rightarrow \int_0^{2\pi} \sin^2(nx)dx = \pi$	$1 \Rightarrow \int_0^{2L} 1dx = 2L$ $\cos(\frac{n\pi x}{L}) \Rightarrow \int_0^{2L} \cos^2(\frac{n\pi x}{L})dx = L$ $\sin\left(\frac{n\pi x}{L}\right) \Rightarrow \int_0^{2L} \sin^2\left(\frac{n\pi x}{L}\right)dx = L$

Fourier series

海大河海系　　　陳正宗

Relation of trigonmetric function: multiplication to sum

$$2sin(A)cos(B) = sin(A+B) + sin(A-B)$$

$$2cos(A)cos(B) = cos(A+B) + cos(A-B)$$

$$2sin(A)sin(B) = cos(A-B) - cos(A+B)$$

Relation of trigonmetric function: sum to multiplication

$$sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$$

$$sin(A-B) = sin(A)cos(B) - cos(A)sin(B)$$

$$cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$$

$$cos(A-B) = cos(A)cos(B) + sin(A)sin(B)$$

Orthogonal properties:

$$\int_{-\pi}^{\pi} cos(mt)sin(nt)dt = 0$$

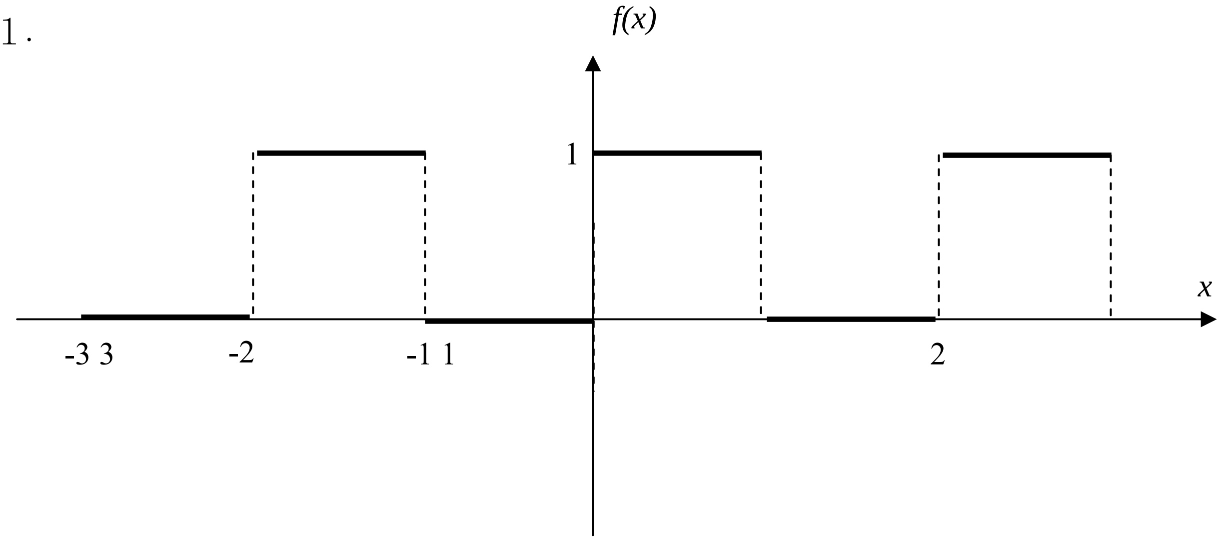
$$\int_{-\pi}^{\pi} cos(mt)cos(nt)dt = \delta_{mn}\pi$$

$$\int_{-\pi}^{\pi} sin(mt)sin(nt)dt = \delta_{mn}\pi$$

年級：＿＿＿＿＿＿　　姓名：＿＿＿＿＿＿＿＿＿　　學號：＿＿＿＿＿＿＿

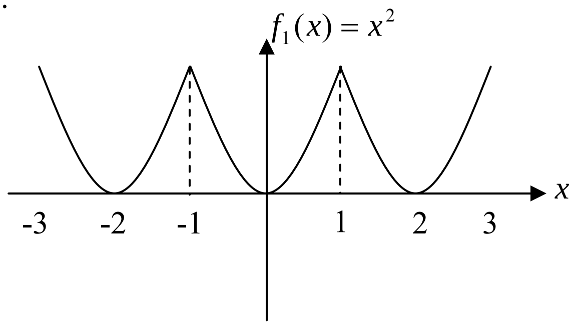
國立台灣海洋大學河海工程學系２００8 工程數學（二）週期看錯與 series sum

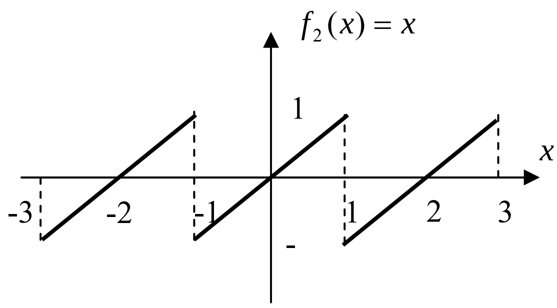
1.



(1) Decompose the function into $y_e(x)$ and $y_o(x)$ and plot $y_e(x)$ and $y_o(x)$.
(2) Expand $y_e(x)$ in terms of Fourier series.
(3) Expand $y_o(x)$ in terms of Fourier series.
(4) Expand $y(x)$ in terms of Fourier series.
(5) If we look function to be period of 4, expand $y(x)$ and compare the one of the period 2.

2.





(1) Expand $f_1(x)$ into Fourier series.
(2) Expand $f_2(x)$ into Fourier series.
(3) $\sum_{n=1}^{\infty} \frac{1}{n^2}$
(4) $\sum_{n=1}^{\infty} \frac{1}{n^4}$
(5) $\sum_{n=1}^{\infty} \frac{1}{n^6}$

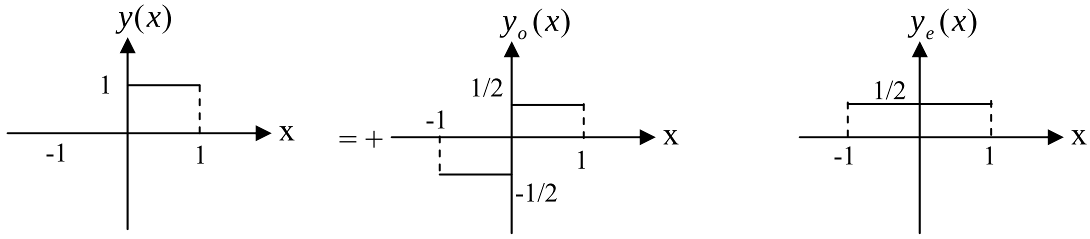
(Hint:Parseval’s theorem)

年級：________ 姓名：____________ 學號：________

國立台灣海洋大學河海工程學系２００8 工程數學（二）週期看錯與 series sum

1.

(1)



(2) $y_e(x) = \frac{1}{2}$

(3) $y_o(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(n\pi x) = \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)\pi x)$

(4) $y(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(n\pi x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)\pi x)$

(5) $a_0 = \frac{1}{4}[\int_{-2}^{-1} dx + \int_0^1 dx] = \frac{1}{2}$, $\quad a_n = \frac{1}{2}[\int_{-2}^{-1} \cos(\frac{n\pi}{2}x)dx + \int_0^1 \cos(\frac{n\pi}{2}x)dx] = 0$

$b_n = \frac{1}{2}[\int_{-2}^{-1} \sin(\frac{n\pi}{2}x)dx + \int_0^1 \cos(\frac{n\pi}{2}x)dx] = \frac{1}{n\pi}[\cos(n\pi) + 1 - 2\cos(\frac{n\pi}{2})]$

$y(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin((2k+1)\pi x)$

T=4 與 T=2 做傅立業展開其結果相同。

2.

(1) $a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$, $a_n = \int_{-1}^{1} x^2 \cos(n\pi x)dx = (\frac{1}{n\pi})^2 4\cos(n\pi)$, $b_n = \int_{-1}^{1} x^2 \sin(n\pi x)dx = 0$

$f_1(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x)$

(2) $a_0 = \frac{1}{2}\int_{-1}^{1} x dx = 0$, $a_n = \int_{-1}^{1} x\cos(n\pi x)dx = 0$, $b_n = \int_{-1}^{1} x\sin(n\pi x)dx = \frac{-2}{n\pi}\cos(n\pi)$

$f_2(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi}(-1)^n \sin(n\pi x)$

(3) $f_1(1) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} = 1$, $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.63498$

(4) $\frac{1}{2}\int_{-1}^{1} x^4 dx = \frac{1}{9} + \sum_{n=1}^{\infty} \frac{8}{n^4\pi^4}$, $\quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.08232$

(5) $\int_0^x f_1(x)dx = \int_0^x (\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x))dx$

$\frac{1}{2}\int_{-1}^{1} (\frac{x^3}{3} - \frac{x}{3})^2 dx = \sum_{n=1}^{\infty} \frac{8}{n^6\pi^6}$, $\quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \approx 1.01734$

年級：______________　姓名：______________________　學號：_________________

國立台灣海洋大學河海工程學系２００8 工程數學（二）series sum

1. $\sum_{n=1}^{\infty}\frac{1}{n^8}=?$

$$\int_0^x(\frac{x^3}{3}-\frac{x}{3})dx=\int_0^x\sum_{n=1}^{\infty}\frac{4}{n^3\pi^3}(-1)^n\sin(n\pi x)dx$$

$$\frac{x^4}{12}-\frac{x^2}{6}=\sum_{n=1}^{\infty}-\frac{4}{n^4\pi^4}(-1)^n\cos(n\pi x)+c$$

方法一：(陳品志)

$$c=\frac{1}{2}\int_{-1}^{1}(\frac{x^4}{12}-\frac{x^2}{6})dx=\frac{-7}{180}$$

方法二：(老師與助教)

$$\frac{1}{12}-\frac{1}{6}=\sum_{n=1}^{\infty}-\frac{4}{n^4\pi^4}+c$$

$$c=\frac{-7}{180}$$

$$\frac{x^4}{12}-\frac{x^2}{6}=\frac{-7}{180}+\sum_{n=1}^{\infty}-\frac{4}{n^4\pi^4}(-1)^n\cos(n\pi x)$$

$$\frac{1}{2}\int_{-1}^{1}(\frac{x^4}{12}-\frac{x^2}{6})dx=(\frac{-7}{180})^2+\sum_{n=1}^{\infty}\frac{1}{2}(-\frac{4}{n^4\pi^4}(-1)^n)^2$$

$$\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9450}\approx 1.004077356$$

方法三：(高聖凱)

$$x^4=\frac{1}{5}+\sum_{n=1}^{\infty}(\frac{8}{n^2\pi^2}-\frac{48}{n^4\pi^4})(-1)^n\cos(n\pi x)$$

$$\int_{-1}^{1}x^8dx=\frac{2}{25}+\sum_{n=1}^{\infty}(\frac{64}{(n\pi)^4}-\frac{768}{(n\pi)^6}+\frac{2304}{(n\pi)^8})$$

$$\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9450}\approx 1.004077356$$

條條大路通羅馬!

Fourier Series: Examples

John Appleby

Contents

1 Important Facts **1**

2 Exercises and Examples **2**

1 Important Facts

1. Suppose $f(x)$ is a periodic function of period 2π which can be represented by a **TRIGONOMETRIC FOURIER SERIES**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

(This means that the series above converges to $f(x)$.)

Then the **Fourier Coefficients** satisfy the **Euler Formulae**, namely:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{for } n = 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{for } n = 1, 2, \ldots \end{aligned}$$

2. A function f is said to be **even** if

$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}$$

and **odd** if

$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}$$

Recall the product of two even functions is even, the product of two odd functions is even and the product of an even and an odd function is odd. Compare the **multiplication** of even and odd **functions** to the **addition** of even and odd **integers**.

3. If f is an **odd** function then

$$\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

while if f is an even function, then

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

2 Exercises and Examples

Example 1. *Let f be a periodic function of period 2π such that*

$$f(x) = \pi^2 - x^2 \quad \text{for } x \in (-\pi, \pi).$$

Supposing that f has a convergent trigonometric Fourier series, show that

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx. \tag{2.1}$$

SOLUTION: The solution can be effected in a number of separate steps:

- Check whether f is even or odd.
- If f is **odd**, all the Fourier coefficients $\mathbf{a_n}$ for $n = 0, 1, 2\ldots$ are **zero**; if f is **even**, all the Fourier coefficients $\mathbf{b_n}$ for $n = 1, 2\ldots$ are **zero**.
- Compute the remaining Fourier coefficients using the Euler Formulae. It is generally a good strategy to use **Integration by Parts**, successively **integrating** $\sin nx$ **and** $\cos nx$ and **differentiating** $f(x)$.
- Replace the expressions for the Fourier coefficients a_n, b_n in

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

STEP 1: $f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$ so f is even.

STEP 2: Since $f(x)$ is even and $\sin nx$ is odd, $f(x)\sin nx$ is odd and hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0.$$

 ©John Appleby, 2001

STEP 3: Since $f(x)$ is even and $\cos nx$ is even, $f(x)\cos nx$ is even, and so

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2\int_0^{\pi} f(x)\cos nx\,dx.$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}2\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx \tag{2.2}$$

As suggested above, we calculate the integral in (2.2) by **Integration by Parts**. Recall the **Integration by Parts** formula:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f'(x)g(x)\,dx \tag{2.3}$$

Let

$$f(x) = \pi^2 - x^2 \quad \text{and} \quad g'(x) = \cos nx$$

so

$$f'(x) = -2x \quad \text{and} \quad g(x) = \int \cos nx\,dx = \frac{1}{n}\sin nx.$$

Using (2.3) and the above, we have

$$\int_0^{\pi} \underbrace{(\pi^2 - x^2)}_{f}\underbrace{\cos nx}_{g'}\,dx \tag{2.4}$$

$$\begin{aligned}
&= \underbrace{(\pi^2 - x^2)}_{f}\underbrace{\frac{1}{n}\sin nx}_{g}\Big|_0^{\pi} - \int_0^{\pi}\underbrace{-2x}_{f'}\underbrace{\frac{1}{n}\sin nx}_{g}\,dx \\
&= (\pi^2 - \pi^2)\frac{1}{n}\sin n\pi - (\pi^2 - 0^2)\frac{1}{n}\sin 0 + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx \\
&= \frac{2}{n}\int_0^{\pi} x\sin nx\,dx. \qquad (2.5)
\end{aligned}$$

Now we calculate this last integral using integration by parts: let

$$f(x) = x \quad \text{and} \quad g'(x) = \sin nx,$$

so

$$f'(x) = 1 \quad \text{and} \quad g(x) = \int \sin nx\,dx = \frac{-\cos nx}{n}.$$

Using (2.3), and remembering that $\cos n\pi = (-1)^n$, $\sin n\pi = 0$ for n an integer, we have

$$\begin{aligned}
\int_0^{\pi}\underbrace{x}_{f}\underbrace{\sin nx}_{g'}\,dx &= \underbrace{x}_{f}\underbrace{\frac{-\cos nx}{n}}_{g}\Big|_0^{\pi} - \int_0^{\pi}\underbrace{1}_{f'}\underbrace{\frac{-\cos nx}{n}}_{g}\,dx \\
&= \pi\frac{-\cos n\pi}{n} - 0\frac{-\cos 0}{n} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx \\
&= -\frac{1}{n}\pi(-1)^n + \frac{1}{n}\frac{\sin nx}{n}\Big|_0^{\pi} = -\frac{1}{n}\pi(-1)^n.
\end{aligned}$$

Using (2.2), (2.5) and the above, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\frac{2}{n}\int_0^\pi x\cos nx\,dx \\ &= \frac{2}{\pi}\frac{2}{n} - \frac{1}{n}\pi(-1)^n = \frac{-4}{n^2}(-1)^n. \end{aligned}$$

It remains to calculate a_0, which is given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}2\int_0^\pi \pi^2 - x^2\,dx \\ &= \frac{1}{\pi}\left(\pi^2 x - \frac{x^3}{3}\right)\Big|_0^\pi = \frac{1}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) = \frac{2\pi^3}{3} \end{aligned}$$

where we use the fact that $f(x) = \pi^2 - x^2$ is even.

STEP 4: Using the formulae obtained above for the Fourier coefficients, we have

$$\pi^2 - x^2 = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx + 0.\sin nx = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx$$

Example 2. *Show that the trigonometric Fourier series of $f(x) = 3x$ for $x \in (-\pi, \pi)$ is given by*

$$\sum_{n=1}^{\infty} \frac{-6}{n}(-1)^n \sin nx.$$

SOLUTION:

STEP 1: $f(-x) = 3. - x = -3x = -f(x)$, so f is an odd function.

STEP 2: Since $f(x)$ is odd and $\cos nx$ is even, it follows that $f(x)\cos nx$ is odd, so

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}.0 = 0.$$

Moreover, since f is odd

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}.0 = 0.$$

STEP 3: We need to calculate the Fourier coefficients using the Euler Formulae. However, noting that $f(x)$ and $\sin nx$ are odd, and therefore that $f(x)\sin nx$ is even we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}2\int_0^\pi f(x)\sin nx\,dx = \frac{6}{\pi}\int_0^\pi x\cos nx\,dx. \quad (2.6)$$

The latter integral is calculated using integration by parts.

 ©John Appleby, 2001

Exercise 2.1. Show that

$$\int_0^\pi x \sin nx\, dx = x\frac{-\cos nx}{n}\Big|_0^\pi - \int_0^\pi \frac{-\cos nx}{n}\, dx = \frac{-\pi}{n}(-1)^n.$$

By virtue of Exercise 2.1, we have, from (2.6)

$$b_n = \frac{6}{\pi}\frac{-\pi}{n}(-1)^n = -\frac{6}{n}(-1)^n.$$

STEP 4: The Fourier series of $f(x) = 3x$ is given by

$$\begin{aligned} a_0 + \sum_{n=1}^\infty a_n \cos nx + b_n \sin nx &= 0 + \sum_{n=1}^\infty 0 \cos nx + -\frac{6}{n}(-1)^n \sin nx \\ &= \sum_{n=1}^\infty -\frac{6}{n}(-1)^n \sin nx. \end{aligned}$$

Now try the following
Exercise 2.2.

(i) Show that $x^3 \cos nx$ is an odd function and $x^3 \sin nx$ is an even function. Hence give the value of

$$\int_{-\pi}^\pi x^3 \cos nx\, dx$$

and write down another expression equal to

$$\int_{-\pi}^\pi x^3 \sin nx\, dx.$$

(ii) By integrating by parts, show that

$$\int_0^\pi x^3 \sin nx\, dx = -\frac{(-1)^n\pi^3}{n} + \frac{3}{n}\int_0^\pi x^2 \cos nx\, dx.$$

Hint: Recall for integer values of n that $\cos n\pi = (-1)^n$.

(iii) Given that

$$\int_0^\pi x^2 \cos nx\, dx = -\frac{2}{n}\int_0^\pi x \sin nx\, dx$$

and

$$\int_0^\pi x \sin nx\, dx = -\frac{\pi}{n}(-1)^n,$$

use part (ii) to prove that

$$\int_0^\pi x^3 \sin nx\, dx = \frac{6\pi}{n^3}(-1)^n - \frac{\pi^3}{n}(-1)^n.$$

(iv) Using parts (i) and (iii), and supposing that the Fourier series converges, show for all $x \in (-\pi, \pi)$ that

$$x^3 = \sum_{n=1}^\infty 2(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right) \sin nx.$$

 ©John Appleby, 2001

Gibbs phenomenon

海大河海系　　　陳正宗

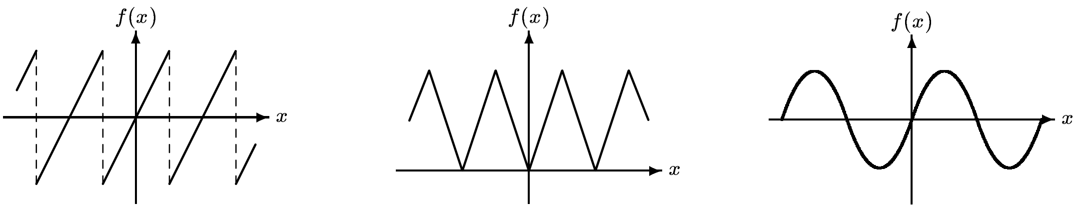
Fourier 級數收斂性與 Gibbs 現象

Fourier 級數的收斂性

1. $f(x)$ 為分段連續函數時 (如下圖)，$f(x)$ 的 Fourier 級數以 $\frac{1}{n}$ 的速度收斂。

2. $f(x)$ 為連續函數而 $f'(x)$ 為分段連續函數時 (如下圖)，$f(x)$ 的 Fourier 級數以 $\frac{1}{n^2}$ 的速度收斂。

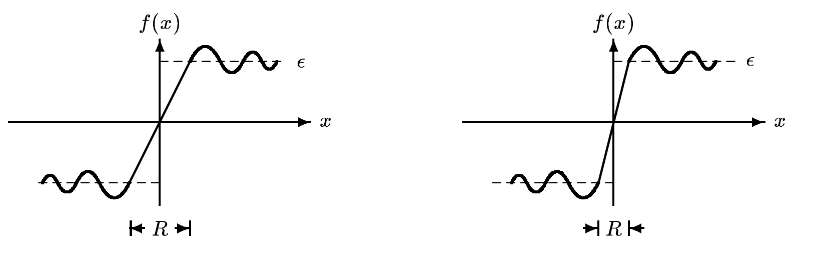
3. $f(x)$ 為連續函數而 $f'(x)$ 亦為連續函數時 (如下圖)，$f(x)$ 的 Fourier 級數以 $\frac{1}{n^3}$ 的速度收斂。



(a)　　　(b)　　　(c)

Gibbs 現象

在 $f(x)$ 的 Fourier 級數展開式中，若只取有限項來近似 $f(x)$ 時，在不連續點的某一鄰域會比原函數值較高，而其範圍會隨所取的項數增加而減少，此種現象稱為 Gibbs 現象 (參見下圖)。



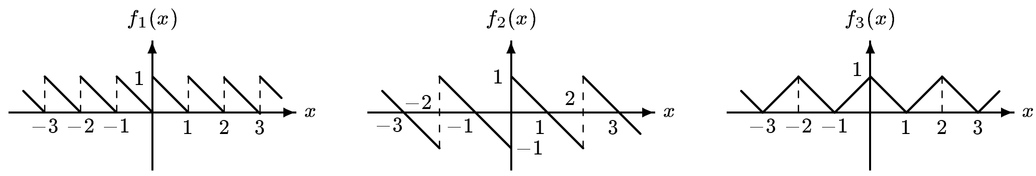
取項數較少時　　　取項數較多時

Gibbs 現象之誤差 ϵ 不會隨所取項數之多寡而改變，但其範圍為 R 會隨項數增加而減少。

例題

設 $f(x) = 1 - x,\ (0 \le x \le 1)$，若將 $f(x)$ 分別視作①一週期函數；②一奇函數；③一偶函數來做 Fourier 級數展開。試比較三者之準確度高低？
(註：不須求出展開式之係數)

解　將此三種情形分別繪圖如下：



1　　2　　3

由於 $f_3(x)$ 在 $(-\infty, \infty)$ 內連續且屬均勻收斂，故其收斂速度最快，準確度最高。 $f_1(x)$ 與 $f_2(x)$ 在 $(-\infty, \infty)$ 內皆為分段收斂，而因 $f_1(x)$ 之不連續點為 $f_2(x)$ 之兩倍，故 $f_2(x)$ 比 $f_1(x)$ 之收斂速度慢，準確度差。

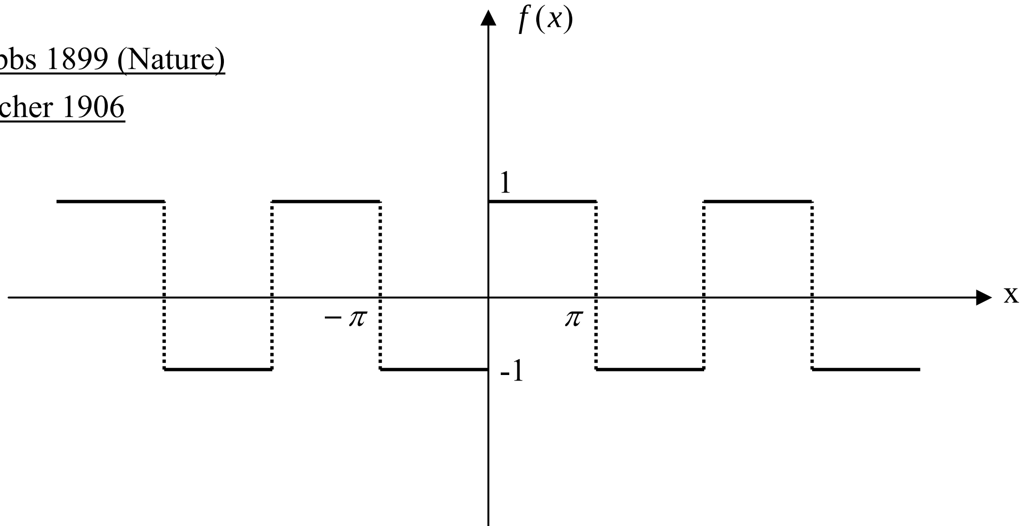
海大河工系陳正宗　工數 (二)

【存檔：c:/ctex/course/math2/gibs.te】【建檔:Apr./13/'97】

國立台灣海洋大學 工數二 Gibbs phenomenon 2008

<u>Gibbs 1899 (Nature)</u>

<u>Bocher 1906</u>



Expand the function into Fourier series and plot.(pointwise convergence instead of uniform convergence)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$f(x)$ 爲偶函數，所以 $a_0 = a_n = 0$

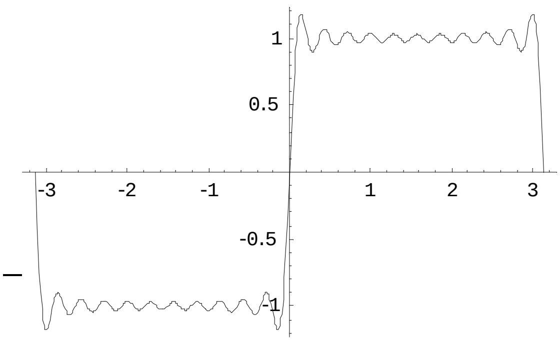
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)dx = \frac{-2}{n\pi}[\cos(n\pi)-1]$$

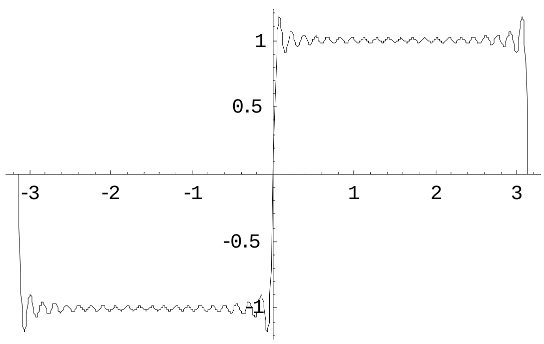
$$f(x) = \sum_{n=1}^{\infty}\frac{-2}{n\pi}[\cos(n\pi)-1]\sin(nx)$$

$$n = 2k+1, \quad b_n = \frac{-2}{n\pi}[\cos(n\pi)-1] = \frac{4}{(2k+1)\pi}$$

$$n = 2k, \quad b_n = \frac{-2}{n\pi}[\cos(n\pi)-1] = 0$$

$$f(x) = \sum_{k=0}^{\infty}\frac{4}{(2k+1)\pi}\sin[(2k+1)x]$$ overshoot 1.09 J (J is the jump)





Fourier series

海大河海系　　陳正宗

Fourier series expansion :

Orthogonal sets: $\{1, cos(nt), sin(nt)\}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(nt) + b_n sin(nt)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n cos(nt + \theta_n)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n cos(nt - \delta_n)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n sin(nt + \alpha_n)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n sin(nt - \beta_n)$$

where

$$c_n^2 = a_n^2 + b_n^2$$

$$\theta_n = tan^{-1}\frac{b_n}{a_n}$$

Orthogonal property for real bases:

$$\int_0^{2\pi} cos(nt)cos(mt)dt = \pi\delta_{mn}$$

$$\int_0^{2\pi} sin(nt)sin(mt)dt = \pi\delta_{mn}$$

$$\int_0^{2\pi} cos(nt)sin(mt)dt = 0$$

Complex Fourier series expansion :

Orthogonal sets: $\{e^{int}\}$

$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{int}$$

where

$$d_{-n} = \frac{1}{2}\{a_n + ib_n\}, n = 1, 2, 3, \cdots$$

$$d_n = \frac{1}{2}\{a_n - ib_n\}, n = 1, 2, 3, \cdots$$

$$d_0 = a_0$$

Orthogonal property for complex bases:

$$\int_0^{2\pi} e^{int}(e^{imt})^* dt = 2\pi\delta_{mn}$$

where $*$ denotes the complex conjugate.

Complex conjugate pair: (e^{int}, e^{-int}) and (d_n, d_{-n})

複數空間講義

	實數空間	複數空間
矩陣	$A = A^T$ 對稱 symmetric $A = A^{-T}$ 反對稱 skew-symmetric	$A = A^H = \bar{A}^T$ (Hermitian) $A = -A^H = -\bar{A}^T$ (skew-Hermitian)
向量	$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = l^2$ $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \underset{\sim}{u} \cdot \underset{\sim}{v}$	$\langle \underset{\sim}{u}, \underset{\sim}{\bar{u}} \rangle = l^2$ $\langle \underset{\sim}{u}, \underset{\sim}{\bar{v}} \rangle =$ 內積
正交性	$\langle \cos nt, \sin mt \rangle = 0 \ , m \neq n$ $\langle \cos nt, \cos mt \rangle = 0 \ , m \neq n$ $\langle \sin nt, \sin mt \rangle = 0 \ , m \neq n$	$\langle e^{int}, e^{imt} \rangle$ $= \int_{-\pi}^{\pi} e^{int} e^{-imt} dt$

(1).Given $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, find $c_n = ? (n = -\infty, \ldots. 0 \ldots., \infty)$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(9t - \frac{\pi}{3}) e^{-int} dt$$

$$c_9 = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(9t - \frac{\pi}{3}) e^{-9ti} dt = \frac{1}{2\pi} (\pi \cos(\frac{\pi}{3}) - i\pi \sin(\frac{\pi}{3})) = \frac{1}{4} - \frac{\sqrt{3}}{4} i$$

$$c_{-9} = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(9t - \frac{\pi}{3}) e^{9ti} dt = \frac{1}{2\pi} (\pi \cos(\frac{\pi}{3}) + i\pi \sin(\frac{\pi}{3})) = \frac{1}{4} + \frac{\sqrt{3}}{4} i$$

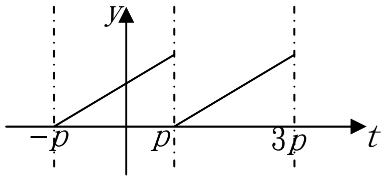
$$f(t) = c_9 e^{9it} + c_{-9} e^{-9it} = (\frac{1}{4} - \frac{\sqrt{3}}{4} i) e^{9it} + (\frac{1}{4} + \frac{\sqrt{3}}{4} i) e^{-9it} = \frac{1}{2} \cos(9t) + \frac{\sqrt{3}}{2} \sin(9t)$$

(2).另解

$$\begin{aligned}
f(t) &= \cos(9t - \frac{\pi}{3}) \\
&= \cos(9t) \cos(\frac{\pi}{3}) + \sin(9t) \sin(\frac{\pi}{3}) \\
&= \frac{1}{2} \frac{1}{2} (e^{i9t} + e^{-i9t}) + \frac{\sqrt{3}}{2} \left(\frac{-i}{2} \right) (e^{i9t} - e^{-i9t}) \\
&= \left(\frac{1}{4} - \frac{\sqrt{3} i}{4} \right) e^{i9t} + \left(\frac{1}{4} + \frac{\sqrt{3} i}{4} \right) e^{-i9t} \\
&= c_9 e^{i9t} + c_{-9} e^{-i9t}
\end{aligned}$$

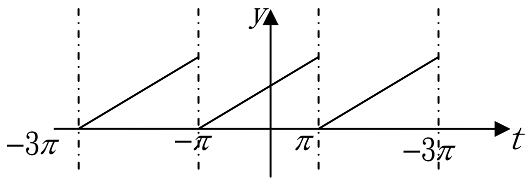
通式與特例 Fourier series

週期$2p$



Real	**Complex**
$a_0 = \frac{1}{2p}\int_{-p}^{p} f(t)dt$ $a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos(\frac{n\pi t}{p})dt$ $b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin(\frac{n\pi t}{p})dt$ 奇函數$\rightarrow a_n = 0$ 偶函數$\rightarrow b_n = 0$	$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-i(\frac{n\pi}{p})t}dt$ 奇函數$\rightarrow c_n$剩虛部 偶函數$\rightarrow c_n$剩實部

週期2π



$\Downarrow$ $2p = 2\pi$ 特例

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$	$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\text{int}}dt$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$c_{-n} = \overline{c_n}$$

複數與工程

賴添盛　技師

複數的起源

1805 年出生於愛爾蘭的哈彌爾頓，在 1843 年 10 月 16 日的深夜，與妻子一同恩愛的散步在布魯赫姆橋上，突發的靈感想通了一個難解的問題：i2=-1…(1-1)，從此以後，人類數學上除了『實數』之外，還增加了『虛數』，兩者結合形成『複數』，改變了人類數學上的解題技術。

『複數』的發明，破解了許多物理數學上原先無解的問題，對人類的科技發展影響深遠。但是，『虛數』畢竟只是一種虛擬的數，在現實的生活中是不存在的，換句話說，『實數』和『虛數』是二種獨立的變數，解題到最後終究必須轉化成實數解，才有物理上的意義。少部分後學者不明白其中奧秘，常常混淆不清。

為了讓讀者認清真相，首先請看一則解一元二次方程式的問題如下：求解：z2+2z+1=0…(2-1)，解：

(一)實數域之解，令 z=x，x 為實數…(2-2)，代入(2-1)式中，得 x2+2x+2=(x+1)2+1≧1…(2-3)，所以(2-1)式無解。

(二)複數域之解，令 z=x+yi，x，y 為實數…(2-4)，代入(2-1)式中，得 z2+2z+2=x2+2xy$i$-y2+2x+2yi+2=(x2+2x-y2+2)+2y(x+1)i=0，得 x2+2x-y2+2=0…(2-5)，2y(x+1)=0…(2-6)，若 y=0，則 x2+2x+2=0，無解。若 x=-1，則 y=±1，得解 z=-1+i，-1-i(共軛複數)…(2-7)

在實數域中無解的問題，因為借用哈彌爾頓虛擬的『虛數』後，卻一定可以在複數域中得到『共軛複數』解，但是這種解只是『虛擬解』，在現實生活中是不存在的。以下再看另一則解結構動力的問題，求解：m(d2v/dt2)+kv=0…(3-1)，v(0)=a…(3-2)，dv(0)/dt=b…(3-3)，解：

(一)複數域之解，令 v=Cest，代入(3-1)式中，得 ms2+k=0…(3-4)，令ω2=k/m，代入(3-4)式中，得 s2+ω2=0…(3-5)，得 s=+ωi，-ωi(共軛複數)，v 之通解為 v=C1eiωt+C2 e-iωt…(3-6)，代入(3-2)，(3-3)式中，得 v(0)=a=C1+C2，dv(0)/dt=b=iω(C1-C2)→C1-C2=b/i

ω，C1=a/2+b/2*i*ω…(3-7)，C2=a/2-b/2*i*ω…(3-8)，代入(3-6)式中，得 v=(a/2)(*ei*ωt+*e*-*i*ωt)+(b/2*i*ω)(*ei*ωt-*e*-*i*ωt)，=a*cos*ωt+(b/ω)*sin*ωt…(3-9)(『虛數』互相抵銷，回到現實)，=ρ*cos*(ωt-θ)…(3-10)，ρ=〔a2+(b/ω)2〕1/2…(3-11)，θ=*tan*-1(b/ωa)…(3-12)，θ為遲滯相位角差，表示當 t=θ/ω時，v 才會達到最大值。

（二)實數域之解，根據 Euler’s 公式，e±*i*ωt=*cos*ωt±*isin*ωt…(3-13)，將(3-6)式之通解改為 v=R1 *cos*ωt+R2*sin*ωt…(3-14)，代入(3-2)，(3-3)式中，得 v(0)=a=R1…(3-15)，dv(0)/dt=b=ωR2→R2=b/ω…(3-16)，代入(3-14)式中，得 v=a*cos*ωt+(b/ω)*sin*ωt…(3-17)，由(3-9)式與(3-17)式得知，實數域與複數域之解完全相同，不因解題技術差異而得到不同解。在解題過程中暫時借用哈彌爾頓所定義的『複數』只是一種技術(手段)，由於複數根一定會以『共軛複數(Conjugate Complex Number)』的型式成雙成對、如影隨形的出現，『虛數』終究會在解題過程中互相抵銷而功成身退，回到現實生活中的實數解。明白這一點，相信你就會愛死『複數』。或許有一天，你會虛擬一種比『虛數』更有意思的『超虛數』，去破解一些目前尚未解決的物理數學問題，你的成就，將會超越哈彌爾頓。

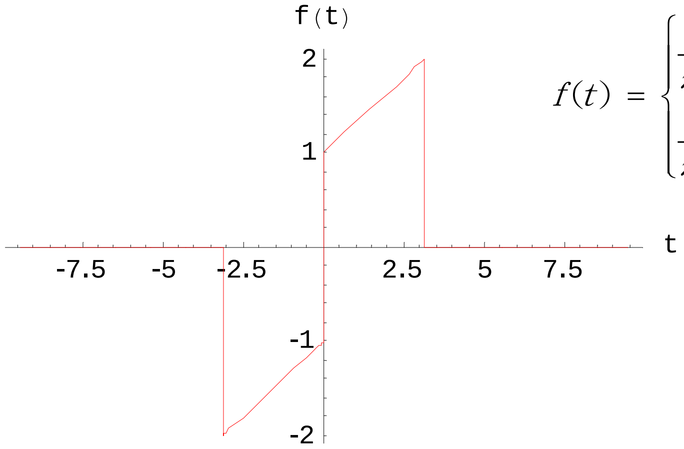
複數的意義

綜合以上說明得知，『虛數』是人類在發展數學上的解題技術時，以人為定義方式發明的一種虛擬的數(相當於『虛擬實境』一樣)。『虛數』顧名思義，是一種虛擬的數，在現實生活中不存在，在實務的商用數學中也用不著。『複數』可以用來破解一些物理數學上的問題，解題到最後經過轉化所得到的實數解，才有物理上的意義，帶有『虛數』的複數解是沒有物理意義的。

資料來源:

技師報 NO.240 (2001/07/14 星期六)—賴添盛 技師 撰

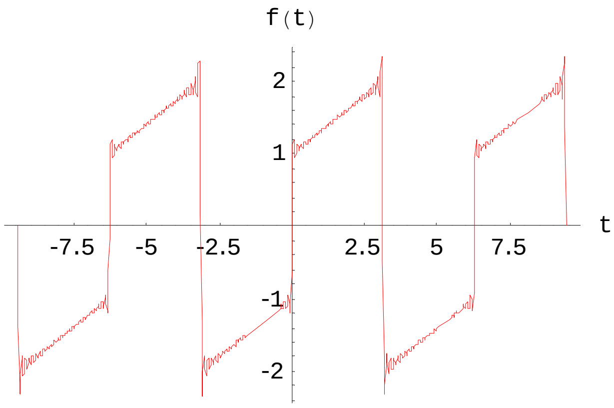
http://www.twce.org.tw/info/技師報/240-3-2.htm



$$f(t) = \begin{cases} \frac{1}{\pi} t + 1, & 0 < t < \pi \\ \frac{1}{\pi} t - 1, & -\pi < t < 0 \end{cases}$$

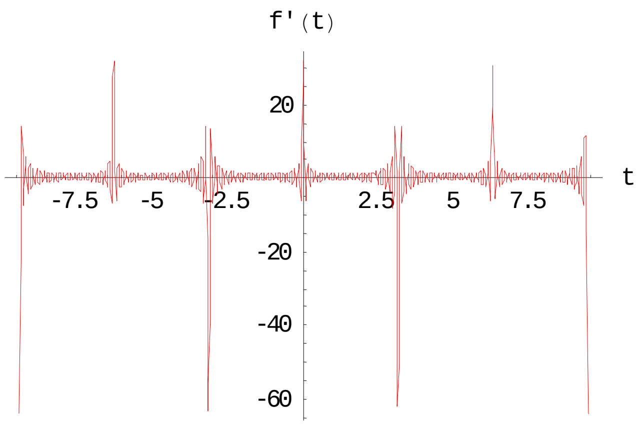
$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin(nt) dt$$



<u>Method 1</u> (硬幹法)

$$f'(t) = \sum_{n=1}^{\infty} n b_n \cos(nt) \quad \text{...(1)}$$



Method 2 (Stokes' 轉換)

猜 $f'(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt)$

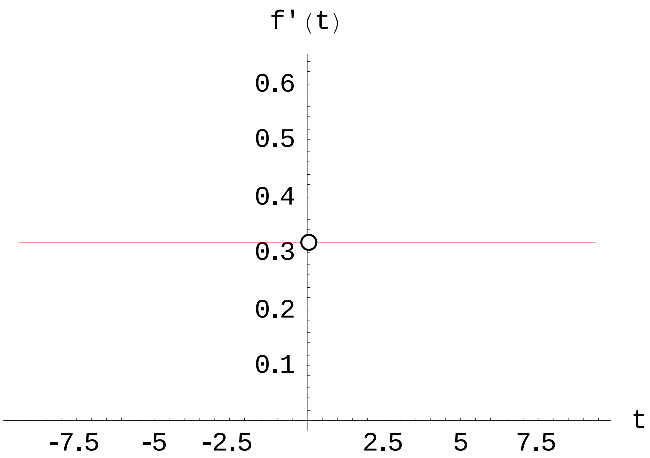
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} f'(t)\cos(nt)dt$$

$$= \frac{2}{\pi}[f(t)\cos(nt)\Big|_{t=0}^{t=\pi} + n\int_{0}^{\pi} f(t)\sin(nt)dt]$$

$$= \frac{2}{\pi}[q(-1)^n - p] + nb_n$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)dt = \frac{1}{\pi}(q - p)$$

$$f'(t) = \frac{1}{\pi}(q - p) + \sum_{n=1}^{\infty}\{\frac{2}{\pi}[q(-1)^n - p] + nb_n\}\cos(nt) \qquad \text{.........................(2)}$$



比較(1)(2)式可看出直接微分會漏項

Alternative series and Stokes' transformation

海大河海系 陳正宗

original function decomposition:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{l}) + b_n sin(\frac{n\pi t}{l})\}$$

derivative of original function decomposition:

$$f'(t) = \sum_{n=1}^{\infty}\{a_n(\frac{-n\pi}{l})sin(\frac{n\pi t}{l}) + b_n(\frac{n\pi}{l})cos(\frac{n\pi t}{l})\} = \frac{1}{2}a'_0 + \sum_{n=1}^{\infty}\{a'_n cos(\frac{n\pi t}{l}) + b'_n sin(\frac{n\pi t}{l})\}$$

where J_k is the jump value at t_k for the function $f(t)$.

relation of a_n, b_n, a'_n and b'_n:

$$a_n = -\frac{1}{n\pi}\sum_{k=1}^{m} J_k sin(\frac{n\pi t_k}{l}) - \frac{l}{n\pi}b'_n,\ n \neq 0 \quad (1)$$

$$b_n = \frac{1}{n\pi}\sum_{k=1}^{m} J_k cos(\frac{n\pi t_k}{p}) + \frac{l}{n\pi}a'_n,\ n \neq 0 \quad (2)$$

Termwise differentiation is only permissible if J_k is zero.

Stokes' transformation:

$$u(x) = \begin{cases} p, & x = 0 \\ \sum_{n=0}^{\infty} b_n sin(n\pi x/l), & 0 < x < l \\ q, & x = l \end{cases} \quad (3)$$

Assume

$$u'(x) = \sum_{n=0}^{\infty} a'_n cos(n\pi x/l),\ 0 \leq x \leq l \quad (4)$$

$$\begin{aligned} a'_n &= \frac{2}{l}\int_0^l u'(x)cos(n\pi x/l)dx \\ &= \frac{2}{l}[u(x)cos(n\pi x/l)]_0^l + \frac{2n\pi}{l^2}\int_0^l u(x)sin(n\pi x/l)dx \\ &= \frac{2}{l}[(-1)^n q - p] + n\pi b_n/l, \end{aligned} \quad (5)$$

and as $n = 0$,

$$a'_0 = \frac{-1}{l}[p - q] \quad (6)$$

Defining

$$r_n \equiv \begin{cases} \frac{-1}{l}[p - q], & n = 0 \\ \frac{2}{l}[(-1)^n q - p], & n = 1, 2, \cdots \end{cases} \quad (7)$$

we have

$$a'_n = r_n + n\pi b_n/l,\ \ n \geq 0 \quad (8)$$

Proof:

$$J_0 = 2p, \ \ at \ \ x = 0$$

$$J_l = 2q, \ \ at \ \ x = -l$$

Alternative series and Stokes' transformation

海大河海系 陳正宗

Table 1: Fourier coefficients for $f(t)$ and $f'(t)$

$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{l})+b_n sin(\frac{n\pi t}{l})\}$	a_0	a_n	b_n
$f'(t)=\frac{1}{2}a'_0+\sum_{n=1}^{\infty}\{a'_n cos(\frac{n\pi t}{l})+b'_n sin(\frac{n\pi t}{l})\}$	a'_0	a'_n	b'_n

relation of a_n, b_n, a'_n and b'_n:

$$a_n = -\frac{1}{n\pi}\sum_{k=1}^{m} J_k sin(\frac{n\pi t_k}{l}) - \frac{l}{n\pi}b'_n,\ n \neq 0 \quad (1)$$

$$b_n = \frac{1}{n\pi}\sum_{k=1}^{m} J_k cos(\frac{n\pi t_k}{p}) + \frac{l}{n\pi}a'_n,\ n \neq 0 \quad (2)$$

relation of a_0 and a'_0 ?

a'_0 by alternative series

$$a'_0 = \frac{1}{l}\int_0 l f'(t)dt = \frac{1}{l}f(t)\ |_{-l}^{0} + \frac{1}{l}f(t)\ |_0^l = \frac{2}{l}\{q-p\}$$

Cesaro sum for Fourier series

海大河海系 陳正宗

Fourier series for original function

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}$$

Fourier series for the derivative of original function

$$f'(t) = \sum_{n=1}^{\infty}\{a_n(\frac{-n\pi}{p})sin(\frac{n\pi t}{p}) + b_n(\frac{n\pi}{p})cos(\frac{n\pi t}{p})\} = \sum_{n=1}^{\infty}\{a'_n cos(\frac{n\pi t}{p}) + b'_n sin(\frac{n\pi t}{p})\}$$

Divergence for Fourier series representation of $f'(t)$ may occur when the termwise differetiation is not permissible.

Two methods can be employed:

(a). Alternative series by considering the jump value of the function. (Stokes' transformation)

(b). Cesaro sum treatment.

The general $C(k,r)$ Cesàro sum is defined as

$$S_k = C(k,r)\{\sum_{n=0}^{k} a_n\} \equiv \frac{C_{r-1}^{k+r-1}s_0 + C_{r-1}^{k+r-2}s_1 + \cdots + C_{r-1}^{r}s_{k-1} + C_{r-1}^{r-1}s_k}{C_r^{k+r}} \quad (1)$$

where $C_r^k = k!/(r!\,(k-r)!)$ and the partial sum is

$$s_k = \sum_{n=0}^{k} a_n \quad (2)$$

The $C(k,1)$ sum reduces to the conventional Cesàro sum:

$$S_k = C(k,1)\{\sum_{n=0}^{k} a_n\} \equiv \frac{s_0 + s_1 + \cdots + s_{k-1} + s_k}{k+1} \quad (3)$$

For the efficiency of computation, the s_i terms are changed to the a_i terms and the equation is thus changed to

$$S_k = C(k,1)\{\sum_{n=0}^{k} a_n\} \equiv \frac{1}{k+1}\sum_{n=0}^{k}(k-n+1)\ a_n \quad (4)$$

Similarly, the $C(k,2)$ Cesàro sum is

$$S_k = C(k,2)\{\sum_{n=0}^{k} a_n\} \equiv \frac{1}{(k+1)(k+2)}\sum_{n=0}^{k}(k-n+1)(k-n+2)\ a_n \quad (5)$$

In the same way, the $C(k,3)$ and $C(k,4)$ Cesàro sums are respectively

$$S_k = C(k,3)\{\sum_{n=0}^{k} a_n\} \equiv \frac{\sum_{n=0}^{k}(k-n+1)(k-n+2)(k-n+3)\ a_n}{(k+1)(k+2)(k+3)}$$

$$S_k = C(k,4)\{\sum_{n=0}^{k} a_n\} \equiv \frac{\sum_{n=0}^{k}(k-n+1)(k-n+2)(k-n+3)(k-n+4)\ a_n}{(k+1)(k+2)(k+3)(k+4)}$$

If the a_0 term is missing, the $C(k,1)$ Cesàro sum reduces to

$$S_k \equiv C(k,1)\{\sum_{n=1}^{k} a_n\} = \frac{1}{k}\sum_{n=1}^{k}(k-n+1)\ a_n \tag{6}$$

Energy conservation

海大河海系　　　陳正宗

Function decomposition: discrete form:

any time function, $f(t)$, with a period $2p$, we have

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{p}) + b_n sin(\frac{n\pi t}{p})\}$$

Energy conservation:

$$\int_0^{2p} f^2(t)dt = \frac{1}{2}a_0^2 p + p\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Parseval's equality:

An alternative method to calculate $\int_0^{2p} f^2(t)dt$ is available since a_n and b_n decay quickly.

Parseval's 定理

$$L^2 = \int_0^{2\pi} \cos^2(mx)dx = \pi$$

$$L = \sqrt{\pi}$$

$$f(x) = \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)] + a_0$$

$$L^2 = \int_0^{2\pi} f(x) f(x) dx$$

$$= \int_0^{2\pi} f^2(x) dx$$

$$= \int_0^{2\pi} \{\sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)] + a_0\} \{\sum_{p=1}^{\infty}[a_p \cos(px) + b_p \sin(px)] + a_p\} dx$$

$$= 2\pi(a_0^2) + \sum_{k=1}^{\infty}[(a_k^2)\pi + (b_k^2)\pi]$$

$$= \pi[2(a_0^2) + \sum_{n=1}^{\infty}[(a_n^2) + (b_n^2)]$$

Fourier coefficient 的新觀點

$$\int_{-\pi}^{\pi}\{f(x)-\{\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]+a_0\}\}^2dx \Rightarrow \text{minimum(最相近)}$$

$$(a-b)^2=a^2-2ab+b^2$$

偏微分 $\frac{\partial F}{\partial a_0}=0$ ， $\frac{\partial F}{\partial a_n}=0$ ， $\frac{\partial F}{\partial b_n}=0$

$$F=\int_{-\pi}^{\pi}f^2(x)dx$$

$$\int_{-\pi}^{\pi}-2f(x)\{\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]+a_0\}dx$$

$$\int_{-\pi}^{\pi}\{\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]+a_0\}^2dx$$

$$=\int_{-\pi}^{\pi}f^2(x)dx$$

$$-2\int_{-\pi}^{\pi}f(x)\{\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]+a_0\}dx$$

$$+2\pi(a_0^2)+\pi\sum_{n=1}^{\infty}[(a_n^2)+(b_n^2)]$$

$$\frac{\partial F}{\partial a_0}=4\pi a_0-2\int_{-\pi}^{\pi}f(x)dx=0 \Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$\frac{\partial F}{\partial a_n}=2\pi a_n-2\int_{-\pi}^{\pi}f(x)\cos(nx)dx=0 \Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx$$

$$\frac{\partial F}{\partial b_n}=2\pi b_n-2\int_{-\pi}^{\pi}f(x)\sin(nx)dx=0 \Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx$$

海洋大學河海工程學系　　工程數學（二）－補救教學練習考 5　　姓名：
學號：

1. 求 $\int_1^e x\ln x dx$ =?

2. 求 $\int e^{ax}\sin bx dx$ =?

3. $f(x)=\cos x+\sqrt{3}\sin x$ 將 $f(x)$ 換成 $r\sin(ax+b)$，求 r,a,b

1. 求$\int_1^e x\ln x dx$ =?

$$\int_1^e x\ln x dx$$

$$= \ln x(\frac{1}{2}x^2) - \int (\frac{1}{x})(\frac{1}{2}x^2)dx$$

$$= \left[\frac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \frac{1}{2}xdx$$

$$= \left[\frac{1}{2}e^2 \ln e\right] - \left[\frac{1}{4}x^2\right]_1^e$$

$$= \frac{1}{2}e^2 - \left[\frac{1}{4}e^2 - \frac{1}{4}\right]$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

u		dv
$(+)\ln x$	$+$ ↘	x
$(-)\frac{1}{x}$	→	$\frac{1}{2}x^2$

2. 求$\int e^{ax}\sin bx dx$ =?

$$\int e^{ax}\sin bx dx$$

$$= e^{ax}(-\frac{1}{b}\cos bx) - ae^{ax}(-\frac{1}{b^2}\sin bx) + \int a^2 e^{ax}(-\frac{1}{b^2}\sin bx)$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx dx$$

$$\Rightarrow (1+\frac{a^2}{b^2})\int e^{ax}\sin bx dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx$$

$$\Rightarrow \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

u		dv
$(+)e^{ax}$		$\sin bx$
	$+$ ↘	
$(-)ae^{ax}$		$-\frac{1}{b}\cos bx$
	$-$ ↘	
$(+)a^2e^{ax}$	→	$-\frac{1}{b}\sin bx$

3. $f(x) = \cos x + \sqrt{3}\sin x$　將 $f(x)$ 換成$r\sin(ax+b)$，求r, a, b

$$= 2(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x)$$

$$= 2(\sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x)$$

$$= 2\sin(\frac{\pi}{6} + x)$$

$$\begin{cases} r = 2 \\ a = 1 \\ b = \frac{\pi}{6} \end{cases}$$

Small test 08-05.doc　小江製

1. $f(t)=t^2+1,\ (-\pi \le t \le \pi)$ ，試求此週期函數的傅立葉級數展開。

1. $f(t)=t^2+1,\ (-\pi\le t\le\pi)$，試求此週期函數的傅立葉級數展開。

$$T=2p=2\pi,\ p=\pi$$

$$f(t)=a_0+\sum_{n=1}^{\infty}[a_n\cos(\frac{n\pi}{p}t)+b_n\sin(\frac{n\pi}{p}t)]$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt=\frac{1}{3}\pi^2+1$$

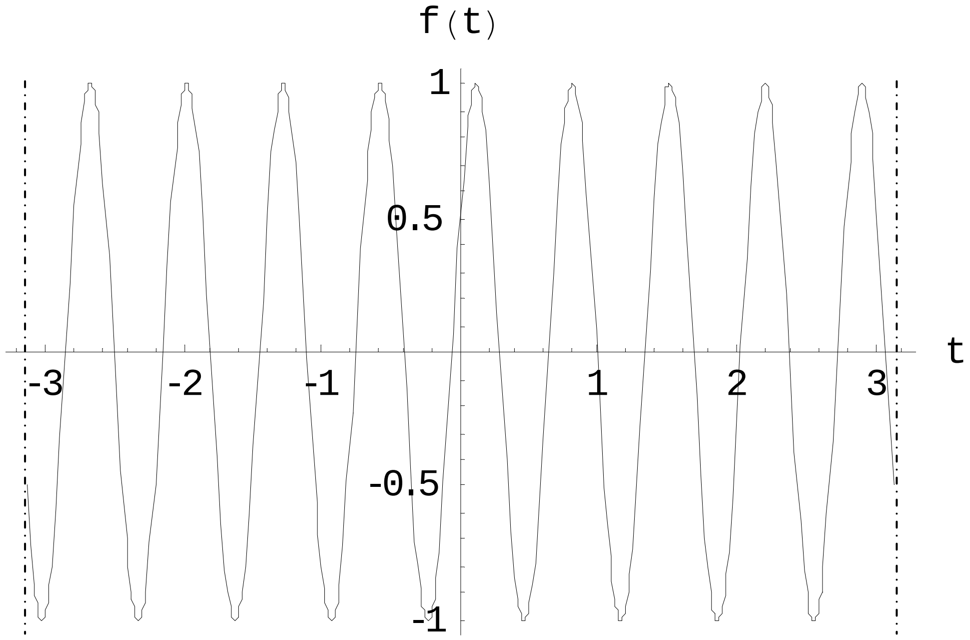
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+1)\cos(nt)dt=\frac{4}{n^2}(-1)^n$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+1)\sin(nt)dt=0$$

$$\Rightarrow f(t)=\frac{1}{3}\pi^2+1+\sum_{n=1}^{\infty}[\frac{4}{n^2}(-1)^n\cos(nt)]$$

Find the Complex Fourier series of

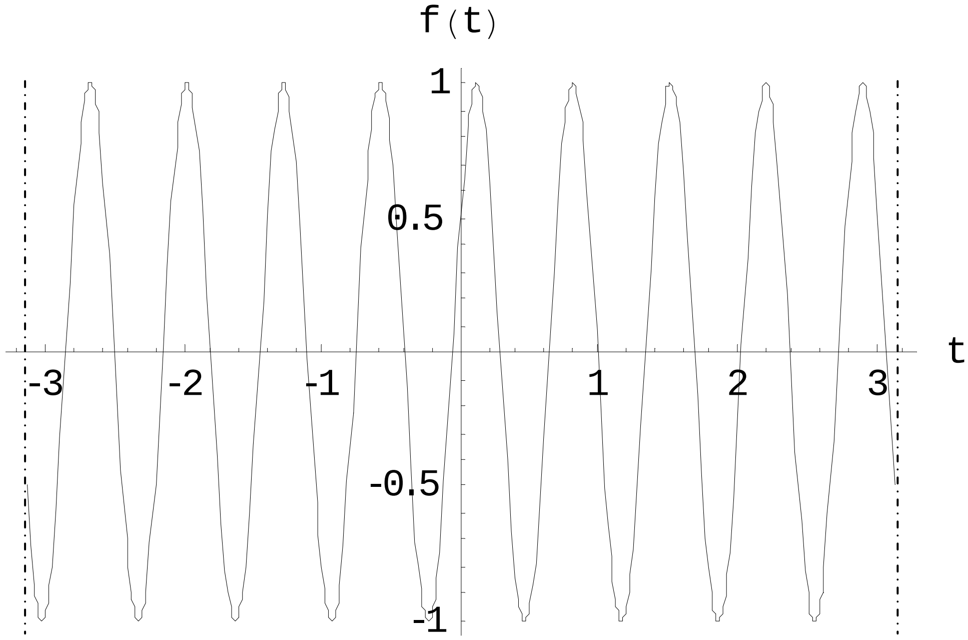
$$f(t) = \cos(9t - \frac{\pi}{3}), -\pi < t < \pi, \quad f(t) = f(t + 2\pi)$$



Given $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-\text{int}}$, find $c_n = ?(n = -\infty, \ldots. 0. \ldots, \infty)$

Find the Complex Fourier series of

$$f(t) = \cos(9t - \frac{\pi}{3}), -\pi < t < \pi, \quad f(t) = f(t + 2\pi)$$



(1).Given $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, find $c_n = ?(n = -\infty, \ldots . 0 . \ldots, \infty)$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(9t - \frac{\pi}{3})e^{-int}dt$$

$$c_9 = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(9t - \frac{\pi}{3})e^{-9ti}dt = \frac{1}{2\pi}(\pi\cos(\frac{\pi}{3}) - i\pi\sin(\frac{\pi}{3})) = \frac{1}{4} - \frac{\sqrt{3}}{4}i$$

$$c_{-9} = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(9t - \frac{\pi}{3})e^{9ti}dt = \frac{1}{2\pi}(\pi\cos(\frac{\pi}{3}) + i\pi\sin(\frac{\pi}{3})) = \frac{1}{4} + \frac{\sqrt{3}}{4}i$$

$$f(t) = c_9 e^{9it} + c_{-9}e^{-9it} = (\frac{1}{4} - \frac{\sqrt{3}}{4}i)e^{9it} + (\frac{1}{4} + \frac{\sqrt{3}}{4}i)e^{-9it} = \frac{1}{2}\cos(9t) + \frac{\sqrt{3}}{2}\sin(9t)$$

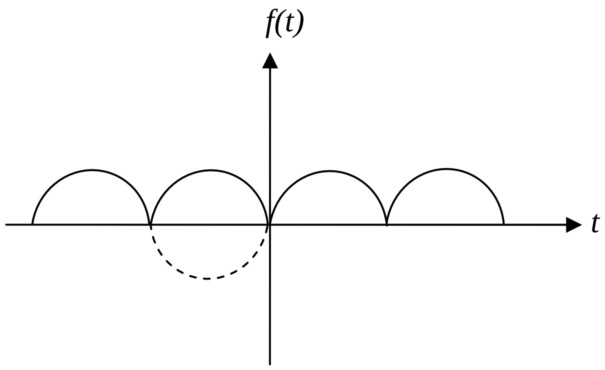
(2).另解

$$\begin{aligned} f(t) &= \cos(9t - \frac{\pi}{3}) \\ &= \cos(9t)\cos(\frac{\pi}{3}) + \sin(9t)\sin(\frac{\pi}{3}) \\ &= \frac{1}{2}\frac{1}{2}(e^{i9t} + e^{-i9t}) + \frac{\sqrt{3}}{2}\left(\frac{-i}{2}\right)(e^{i9t} - e^{-i9t}) \\ &= \left(\frac{1}{4} - \frac{\sqrt{3}i}{4}\right)e^{i9t} + \left(\frac{1}{4} + \frac{\sqrt{3}i}{4}\right)e^{-i9t} \\ &= c_9 e^{i9t} + c_{-9}e^{-i9t} \end{aligned}$$

國立臺灣海洋大學　　　　河海工程學系 2B　　　　工程數學(二)

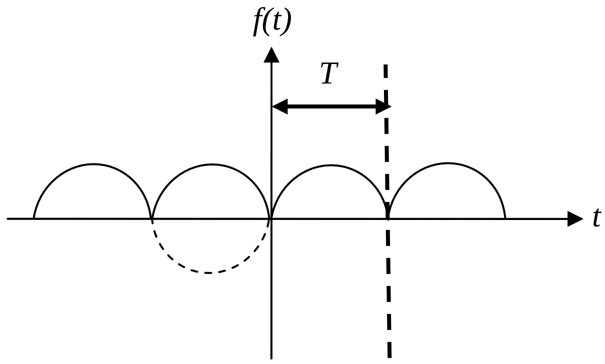
日期:2008 年五 5 月 5 日　姓名:__________ 學號:__________

Find the complex Fourier coefficient of the follow $f(t) = |sin(\lambda t)|$



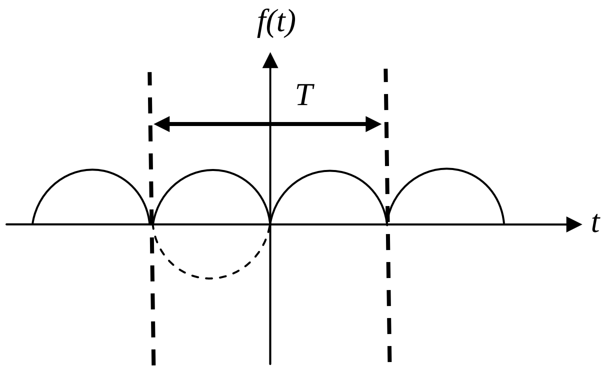
(1) 單數號的同學把週期看成 π/λ

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{2in\lambda t}$$



(2) 雙數號的同學把週期看成 $2\pi/\lambda$

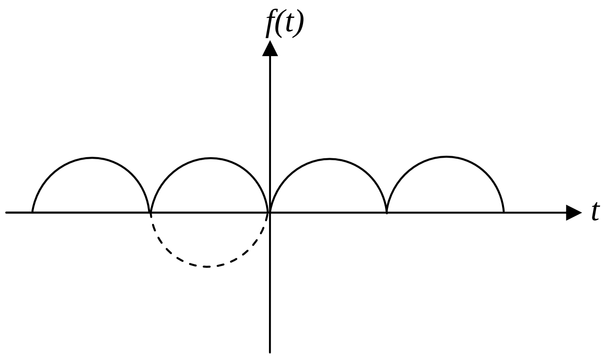
$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{in\lambda t}$$



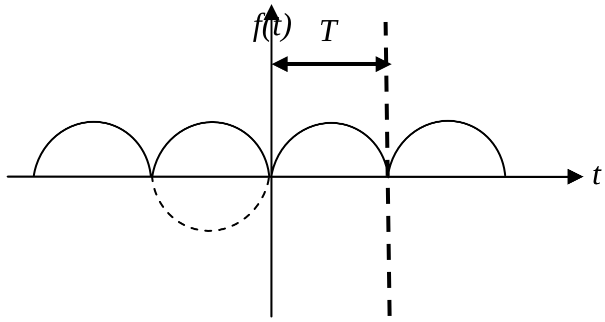
(3) C_n 和 d_n 有關嗎?(魯蛋評論)

日期:2008 年五 5 月 5 日　姓名:________________ 學號:________________

Find the complex Fourier coefficient of the follow $f(t)=|sin(\lambda t)|$



(1) 單數號的同學把週期看成 π/λ



$$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{2in\lambda t}$$

解一：

$$C_n = Re\left\{\frac{\lambda}{\pi}\cdot 2\int_0^{\frac{\pi}{2\lambda}} sin(\lambda t)e^{-2in\lambda t}dt\right\}$$

$$=\frac{2\lambda}{2\pi i}\int_0^{\frac{\pi}{2\lambda}}(e^{i\lambda t}-e^{-i\lambda t})e^{-2in\lambda t}dt$$

$$=\frac{2\lambda}{2\pi i}\int_0^{\frac{\pi}{2\lambda}}(e^{(1-2n)i\lambda t}-e^{-(1+2n)i\lambda t})dt$$

$$=\frac{2}{2\pi i}\left(\left.\frac{e^{(1-2n)i\lambda t}}{(1-2n)i}\right|_0^{\frac{\pi}{2\lambda}}+\left.\frac{e^{-(1+2n)i\lambda t}}{(1+2n)i}\right|_0^{\frac{\pi}{2\lambda}}\right)$$

$$=\frac{-1}{\pi}\left(\frac{-1}{(1-2n)}+\frac{-1}{(1+2n)}\right)$$

$$=\frac{2}{\pi}\frac{1}{1-4n^2}$$

解二：

$$C_n=\frac{\lambda}{\pi}\int_0^{\frac{\pi}{\lambda}} sin(\lambda t)cos(2n\lambda t)dt$$

$$=\frac{\lambda}{2\pi}\left[\int_0^{\frac{\pi}{\lambda}} sin(1+2n)\lambda t+sin(1-2n)\lambda t dt\right]$$

$$=\frac{\lambda}{2\pi}\left[\frac{-cos(1+2n)\lambda t}{(1+2n)\lambda}-\frac{cos(1-2n)\lambda t}{(1-2n)\lambda}\bigg|_0^{\frac{\pi}{\lambda}}\right]$$

$$=\frac{2cos^2(n\pi)}{\pi-4n^2\pi}$$

$$=\frac{2}{\pi}\frac{1}{1-4n^2}$$

解三：

$$C_n=\frac{\lambda}{\pi}\int_0^{\frac{\pi}{\lambda}} sin(\lambda t)e^{-2in\lambda t}dt$$

$$=\frac{\lambda}{\pi}\left[\frac{-1}{\lambda}cos(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}-\int_0^{\frac{\pi}{\lambda}}2in\cdot cos(\lambda t)e^{-2in\lambda t}dt\right]$$

$$=\frac{\lambda}{\pi}\left[\frac{-1}{\lambda}cos(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}-\left[\frac{2in}{\lambda}sin(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}-\int_0^{\frac{\pi}{\lambda}}4n^2\cdot sin(\lambda t)e^{-2in\lambda t}dt\right]\right]$$

移項

$$\Rightarrow\frac{\lambda}{\pi}(1-4n^2)\int_0^{\frac{\pi}{\lambda}} sin(\lambda t)e^{-2in\lambda t}dt$$

$$=\frac{\lambda}{\pi}\left[\frac{-1}{\lambda}cos(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}-\frac{2in}{\lambda}sin(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}\right]$$

同除$(1-4n^2)$

$$\Rightarrow C_n=\frac{\lambda}{\pi}\int_0^{\frac{\pi}{\lambda}} sin(\lambda t)e^{-2in\lambda t}dt$$

$$=\frac{1}{(1-4n^2)}\frac{\lambda}{\pi}\left[\frac{-1}{\lambda}cos(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}-\frac{2in}{\lambda}sin(\lambda t)e^{-2in\lambda t}\bigg|_0^{\frac{\pi}{\lambda}}\right]$$

$$=\frac{2}{(1-4n^2)}\frac{1}{\pi}$$

解四：

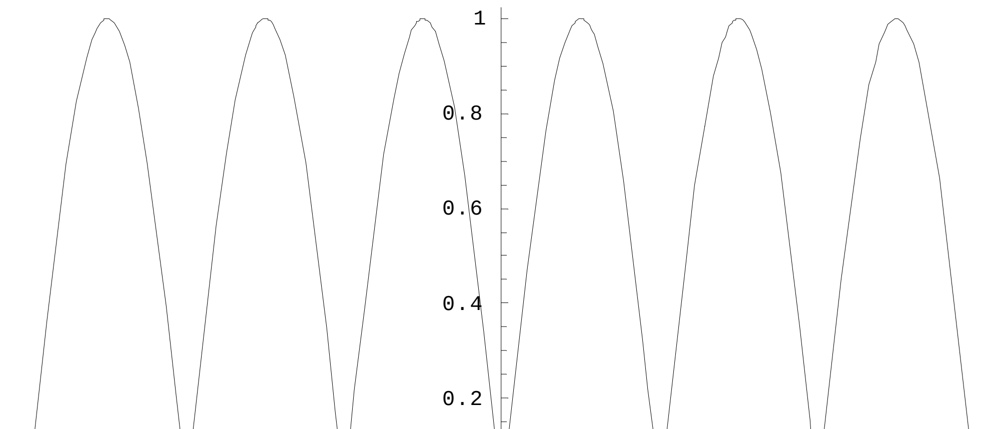
$$C_n = \frac{\lambda}{\pi}\int_0^{\frac{\pi}{\lambda}} sin(\lambda t)e^{-2in\lambda t}dt$$
$$= \frac{\lambda}{2\pi i}\int_0^{\frac{\pi}{\lambda}}(e^{i\lambda t} - e^{-i\lambda t})e^{-2in\lambda t}dt$$
$$= \frac{\lambda}{2\pi i}\left[\int_0^{\frac{\pi}{\lambda}} e^{(1-2n)i\lambda t}dt - \int_0^{\frac{\pi}{\lambda}} e^{-(1+2n)i\lambda t}dt\right]$$
$$= \frac{\lambda}{2\pi i}\left[\frac{1}{(1-2n)i\lambda}(e^{(1-2n)i\pi} - 1) + \frac{1}{(1+2n)i\lambda}(e^{-(1+2n)i\pi} - 1)\right]$$
$$= \frac{1}{2\pi}\left[\frac{1}{1-2n} - \frac{e^{(1-2n)i\pi}}{1-2n} + \frac{1}{1+2n} - \frac{e^{-(1+2n)i\pi}}{1+2n}\right]$$

其中

$$e^{(1-2n)i\pi} = cos(1-2n)\pi + i\,sin(1-2n)\pi = -1$$
$$e^{-(1+2n)i\pi} = cos(1+2n)\pi - i\,sin(1+2n)\pi = -1$$
$$\Rightarrow C_n = \frac{1}{2\pi}\left[\frac{1}{1-2n} + \frac{1}{1-2n} + \frac{1}{1+2n} + \frac{1}{1+2n}\right]$$
$$= \frac{2}{\pi}\frac{1}{1-4n^2}$$

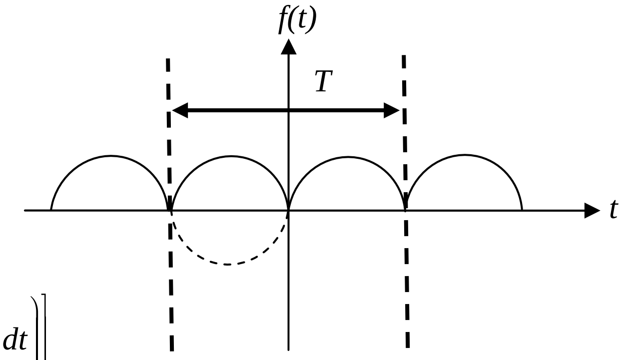
Mathematica 指令與圖形

單數號 → $\text{Plot}\left[\sum_{n=-100}^{100} \frac{2}{\pi * (1 - 4 * n^2)} * e^{2*i*t*n}, \{t, -3\pi, 3\pi\}\right]$



【 工程數學(二)　補救教學第八次課堂小考　小老師-李家瑋 】　謝祥志製
小江改

⑵ 雙數號的同學把週期看成 $2\pi/\lambda$



$$f(t)=\sum_{n=-\infty}^{\infty} d_n e^{in\lambda t}$$

Sol:

$$D_n=\frac{\lambda}{2\pi}\left[\int_0^{\frac{\pi}{\lambda}} sin(\lambda t)e^{-in\lambda t}dt-\left(\int_{-\frac{\pi}{\lambda}}^{0} sin(\lambda t)e^{-in\lambda t}dt\right)\right]$$

$$=\frac{\lambda}{4\pi i}\left[\int_0^{\frac{\pi}{\lambda}}\left(e^{i\lambda t}-e^{-i\lambda t}\right)e^{-in\lambda t}dt-\left(\int_{-\frac{\pi}{\lambda}}^{0}\left(e^{i\lambda t}-e^{-i\lambda t}\right)e^{-in\lambda t}dt\right)\right]$$

$$=\frac{\lambda}{4\pi i}\left[\int_0^{\frac{\pi}{\lambda}} e^{(1-n)i\lambda t}dt-\int_0^{\frac{\pi}{\lambda}} e^{-(1+n)i\lambda t}dt-\left(\int_{-\frac{\pi}{\lambda}}^{0} e^{(1-n)i\lambda t}dt-\int_{-\frac{\pi}{\lambda}}^{0} e^{-(1+n)i\lambda t}dt\right)\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{1-n}+\frac{1}{1+n}+\frac{(-1)^n}{1-n}+\frac{(-1)^n}{1+n}\right]$$

$$=\frac{1+(-1)^n}{\left(1-n^2\right)\pi}$$

其中

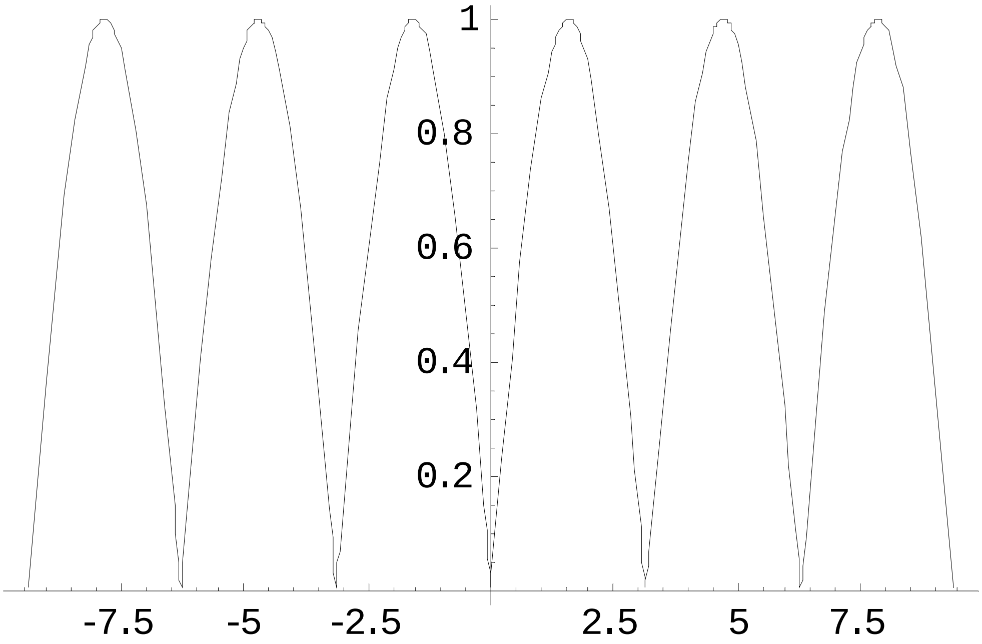
$$e^{(1-n)i\pi}=cos(1-n)\pi+i\,sin(1-n)\pi=\left(-1\right)^{1-n}$$

$$e^{-(1+n)i\pi}=cos(1+n)\pi-i\,sin(1+n)\pi=\left(-1\right)^{1+n}$$

D_n 也可如上列 C_n 之求解方法進行求解

Mathematica 指令與圖形

雙數號 → $\text{Plot}\left[\sum_{n=-100}^{-2}\frac{1+\text{Cos}[n*\pi]}{\pi-n^2*\pi}*e^{i*t*n}+\sum_{n=2}^{100}\frac{1+\text{Cos}[n*\pi]}{\pi-n^2*\pi}*e^{i*t*n}+\frac{2}{\pi},\{t,-3\pi,3\pi\}\right]$



(3) C_n 和 d_n 有關嗎?(魯蛋評論)

Sol: 比較 C_n 與 D_m

$$f(t)=\sum_{n=-\infty}^{\infty}C_n e^{i(2n)\lambda t},\ where\ \ C_n=\frac{2}{\pi}\frac{1}{1-4n^2}..........(1)$$

$$f(t)=\sum_{m=-\infty}^{\infty}D_m e^{im\lambda t},\ where\ \ D_m=\frac{1+(-1)^m}{\left(1-m^2\right)\pi}...........(2)$$

由(1)(2)可知

$\Rightarrow$(1)式基底: $e^0, e^{2i\lambda t}, e^{4i\lambda t}$......

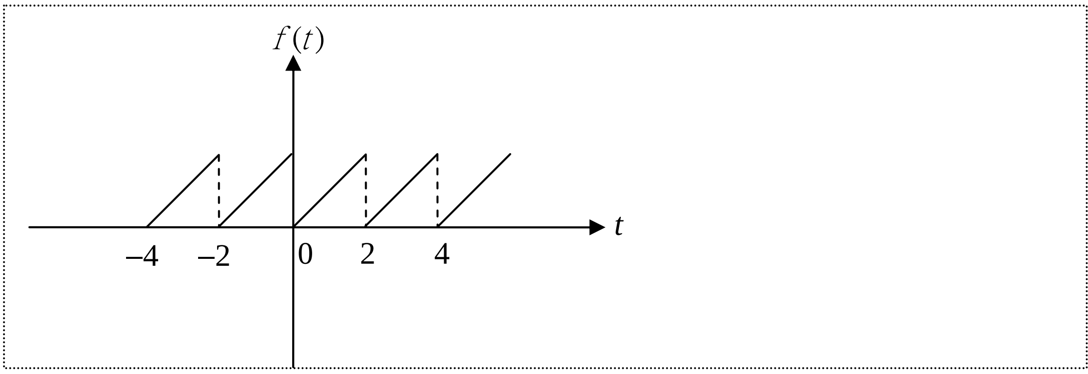
(2)式基底: $e^0, e^{i\lambda t}, e^{2i\lambda t}, e^{3i\lambda t}$........

其中 $\begin{cases} D_{2k} = \dfrac{2}{(1-4k^2)}\dfrac{1}{\pi} \\ D_{2k+1} = 0 \end{cases}$,可得知（2）式中奇數項係數爲0

進而得知（1）（2）式，爲等價

亦可視成 $m=2k$ 時，$D_{2k}=\dfrac{2}{(1-4k^2)}\dfrac{1}{\pi}e^{i(2k)t} \Leftrightarrow \dfrac{2}{(1-4n^2)}\dfrac{1}{\pi}e^{i(2n)t}=C_n$

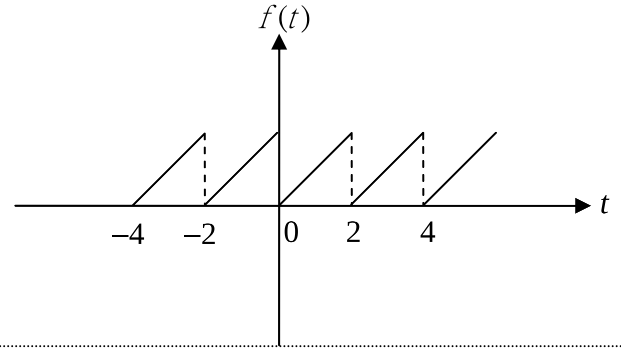
$f(t)=2t\,,0\le t\le 2\quad f(t+2)=f(t)\quad T=2=2p$



請以複數的傅立葉級數展開之。

($f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\lambda t}$)

$f(t)=2t\,,0\le t\le 2 \quad f(t+2)=f(t) \quad T=2=2p$



請以複數的傅立葉級數展開之。

($f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\lambda t}$)

sol

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi t}$$

$$=\frac{1}{2p}\int_0^{2p} f(t)e^{-in\pi t}dt$$

$$=\frac{1}{2}\int_0^{2} 2te^{-in\pi t}dt$$

$$=-\frac{t}{in\pi}e^{-\text{in}\pi\text{t}}\Big|_0^2+\left[\int_0^2\frac{1}{\text{in}\pi}e^{-in\pi t}dt\right]$$

$$=-\frac{t}{in\pi}e^{-\text{in}\pi\text{t}}\Big|_0^2+\left[\frac{-1}{\text{n}^2\pi^2}e^{-in\pi\text{t}}\right]_0^2$$

$$=-\frac{2}{in\pi}e^{-2\text{in}\pi}+\left[\frac{-1}{\text{n}^2\pi^2}e^{-2in\pi}-(\frac{-1}{n^2\pi^2})\right]$$

$$=-\frac{2}{in\pi}e^{-2\text{in}\pi}+\frac{1}{\text{n}^2\pi^2}e^{-2\text{in}\pi}-\frac{1}{n^2\pi^2}$$

$$=\frac{2}{in\pi}(-1)^{2\text{n}}+\frac{1}{\text{n}^2\pi^2}(-1)^{2\text{n}}-\frac{1}{n^2\pi^2}$$

$$=-\frac{2}{in\pi}=\frac{2i}{n\pi}$$

$$\text{c}_0=\frac{1}{2p}\int_0^{2p} f(t)dt=\frac{1}{2}\int_0^2 2tdt=2$$

$$\text{f(t)}=\text{c}_0+\sum_{n=-\infty}^{\infty} c_n e^{in\pi t}=2+\sum_{n=-\infty}^{\infty}\frac{2i}{n\pi}e^{in\pi t}$$

第三章 傳立葉積分與轉換

3.1 Fourier integral and Fourier transform 大綱------------------------178

3.2 Fourier integral

譜的定義---179

From Fourier series to Fourier integral------------------------------------180

From Fourier series to Fourier integral 之魯蛋講義-------------------------181

3.3 Fourier transform

Fourier transform-Fourier 積分式-----------------------------------182

Fourier Transform---183

Convolution 褶積---184

Comparsions of time domain and frequency domain approaches---------------188

Parseval's theorem--189

工數補救教學 Energy form--190

Some interesting properties of operators-------------------------------196

因果函數 Fourier 轉換特性---197

Time Domain versus Frequency Domain------------------------------------198

Constraints in Frequency Domain---------------------------------------199

Geometric Meaning for Fourier Transform--------------------------------200

Example for Fourier transform---201

Properties of Fourier transform---------------------------------------203

Table of Fourier integral---204

Gibbs phenomenon by Fourier transform----------------------------------205

Fourier transform-Fourier 轉換--206

3.4 小考

小考與解答--207

3.5 大考

大考與解答--211

Fourier Integral and Fourier Transform

- Fourier integral

Introduction 何謂譜
From Fourier series to Fourier integral(魯蛋版)

- Fourier transform

	Fourier transform(ii)	
	Fourier Transform(ppt)	
摺積	Convolution(ppt file) convolution 動畫 correlation 動畫	影片區(youtube) Convolution
	Fourier transform(iii)	
Parseval's 定理	Parseval's theorem	
	Energy form(雅鈞)	
	Some interesting properties of operators	
Hilbert 轉換	因果函數 Fourier 轉換特性	影片區(youtube) Hilbert transform
	Hilbert transform 1	
	Hilbert transform 2	
	Geometric Meaning for Fourier Transform	
例題	Example for Fourier Transform	
	Properties of Fourier transform	
	Table of Fourier integral	
Gibbs 現象	Gibbs phenomenon by Fourier transform	
	Fourier transform(i)	

Fourier integral and Fourier transform

海大河海系　　陳正宗

From Fourier series to Fourier integral

Gibbs phenomenon

Properties of Fourier transform

Applications of Fourier transform

Singular function and its Fourier transform

Fourier transform to Laplace transform

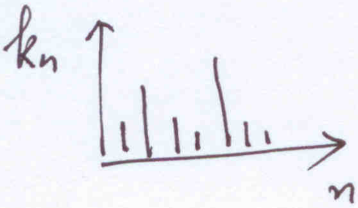
【存檔：c:/ctex/course/math2/fint1.te】【建檔:Mar./3/'97】

列印 列印

	【譜】言-12-19
注音	ㄆㄨˇ　　　　祖譜
解釋	(1)記載人物事蹟圖文的文獻冊子。如:「家譜」、「圖譜」。 (2)記錄樂曲旋律符號的表式。如:「樂譜」、「歌譜」。 (3)記錄一些準則與範例,讓人模仿學習的書冊。如:「書譜」、「棋譜」、「食譜」。 (4)大致的規矩或原則。如:「沒個譜兒」、「這話愈說愈離譜。」、「經你一說明,我心裡大概有個譜了。」 (5)按歌詞編寫樂曲。如:「譜曲」。　　臉譜
附圖	點選開啟全文檔

Fourier series 要叙的是

波(數)譜 與 頻(率)譜



April 30/2008 J. T. Chen

From Fourier series to Fourier integral

海大河海系　　陳正宗

試由複數 Fourier 展開式導出複數型 Fourier 積分。

解　設 $f(x)$ 為週期 T 之週期函數，則其複數型 Fourier 級數為

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}, \quad 其中\ c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\, e^{-i\omega_n \tau}\, d\tau, \quad \omega_n = \frac{2n\pi}{T}$$

將 c_n 併入 $f(x)$ 內，得 $f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau)\, e^{-i\omega_n(\tau - x)}\, d\tau$ .... ①

而由 $\omega_n = \frac{2n\pi}{T}$ 可得

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{2(n+1)\pi}{T} - \frac{2n\pi}{T} = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\Delta\omega}{2\pi}$$

當 $T \to \infty$ 時, $\Delta\omega \to d\omega$, $\omega_n \to \omega$, 代入①式，得

$$\begin{aligned}
f(x) &= \lim_{p\to\infty} \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\tau) e^{-i\omega_n(\tau-x)}\, d\tau \\
&= \int_{-\infty}^{\infty} \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau) e^{-i\omega(\tau-x)}\, d\tau\Big]\, d\omega \quad \text{Riemann sum to integral} \\
&= \int_{-\infty}^{\infty} \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau}\, d\tau\Big]\, e^{i\omega x}\, d\omega \\
&= \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega
\end{aligned}$$

其中 $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$

Fourier transform:

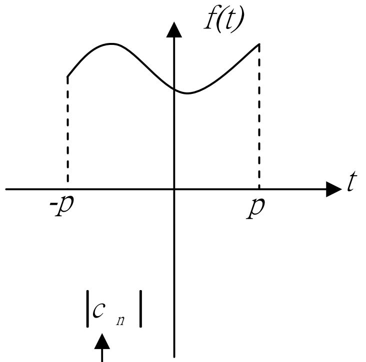
$$F(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

Inverse Fourier transform:

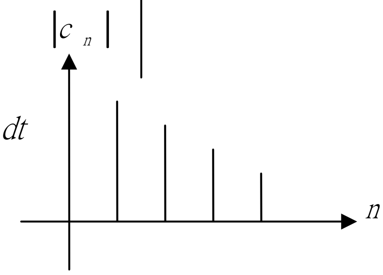
$$F_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau) e^{i\omega\tau}\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$$

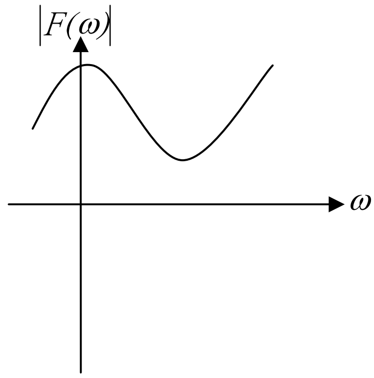
Fourier series -> Fourier integral

$T=2p$



$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{i(\frac{n\pi}{p})t}$$





$$c_n=\frac{1}{2p}\int_{-p}^{p} f(t)e^{-i(\frac{n\pi}{p})t}dt$$

把 c_n 代入原式

$$\Rightarrow f(t)=\sum_{n=-\infty}^{\infty}\left[\frac{1}{2p}\int_{-p}^{p} f(\tau)e^{-i(\frac{n\pi}{p})\tau}d\tau\right]e^{i(\frac{n\pi}{p})t}$$

現在若 $f(t)$ 在 $(-\infty,\infty)$ 都有定義，且不是一個週期函數了。爲了處理此種情況，我們可以將 $f(t)$ 視爲一個週期函數，只不過 $p\rightarrow\infty$。也就是整個週期 $T=2p$ 在 $(-\infty,\infty)$ 區間。(解放週期)

令 $\omega_n=\dfrac{n\pi}{p}$ ，$\omega_{n+1}=\dfrac{(n+1)\pi}{p}$ $\Delta\omega=\omega_{n+1}-\omega_n=\dfrac{\pi}{p}(\dfrac{1}{p}=\dfrac{\Delta\omega}{\pi})$

$$\Rightarrow f(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-p}^{p} f(\tau)e^{-i\omega_n\tau}d\tau\right]e^{i\omega_n t}\Delta W$$

$$\Rightarrow \lim_{p\rightarrow\infty} f(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)e^{-i\omega_n\tau}d\tau\right]e^{i\omega_n t}\Delta\omega$$

$$\Rightarrow f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau\right]e^{i\omega t}d\omega$$ Fourier integral

$$\therefore F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$ Fourier transform

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$ inverse Fourier transform

$$\int_{-\infty}^{\infty} F(\omega)d\omega=\sum_{n=-\infty}^{\infty} F(\omega_n)\Delta\omega$$ Riemann 和化積分

Fourier transform

海大河海系　　　陳正宗

Fourier 積分式

Fourier 積分是非週期性函數的 Fourier 展開式。當非週期性函數 $f(x)$ 為分段連續，且 $\int_{-} |f(x)|\, dx$ 存在時

$$\int_{0} [\, A(\omega)\cos\omega x + B(\omega)\sin\omega x \,]\, d\omega = \begin{cases} f(x) & x \text{ 為連續點} \\ \frac{1}{2}[f(x^{+}) + f(x^{-})] & x \text{ 為跳躍點} \end{cases} \tag{1}$$

其中 $A(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-} f(x)\cos\omega x\, dx,\ B(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-} f(x)\sin\omega x\, dx,$ 此稱為 Fourier 全三角積分式。

I. 如 $f(x)$ 具對稱性，則 $A(\omega)$ 與 $B(\omega)$ 中有一個為零。

(1) $f(x) = f(-x)$ 時，得 $B(\omega) = 0$

$$\Rightarrow f(x) = \int_{0} A(\omega)\cos\omega x\, d\omega, \quad \text{其中 } A(\omega) = \frac{2}{\pi}\int_{0} f(x)\cos\omega x\, dx$$

此稱為 Fourier 餘弦積分。

(2) $f(x) = -f(-x)$ 時，得 $A(\omega) = 0$

$$\Rightarrow f(x) = \int_{0} B(\omega)\sin\omega x\, d\omega, \quad \text{其中 } B(\omega) = \frac{2}{\pi}\int_{0} f(x)\sin\omega x\, dx$$

此稱為 Fourier 正弦積分。

II. 複數型 Fourier 積分：

$$f(x) = \int_{-} F(\omega)\, e^{i\omega x}\, d\omega, \quad \text{其中 } F(\omega) = \frac{1}{2\pi}\int_{-} f(x)\, e^{-i\omega x}\, dx \tag{2}$$

III. Fourier 積分之 Parseval's 恆等式：

$$\text{實數型：} \frac{1}{\pi}\int_{-} f^{2}(x)\, dx = \int_{0} [\, A^{2}(\omega) + B^{2}(\omega) \,]\, d\omega \tag{3}$$

$$\text{複數型：} \frac{1}{2\pi}\int_{-} f^{2}(x)\, dx = \int_{-} |F(\omega)|^{2}\, d\omega \tag{4}$$

IV. Fourier 級數展開與 Fourier 積分之不同在於：

(1) 前者是對週期函數，後者是對無週期或只有一週期之函數；

(2) 前者是斷續頻相加，後者是連續頻相加。

Fourier Transform

- Suppose $\int_{-\infty}^{\infty} |f(x)| dx$ converges. Then the Fourier transform of f is defined to be the function

$$\mathfrak{F}[f](\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

- Let a be a positive constant. Then

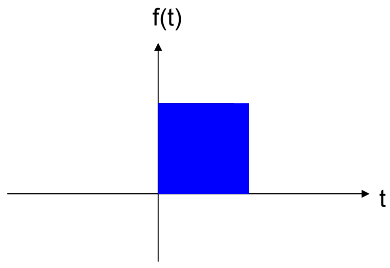
$$\mathfrak{F}\left[e^{-a|t|}\right](\omega) = \frac{2a}{a^2 + \omega^2}$$

Convolution褶積

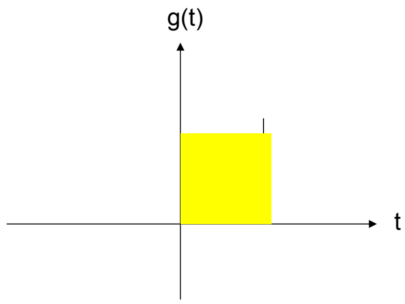
- t在不同範圍f(t)和g(t)所形成的圖
- 以及t在不同範圍h(t)所形成的圖

$$h(t)=\int_{-\infty}^{\infty} f(u)g(t-u)du$$

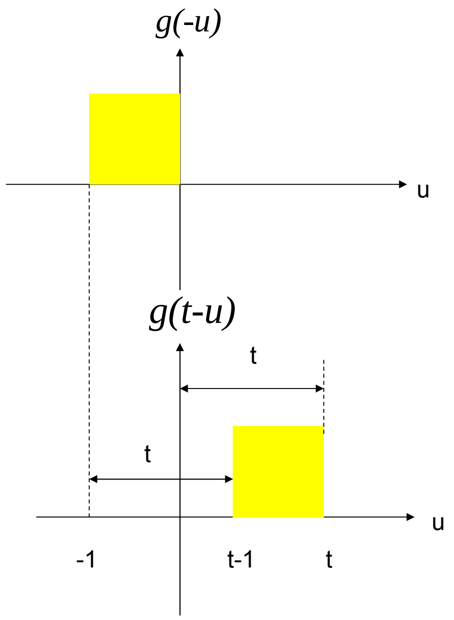
•



$$f(t)=\begin{cases}1 & , 0 \le t \le 1 \\ 0 & , otherwise\end{cases}$$

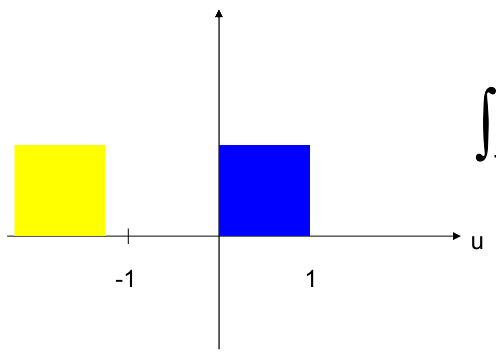


$$g(t)=\begin{cases}1 & , 0 \le t \le 1 \\ 0 & , otherwise\end{cases}$$



$$\int_{-\infty}^{\infty} f(u)g(t-u)du \quad , t<0$$

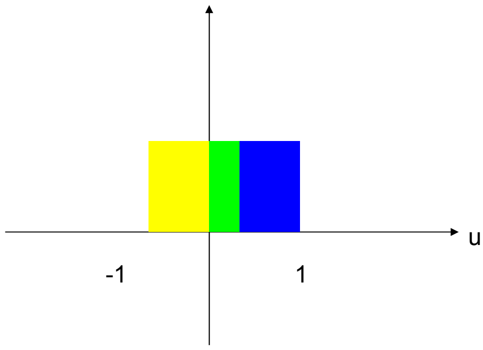
•



$$\int_{-\infty}^{\infty} f(u)g(t-u)du=0$$

$$\int_{-\infty}^{\infty} f(u)g(t-u)du \;,\; 0 \le t \le 1$$

•



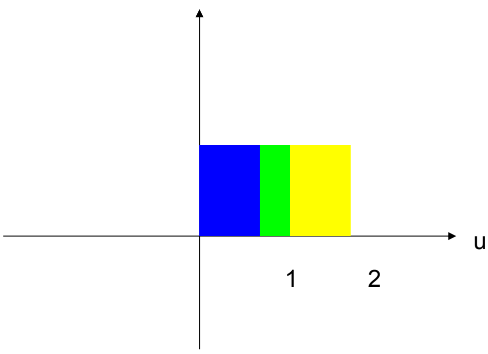
$$\int_{-\infty}^{\infty} f(u)g(t-u)du$$

$$= \int_{0}^{t} 1\, dt$$

$$= t$$

$$\int_{-\infty}^{\infty} f(u)g(t-u)du \;,\; 1 \le t \le 2$$

•



$$\int_{-\infty}^{\infty} f(u)g(t-u)du$$

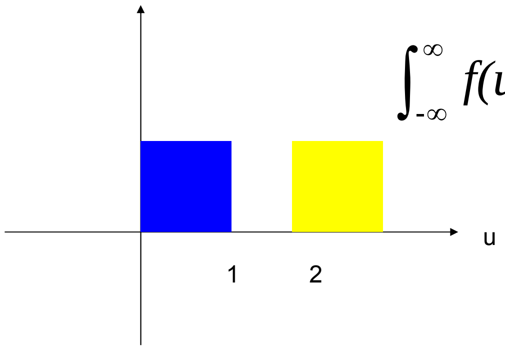
$$= \int_{t-1}^{1} 1\, du$$

$$= 1-(t-1)$$

$$= 2-t$$

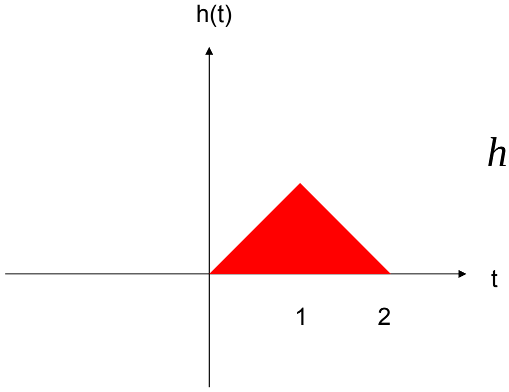
$$\int_{-\infty}^{\infty} f(u)g(t-u)du \ , t>2$$

•



$$\int_{-\infty}^{\infty} f(u)g(t-u)du=0$$

$$h(t)=\int_{-\infty}^{\infty} f(u)g(t-u)du$$



$$h(t)=\begin{cases} t \ , 0 \le t<1 \\ 2-t \ , 1 \le t<2 \\ 0 \ , t<0 \ \& \ t>2 \end{cases}$$

Table 1: Comparsions of time domain and frequency domain approaches

time domain	frequency domain
$x(t), p(t)$	$\bar{X}(\bar{\omega}) = \mathcal{F}\{x(t)\}, \bar{P}(\bar{\omega}) = \mathcal{F}\{p(t)\}$
$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = p(t)$	$(-\bar{\omega}^2 + 2\xi\omega\bar{\omega}i + \omega^2)\bar{X} = \bar{P}$
$h(t), \delta(t)$	$H(\bar{\omega}, \omega) = \mathcal{F}\{h(t)\}, 1 = \mathcal{F}\{\delta(t)\}$
$\ddot{h} + 2\xi\omega\dot{h} + \omega^2 h = \delta(t)$ (implulse function)	$(-\bar{\omega}^2 + 2\xi\omega\bar{\omega}i + \omega^2)\bar{X} = 1$ $\bar{X} = H(\bar{\omega}, \omega) = \frac{1}{(\omega^2 - \bar{\omega}^2 + i2\xi\omega\bar{\omega})}$
$\ddot{x}_r + 2\xi\omega\dot{x}_r + \omega^2 x_r = cos(\alpha t)$	$\bar{X}/H(\bar{\omega}, \omega) = \frac{1}{2}(\delta(\bar{\omega} - \alpha) + \delta(\bar{\omega} + \alpha))$ $x_r = \frac{1}{2}\{H(\alpha, \omega)e^{i\alpha t} + H(-\alpha, \omega)e^{-i\alpha t}\} = Re\{H(\alpha, \omega)e^{i\alpha t}\}$
$\ddot{x}_i + 2\xi\omega\dot{x}_i + \omega^2 x_i = sin(\alpha t)$	$\bar{x}/H(\bar{\omega}, \omega) = \frac{1}{2}(\delta(\bar{\omega} - \alpha) - \delta(\bar{\omega} + \alpha))$ $x_i = \frac{1}{2}\{H(\alpha, \omega)e^{i\alpha t} - H(-\alpha, \omega)e^{-i\alpha t}\} = Im\{H(\alpha, \omega)e^{i\alpha t}\}$

Fourier transform:

$$\bar{X}(\bar{\omega}) = \int_{-\infty}^{\infty} x(t)e^{-i\bar{\omega}t}dt$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{X}(\bar{\omega})e^{i\bar{\omega}t}d\bar{\omega}$$

Find $h(t)$ by contour integration:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{H}(\bar{\omega}, \omega)e^{i\bar{\omega}t}d\bar{\omega}$$

case.1: $H(\bar{\omega}, \omega) = \frac{1}{(\omega^2 - \bar{\omega}^2 + i2\xi\omega\bar{\omega})}$

case.2: $H(\bar{\omega}, \omega) = \frac{1}{(\omega^2 - \bar{\omega}^2)}, \xi = 0$

case.3: $H(\bar{\omega}, \omega) = \frac{1}{(-\bar{\omega}^2 + \omega^2(1 \pm i\eta))}, +\ if\ \bar{\omega} > 0, -\ if\ \bar{\omega} < 0$

case.4: $H(\bar{\omega}, \omega) = \frac{1}{(-\bar{\omega}^2 + \omega^2)}, \eta = 0$

case.5: $H(\bar{\omega}, \omega) = \frac{1}{i\bar{\omega}}$, or $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{it}e^{i\bar{\omega}t}dt = ?$

Can the solutions of (2) and (4) be derived by the limiting process ?

Parseval's theorem

海大河海系　　陳正宗

Given

$$F(\omega) = \int_{-} f(t)e^{-i\omega t}dt, \;\; G(\omega) = \int_{-} g(t)e^{-i\omega t}dt$$

$$f(t) * g(t) = \int_{-} f(u)g(t-u)du$$

Prove that

$$\mathcal{F}\{f(t) * g(t)\} = F(\omega)G(\omega)$$

Proof:

$$\int_{-} f(t)e^{-i\omega t}dt \int_{-} g(\tau)e^{-i\omega\tau}d\tau = \int_{-}\int_{-} f(t)g(\tau)e^{-i\omega(t+\tau)}dtd\tau$$

By changing the variables, we have $(t,\tau) \to (t,u)$,

$$t+\tau = u, \;\; t = t$$

Therefore,

$$dudt = J(u,t;\tau,t)d\tau dt$$

where Jacobian $J = 1$. We have

$$\int_{-}\int_{-} f(t)g(u-t)e^{-i\omega u}dtdu = \int_{-} \{\int_{-} f(t)g(u-t)dt\}e^{-i\omega u}du$$

That is

$$\int_{-} \{\int_{-} f(t)g(u-t)dt\}e^{-i\omega u}du = \int_{-} \{f(u) * g(u)\}e^{-i\omega u}du$$

Therefore,

$$F(\omega)G(\omega) = \mathcal{F}\{f * g\}$$

By choosing special case, we have

$$f(t) = f(t)$$

$$g(t) = f(-t)$$

$$f * g = \int_{-} f(u)g(t-u)du = \int_{-} f(u)f(-t+u)du$$

$$F(\omega)F(-\omega) = | F(\omega) |^2$$

$$\frac{1}{2\pi}\int | F(\omega) |^2 e^{i\omega\tau}d\omega = \int_{-} f(u)f(-\tau+u)du$$

By choosing special case, $\tau = 0$, we have

$$\frac{1}{2\pi}\int | F(\omega) |^2 d\omega = \int f^2(u)du$$

工數補救教學
Energy form

投影片製作：亞軍
指導學長：班主任&魯蛋
指導教授：陳正宗終身特聘教授
時間：2008/06/16

週期=2T

$$f(t)= a_0+\sum_{n=1}^{\infty} a_n cos(\frac{n\pi t}{T})+b_n sin(\frac{n\pi t}{T})$$

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi t}{T}}$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega, F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$$

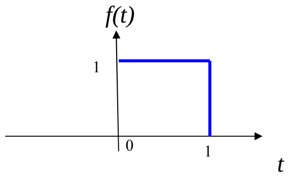
$$\int_{-T}^{T} f^2(t)\,dt =$$

$$\begin{cases}
(1)\ f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \\
\qquad \xrightarrow{period=2T} \int_{-T}^{T} f^2(t)\,dt = 2T a_0^2 + T\left(\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right) \\
(2)\ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n\pi t}{T}} \\
\qquad \xrightarrow{period=2T} \int_{-T}^{T} f^2(t)\,dt = 2T \sum_{n=-\infty}^{\infty} |c_n|^2 \\
(3)\ f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\,d\omega,\ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,dt \\
\qquad \to \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega
\end{cases}$$

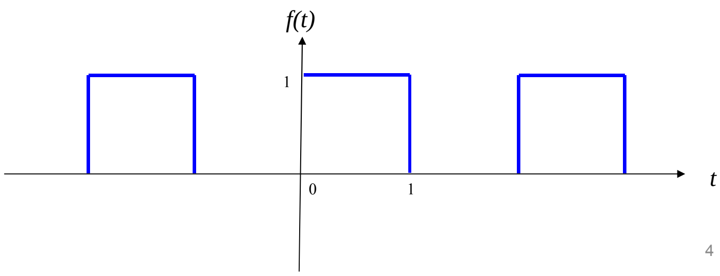
2012/2/15 3

範例

以此(週期)函數爲例
三種都算算看



⇓



2012/2/15 4

第一種

$$T = 1$$

$$a_0 = \frac{1}{2}\int_0^1 1\,dt = \frac{1}{2}$$

$$a_n = \int_0^1 1\cos(n\pi t)\,dt = 0$$

$$b_n = \int_0^1 1\sin(n\pi t)\,dt = \frac{1}{n\pi}\left((-1)^{n+1}+1\right)$$

$$\Rightarrow f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{\left((-1)^{n+1}+1\right)}{n\pi}\sin(n\pi t)$$

2012/2/15 5

$$\int_0^1 f^2(t)\,dt = 2T a_0^2 + T\left(\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right)$$

$$= 2\times 1\times\left(\frac{1}{2}\right)^2 + \sum_{n=1}^{\infty}\frac{2+2(-1)^{n+1}}{n^2\pi^2}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}$$ (n爲奇數)

2012/2/15 6

第二種

$$T=1$$

$$c_n=\frac{1}{2}\int_0^1 1e^{-in\pi t}dt$$

$$=\frac{1}{-2in\pi}\left(e^{-in\pi}-1\right)$$

$$=\frac{1}{-2in\pi}\left((-1)^n-1\right)$$

$$c_0=a_0=\frac{1}{2}$$

$$\Rightarrow f(t)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{-2in\pi}\left((-1)^n-1\right)e^{in\pi t}$$

2012/2/15 7

$$\int_{-1}^{1}f^2(t)\,dt=2T\left(c_0{}^2+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left|c_n\right|^2\right)$$ (因為 c_0 被提出來另外討論)

$$=\frac{1}{2}+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1+(-1)^{n+1}}{2n^2\pi^2}$$

$$=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{2}{n^2\pi^2}$$ (n為奇數)

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}$$ (n為奇數)

2012/2/15 8

第三種

$$F(\omega)=\frac{1}{-i\omega}(e^{-i\omega}-1)$$
$$=\frac{i}{\omega}(\cos\omega-i\sin\omega-1)$$
$$=\left(\frac{\cos\omega-1}{\omega}\right)i+\frac{\sin\omega}{\omega}$$

$$|F(\omega)|^2=F(\omega)\bar{F}(\omega)$$
$$=\left(\frac{\sin\omega}{\omega}+i\frac{\cos\omega-1}{\omega}\right)\left(\frac{\sin\omega}{\omega}-i\frac{\cos\omega-1}{\omega}\right)$$
$$=\frac{\sin^2\omega}{\omega^2}+\frac{\cos^2\omega-2\cos\omega+1}{\omega^2}$$
$$=\frac{2(1-\cos\omega)}{\omega^2}$$

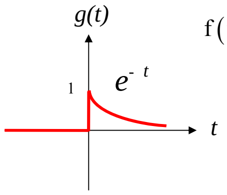
2012/2/15 9

$$\int_{-\infty}^{\infty} f^2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2(1-\cos\omega)}{\omega^2}d\omega$$
$$=?$$

就大二工數所學無法解此式的積分

2012/2/15 10

換個例題



$$f(t)=\begin{cases} e^{-t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f^2(t)dt=\int_0^{\infty} e^{-2t}dt=\frac{1}{2}$$

$$F(\omega)=\frac{1}{1+i\omega}$$

$$|F(\omega)|^2=F(\omega)\bar{F}(\omega)=\frac{1}{1+\omega^2}$$

$$\int_{-\infty}^{\infty} f^2(t)dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}d\omega$$

令 $\omega=\tan\theta$, $d\omega=\sec^2 d\theta$

$$=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{sec^2\theta}{1+tan^2\theta}d\theta$$

$$=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{sec^2\theta}{\sec^2\theta}d\theta$$

$$=\frac{1}{2}$$

驗證成功!!

2012/2/15 11

Some interesting properties of operators

①. $r^4 = 1$, $r = 1, -1, i, -i$

②. $\mathrm{FF}(f(t)) = 2\pi f(-t)$, $\mathrm{F}(f(t)) = 4\pi^2 f(t)$

where F is Fourier transform.

③. HH $(y(t)) = -y(t)$

where H is the Hilbert transform.

④. $HH\, \underset{\sim}{y} = \underset{\sim}{y}$, $H^2 = I$

where H is Householder matrix.

⑤. MM $(\cos m\theta) = -\pi^2 \dfrac{d^2}{d\theta^2}(\cos m\theta)$

where M is the integral operator of $M(s,x)$ kernel.

⑥. UU$(\cos m\theta) = -\pi^2 \iint (\cos m\theta) d\phi d\theta$

where U is the integral operator of $U(s,x)$ kernel.

⑦. $\mathrm{T}^i\mathrm{T}^e = \mathrm{UM}$, $\mathrm{L}^i\mathrm{L}^e = \mathrm{MU}$

⑧. $i^2 = -1$

⑨. $C^3 = I$

where $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is a circulant matrix.

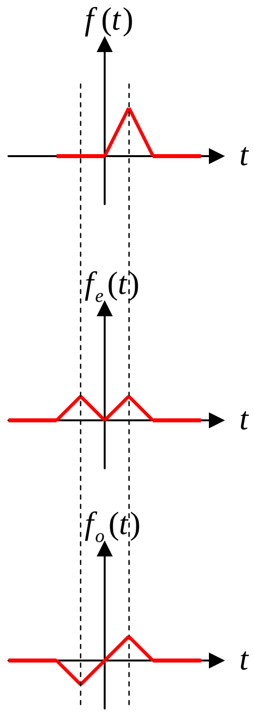
⑩. $I^2 = I$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

⑪. $J^2 = -J$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

⑫. LL $(at^2 y''(t) + bty'(t) + cy(t)) = at^2 y''(t) + bty'(t) + cy(t)$

where L is the Laplace transform.

因果函數 Fourier 轉換特性



Let $f(t)=f_e(t)+f_o(t)$ and $F(\omega)=F_R(\omega)+iF_I(\omega)$

where $f_o(t)=f_e(t)\cdot\mathrm{sgn}(t)$, $f_e(t)=f_o(t)\cdot\mathrm{sgn}(t)$,

$$F(f_e(t))=F_R(\omega),\quad F(f_o(t))=iF_I(\omega),\ \text{and}\ \ F(\mathrm{sgn}(t))=S(\omega)=\frac{-2i}{\omega}$$

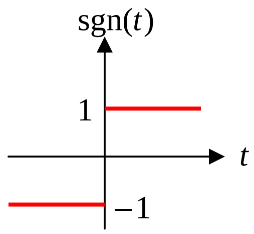
$$F(f(t)*g(t))=F(\omega)\cdot G(\omega)\qquad \Rightarrow F(F(\omega)*G(\omega))=4\pi^2 f(-t)\cdot g(-t)$$

(1) $F_R(\omega)=\displaystyle\int_{-\infty}^{\infty}\frac{F_I(u)}{\pi(\omega-u)}du$

$$f_e(t)=f_o(t)\cdot\mathrm{sgn}(t)=(-f_o(-t))\cdot(-\mathrm{sgn}(-t))=f_o(-t)\cdot\mathrm{sgn}(-t)$$

$$\begin{aligned}\Rightarrow F(iF_I(\omega)*S(\omega))&=2\pi f_o(-t)\cdot 2\pi\,\mathrm{sgn}(-t)\\&=2\pi\cdot 2\pi f_e(-t)\\&=2\pi F(F_R(\omega))\end{aligned}$$

$$\Rightarrow F_R(\omega)=\frac{1}{2\pi}(iF_I(\omega)*S(\omega))=\int_{-\infty}^{\infty}\frac{iF_I(u)}{2\pi}\cdot S(\omega-u)du=\int_{-\infty}^{\infty}\frac{iF_I(u)}{2\pi}\cdot\frac{-2i}{(\omega-u)}du=\int_{-\infty}^{\infty}\frac{F_I(u)}{\pi(\omega-u)}du$$



(2) $F_I(\omega)=\displaystyle\int_{-\infty}^{\infty}\frac{-F_R(u)}{\pi(\omega-u)}du$

$$f_o(t)=f_e(t)\cdot\mathrm{sgn}(t)=(f_e(-t))\cdot(-\mathrm{sgn}(-t))=-f_e(-t)\cdot\mathrm{sgn}(-t)$$

$$\begin{aligned}\Rightarrow F(F_R(\omega)*S(\omega))&=2\pi f_e(-t)\cdot 2\pi\,\mathrm{sgn}(-t)\\&=2\pi\cdot 2\pi\cdot f_o(-t)\\&=2\pi\cdot F(iF_I(\omega))\\&=2\pi i\cdot F(F_I(\omega))\end{aligned}$$

$$\Rightarrow F_I(\omega)=\frac{1}{2\pi i}(F_R(\omega)*S(\omega))=\int_{-\infty}^{\infty}\frac{F_R(u)}{2\pi i}\cdot S(\omega-u)du=\int_{-\infty}^{\infty}\frac{F_R(u)}{2\pi i}\cdot\frac{-2i}{(\omega-u)}du=-\int_{-\infty}^{\infty}\frac{F_R(u)}{\pi(\omega-u)}du$$

Time Domain versus Frequency Domain

$$f(t) \xrightarrow[\text{Transform}]{\text{Fourier}} F(\omega) = F_R(\omega) + iF_I(\omega),\ F(-\omega) = F^*(\omega)$$

Even function $\dfrac{f(t)+f(-t)}{2}$ ← Inverse Fourier Transform

Odd function $\dfrac{f(t)-f(-t)}{2}$ ← Inverse Fourier Transform

Special case: casual function

f(t)= $\begin{cases} f^{+}(t), t \geq 0 \\ 0, t < 0 \end{cases}$ → Only $F_R(\omega)$ or - $F_I(\omega)$ can determine $f(t)$

Hilbert transform pair ← Any constraint between $F_R(\omega)$ and $-F_I(\omega)$

海大河工系 工數與 Bem 教材 陳正宗特聘教授 Hilbert1.doc 吳建和再製

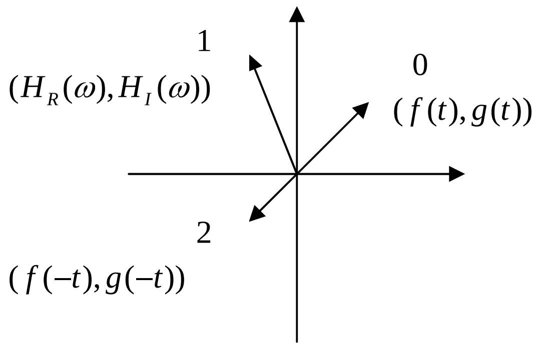
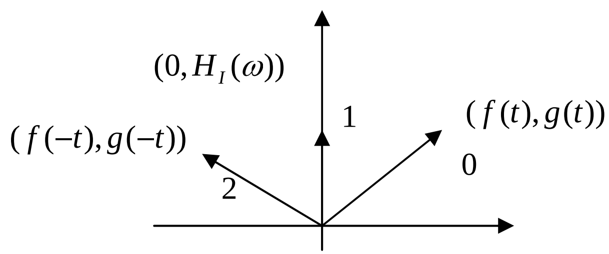
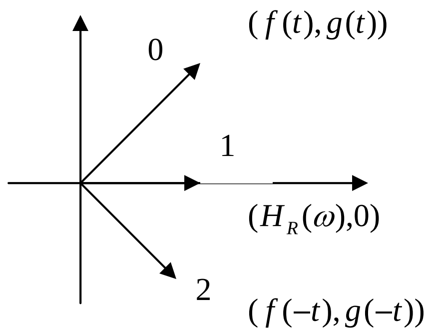
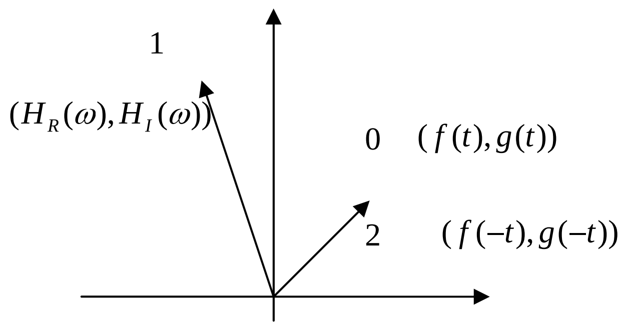
Constraints in Frequency Domain

$$f(t)+ig(t) \rightarrow H_R(\omega)+iH_I(\omega)=H(\omega) \rightarrow f(-t)+ig(-t)$$

f(t) \ g(t)	odd	even	0	Casual function
odd	$F\{F(f(t)+ig(t))\}$ $=-f(t)-ig(t)$	$H_R(\omega)=0$	$H(-\omega)=H^*(\omega)$	
even	$H_I(\omega)=0$	$F\{F(f(t)+ig(t))\}$ $=f(t)+ig(t)$	$H(-\omega)=H^*(\omega)$	
0	$H(-\omega)=-H^*(\omega)$	$H(-\omega)=-H^*(\omega)$	╳	$H_R(\omega)=\int_{-\infty}^{\infty}\frac{H_I(u)}{\pi(\omega-u)}du$
Casual function			$-H_I(\omega)=\int_{-\infty}^{\infty}\frac{H_R(u)}{\pi(\omega-u)}du$	$H_R(\omega)=\int_{-\infty}^{\infty}\frac{H_I(u)}{\pi(\omega-u)}du$ $-H_I(\omega)=\int_{-\infty}^{\infty}\frac{H_R(u)}{\pi(\omega-u)}du$

Geometric Meaning for Fourier Transform

$$f(t) + ig(t) \rightarrow H_R(\omega) + iH_I(\omega) = H(\omega) \rightarrow f(-t) + ig(-t)$$

f(t) \ g(t)	**odd**	**even**
odd		
even		

海大河工系 工數與 Bem 教材 陳正宗特聘教授 Hilbert3.doc 吳建和再製

Example for Fourier transform

海大河海系　　陳正宗

例題

設 $f(t) = \begin{cases} 0.5 & , t > 0 \\ -0.5 & , t < 0 \end{cases}$

(1)求 $f(t)$ 的 Fourier 轉換？並說明為何沒有實部。

(2)利用(1)之結果，求 $\int_{-} \frac{\sin\omega}{\omega}\,d\omega = ?$

(3)利用 Parseval 定理，求 $\int_{-} \left|\frac{1}{a+ib\omega}\right|^2 d\omega = ?$ 其中 $a,\, b$ 為常數

(4)利用留數定理，求 $\int_{-} \left|\frac{1}{a+ib\omega}\right|^2 d\omega = ?$

(5)若(3)與(4)之結果相同，請說明其物理意義。　　(　海洋河工)

解　(1)利用 Fourier 轉換之定義

$$
\begin{aligned}
F(\omega) = \mathscr{F}\{\, f(t)\,\} &= \int_{-} f(t)\, e^{-i\omega t}\, dt \\
&= \int_{-}^{0} (-0.5)\, e^{-i\omega t}\, dt + \int_{0} (0.5)\, e^{-i\omega t}\, dt \\
&= 2\,(0.5) \int_{0} e^{-i\omega t}\, dt = \frac{e^{-i\omega t}}{-i\omega}\Big|_{0} = \frac{1}{i\omega}
\end{aligned}
$$

沒有實部乃因 $f(t)$ 為奇函數。

(2)利用逆 Fourier 轉換之定義

$$
\begin{aligned}
f(t) = \mathscr{F}^{-1}\{\, F(\omega)\,\} &= \frac{1}{2\pi}\int_{-} \frac{1}{i\omega}\, e^{i\omega t}\, d\omega \\
&= \frac{1}{2\pi}\int_{-} \frac{\cos\omega t + i\sin\omega t}{i\omega}\, d\omega \\
&= \frac{1}{2\pi}\left[(-i)\int_{-} \frac{\cos\omega t}{\omega}\, dt + \int_{-} \frac{\sin\omega t}{\omega}\, d\omega\right] = \frac{1}{2\pi}\int_{-} \frac{\sin\omega t}{\omega}\, d\omega
\end{aligned}
$$

$$
(\text{因 } \frac{\cos\omega t}{\omega} \text{ 為奇函數，} \int_{-} \frac{\cos\omega t}{\omega} = 0)
$$

上式中取 $t = 1$，則得 $0.5 = \frac{1}{2\pi}\int_{-} \frac{\sin\omega}{\omega}\, d\omega$

於是 $\int_{-} \frac{\sin\omega}{\omega}\, d\omega = \pi$

(3)取 $g(t) = \begin{cases} \dfrac{e^{-\frac{a}{b}t}}{b} & , \ t > 0 \\ 0 & , \ t < 0 \end{cases} \quad (b \neq 0)$，則

$$\begin{aligned} G(\omega) &= \mathscr{F}\{\, g(t)\,\} = \int_{-} g(t)\, e^{-i\omega t}\, dt = \int_0 \frac{e^{-\frac{a}{b}t}}{b}\, e^{-i\omega t}\, dt \\ &= \frac{1}{b}\int_0 e^{-(\frac{a}{b}+i\omega)\,t}\, dt = \frac{1}{b}\,\frac{e^{-(\frac{a}{b}+i\omega)\,t}}{-\frac{a}{b}-i\omega}\Big|_0 = \frac{1}{b}\,\frac{1}{\frac{a}{b}+i\omega} \\ &= \frac{1}{a+ib\omega} \end{aligned}$$

亦即 $\dfrac{1}{a+ib\omega}$ 是 $g(t)$ 的 Fourier 轉換。

代入 Fourier 轉換的 Parseval 定理：

$$\int_0 g^2(x)\, dx = \frac{1}{2\pi}\int_{-} |G(\omega)|^2\, d\omega$$

$$\begin{aligned} \text{得} \int_{-} \left|\frac{1}{a+ib\omega}\right|^2 d\omega &= 2\pi\int_0 \left|\frac{e^{-\frac{a}{b}t}}{b}\right|^2 dt = \frac{2\pi}{b^2}\int_0 e^{-\frac{2a}{b}t}\, dt \\ &= \frac{2\pi}{b^2}\,\frac{e^{-\frac{2a}{b}t}}{-\frac{2a}{b}}\Big|_0 = \frac{\pi}{ab} \end{aligned}$$

(4)因 $\left|\dfrac{1}{a+ib\omega}\right|^2 = \dfrac{1}{a^2+b^2\omega^2}$，故

$$\int_{-} \left|\frac{1}{a+ib\omega}\right|^2 d\omega = \int_{-} \frac{1}{a^2+b^2\omega^2}\, d\omega$$

而根據留數定理

$$\int_{-} h(x)\, dx = 2\pi i \sum_{\substack{\text{upper}\\ \text{plane}}} \text{Res}\{\, h(z)\,\}$$

所以

$$\begin{aligned} \int_{-} \frac{1}{a^2+b^2\omega^2}\, d\omega &= 2\pi i \operatorname*{Res}_{z=i\frac{a}{b}} \frac{1}{a^2+b^2z^2} = 2\pi i\,\frac{1}{2abi} \\ &= \frac{\pi}{ab} \end{aligned}$$

(5)其物理意義是能量守恒。

Properties of Fourier transform

海大河海系　　陳正宗

1 Definition of Fourier transform($f(t)$ is real):

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \;\; F(-\omega) = F^*(\omega)$$

2 Definition of Inverse Fourier transform:

$$u(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

3 Linear property:

$$\mathcal{F}\{a\ f(t) + b\ g(t)\} = a\ F(\omega) + b\ G(\omega)$$

4 Double Fourier transform:

$$\mathcal{F}\{\mathcal{F}\{f(t)\}\} = 2\pi f(-t)$$

5 Time shifting:

$$\mathcal{F}\{f(t-a)\} = e^{-i\omega a}F(\omega)$$

6 Frequency shifting:

$$\mathcal{F}\{e^{i\omega_0 t}f(t)\} = F(\omega - \omega_0)$$

7 Time derivative:

$$\mathcal{F}\{f'(t)\} = i\omega\, F(\omega)$$

8 Frequency derivative:

$$F'(\omega) = \mathcal{F}\{-itf(t)\}$$

9 Convolution:

$$\mathcal{F}\{f(t) * g(t)\} = F(\omega)G(\omega)$$

where

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(u)g(t-u)du$$

10 Energy conservation:

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mid F(\omega) \mid^2 d\omega$$

Table of Fourier integral

海大河海系　　陳正宗
(file:tabfor.te)

常見函數之 Fourier 轉換

$f(x)$	$F(\omega)$
$\frac{1}{2}$ for $\|x\| < a$; 0 for $\|x\| > a$	$\frac{\sin a\omega}{\omega}$
$\frac{1}{\sqrt{2\pi\|x\|}}$	$\frac{1}{\sqrt{\|\omega\|}}$
$\frac{2a-x}{4}$ for $\|x\| < 2a$; 0 for $\|x\| > 2a$	$\frac{\sin^2 a\omega}{\omega^2}$
$\frac{a}{\pi(x^2+a^2)}$	$e^{-a\|\omega\|}$
$\frac{e^{-x^2/4a}}{2\sqrt{\pi a}}$	$e^{-a\omega^2}$
$\frac{e^{-a\|x\|}}{2a}$	$\frac{1}{\omega^2+a^2}$
$\frac{K_0(a\|x\|)}{\pi}$	$\frac{1}{\sqrt{\omega^2+a^2}}$
ie^{iax} for $x > 0$; 0 for $x < 0$	$\frac{1}{\omega-a}$ (for $\mathscr{I}\, a > 0$)
$f^{(n)}(x)$	$(i\omega)^n F(\omega)$

1

Gibbs phenomenon by Fourier transform

海大河海系　　陳正宗

Given a function $f(t)$, we have

$$f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-} \int_{-} f(t)e^{-i\omega t}dte^{i\omega u}d\omega \;\rightarrow\; f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-} \int_{-} f(t)e^{-i\omega(t-u)}dtd\omega$$

$$f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-} \int_{-} f(t)e^{-i\omega(t-u)}d\omega dt \;\rightarrow\; f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-} f(t)\frac{e^{-i\omega(t-u)}}{-i(t-u)}\Big|_{-\omega_0}^{\omega_0} dt$$

$$f_{\omega_0}(u) = \int_{-} \frac{sin(\omega_0(t-u))}{(t-u)}f(t)dt \;\rightarrow\; f_{\omega_0}(u) = \int_{-} R_{\omega_0}(u,t)f(t)dt$$

where reproducing kernel $R_{\omega_0}(u,t) = \frac{sin(\omega_0(t-u))}{(t-u)}$

After substituting the Heaviside function into $f(t)$, we have

$$f_{\omega_0}(u) = \int_0 \frac{sin(\omega_0(t-u))}{(t-u)}dt$$

By changing the variable, we have

$$f_{\omega_0}(u) = \int_{-}^{\omega_0 u} \frac{1}{\pi}\frac{sin(x)}{(x)}dx = \int_{-}^{0} \frac{1}{\pi}\frac{sin(x)}{(x)}dx + \int_0^{\omega_0 t} \frac{1}{\pi}\frac{sin(x)}{(x)}dx$$

$$f_{\omega_0}(u) = 0.5 + \int_0^{\omega_0 u} \frac{1}{\pi}\frac{sin(x)}{(x)}dx$$

$$f_{\omega_0}(u) = 0.5 + \frac{1}{\pi}Si(\omega_0 u)$$

where Si function is defined by

$$Si(z) = \int_0^z \frac{sin(x)}{x}dx$$

The maximum value occurs as the derivative is zero, *i.e.*,

$$\frac{sin(z)}{z} = 0$$

The maximum may occur as $z = 0, \pi, 2\pi, \cdots$.

The maximum of 1.09 can be obtained by substituting $u = \pi/\omega_0$

Fourier transform

海大河海系　　陳正宗

Fourier 轉換

若函數 $f(x)$ 為分段連續且 $\int_{-} |f(x)|$ 存在，則以下轉換關係成立

$$\begin{cases} F(\omega) = \mathscr{F}\{f(x)\} = \int_{-} f(x)\, e^{-i\omega x}\, dx & (\text{Fourier 轉換}) \\ f(x) = \mathscr{F}^{-1}[F(\omega)] = \dfrac{1}{2\pi}\int_{-} F(\omega)\, e^{i\omega x}\, d\omega & (\text{逆 Fourier 轉換}) \end{cases} \tag{1}$$

其中 $e^{-\omega x}$, $e^{\omega x}$ 為 Fourier 轉換的基底，或稱核函數 (Kernel function).

當 $f(x)$ 為偶函數時

$$\begin{cases} F_c(\omega) = \mathscr{F}\{f(x)\} = 2\int_0 f(x)\cos\omega x\, dx & (\text{Fourier 餘弦轉換}) \\ f(x) = \mathscr{F}^{-1}[F(\omega)] = \dfrac{1}{\pi}\int_0 F_c(\omega)\cos\omega x\, d\omega & (\text{逆 Fourier 餘弦轉換}) \end{cases} \tag{2}$$

當 $f(x)$ 為奇函數時

$$\begin{cases} F_s(\omega) = \mathscr{F}\{f(x)\} = 2\int_0 f(x)\sin\omega x\, dx & (\text{Fourier 正弦轉換}) \\ f(x) = \mathscr{F}^{-1}[F(\omega)] = \dfrac{1}{\pi}\int_0 F_s(\omega)\sin\omega x\, d\omega & (\text{逆 Fourier 正弦轉換}) \end{cases} \tag{3}$$

Fourier 轉換的 Parseval's 恆等式：

$$\int_0 f^2(x)\, dx = \frac{1}{2\pi}\int_{-} |F(\omega)|^2\, d\omega \tag{4}$$

不過為求對稱性，有些書將上述轉換關係定成

$$\begin{cases} F(\omega) = \dfrac{1}{\sqrt{2\pi}}\int_{-} f(x) e^{i\omega x}\, dx & (\text{Fourier 轉換}) \\ f(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-} F(\omega)\, e^{i\omega x}\, d\omega & (\text{逆 Fourier 轉換}) \end{cases} \tag{5}$$

當 $f(x)$ 為偶函數時

$$\begin{cases} F_c(\omega) = \sqrt{\dfrac{2}{\pi}}\int_{-} f(x)\cos\omega x\, dx & (\text{Fourier 餘弦轉換}) \\ f(x) = \sqrt{\dfrac{2}{\pi}}\int_{-} F_c(\omega)\cos\omega x\, d\omega & (\text{逆 Fourier 正弦轉換}) \end{cases} \tag{6}$$

當 $f(x)$ 為奇函數時

$$\begin{cases} F_s(\omega) = \sqrt{\dfrac{2}{\pi}}\int_{-} f(x)\sin\omega x\, dx & (\text{Fourier 餘弦轉換}) \\ f(x) = \sqrt{\dfrac{2}{\pi}}\int_{-} F_s(\omega)\sin\omega x\, d\omega & (\text{逆 Fourier 正弦轉換}) \end{cases} \tag{7}$$

1. $f(t)=\begin{cases} e^t, & if\ b<t<c \\ 0, & otherwise \end{cases}$, 試求其傅立業轉換。

2. 試利用上題結果，及尺度變換性質，求 $f(at), (a>0)$ 之傅立業轉換。

3. $f(t)=\begin{cases} 6, & -2\le x\le 2 \\ 0, & otherwise \end{cases}$, $g(t)=\begin{cases} 6, & 3\le x\le 7 \\ 0, & otherwise \end{cases}$, 試利用 $f(t)$ 的傅立葉轉換，及平移性質，求 $g(t)$ 的傅立業轉換。

1. $f(t)=\begin{cases} e^t,\ if\ b<t<c \\ 0,\quad otherwise \end{cases}$，試求其傅立業轉換。

$$F(\omega)=\int_b^c e^t e^{-i\omega t}dt=\int_b^c e^{-t(1-i\omega t)}dt=\frac{1}{1-i\omega}e^{t(1-i\omega)}\Big|_b^c=\frac{1}{1-i\omega}\left(e^{c(1-i\omega)}-e^{b(1-i\omega)}\right)$$

2. 試利用上題結果，及尺度變換性質，求 $f(at),(a>0)$ 之傅立業轉換。

$$F\{f(at)\}=\frac{1}{|a|}F(\frac{\omega}{a})=\frac{1}{|a|}\frac{1}{1-i\dfrac{\omega}{a}}\left(e^{c(1-i\frac{\omega}{a})}-e^{b(1-i\frac{\omega}{a})}\right)=\frac{1}{a-i\omega}\left(e^{c(1-i\frac{\omega}{a})}-e^{b(1-i\frac{\omega}{a})}\right)$$

3. $f(t)=\begin{cases} 6,\ -2\le x\le 2 \\ 0,\quad otherwise \end{cases}$，$g(t)=\begin{cases} 6,\quad 3\le x\le 7 \\ 0,\quad otherwise \end{cases}$，試利用 $f(t)$ 的傅立葉轉換，及平移性質，求 $g(t)$ 的傅立業轉換。

$$F\{f(t)\}=\int_{-2}^{2}6e^{-i\omega t}dt=\frac{-6}{i\omega}e^{-i\omega t}\Big|_{-2}^{2}=\frac{6}{i\omega}\left(e^{2i\omega}-e^{-2i\omega}\right)\ or\ \frac{12}{\omega}\sin(2\omega)$$

$$F\{g(x)\}=F\{f(x-5)\}=e^{-5i\omega}F\{f(x)\}=\frac{6}{i\omega}\left(e^{-3i\omega}-e^{-7i\omega}\right)\ or\ \frac{12}{\omega}\sin(2\omega)e^{-5i\omega}$$

若$\mathcal{F}\{f(t)\}=F(\omega)$且$\mathcal{F}\{g(t)\}=G(\omega)$，捲積（$*$, convolution）定義為

$$f(t)*g(t)=\int_{-\infty}^{\infty} f(u)g(t-u)du$$

可得證$\mathcal{F}\{f(t)*g(t)\}=F(\omega)G(\omega)$。

今定義 correlation 為

$$c(t)=f(t)\circ g(t)=\int_{-\infty}^{\infty} f(u)g(t+u)du$$

(1) Plot $c(t)$ versus t

(2) 試問$\mathcal{F}\{f(t)\circ g(t)\}$與原函數（$f(t),g(t)$）作傅力葉轉換（$F(\omega),G(\omega)$）之關係。

若 $\mathcal{F}\{f(t)\}=F(\omega)$ 且 $\mathcal{F}\{g(t)\}=G(\omega)$，捲積（$*$, convolution）定義為

$$f(t)*g(t)=\int_{-\infty}^{\infty} f(u)g(t-u)du$$

可得證 $\mathcal{F}\{f(t)*g(t)\}=F(\omega)G(\omega)$。

今定義 correlation 為

$$c(t)=f(t)\circ g(t)=\int_{-\infty}^{\infty} f(u)g(t+u)du$$

試問 $\mathcal{F}\{f(t)\circ g(t)\}$ 與原函數（$f(t), g(t)$）作傅力葉轉換（$F(\omega), G(\omega)$）之關係。

$$c(t)=f(t)\circ g(t)=\int_{-\infty}^{\infty} f(u)g(t+u)du$$

令 $u=-v, du=-dv$

可將上式寫成 $c(t)=f(t)\circ g(t)=\int_{-\infty}^{\infty} f(-v)g(t-v)dv=f(-t)*g(t)$

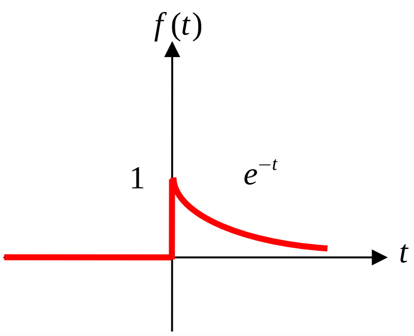
所以 $\mathcal{F}\{f(t)\circ g(t)\}=\mathcal{F}\{f(-t)*g(t)\}=F(-\omega)G(\omega)$

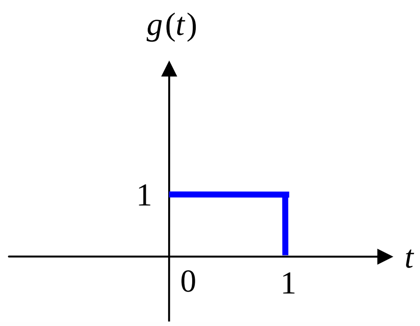
若改成 $c(t)=g(t)\circ f(t)=\int_{-\infty}^{\infty} g(u)f(t+u)du$

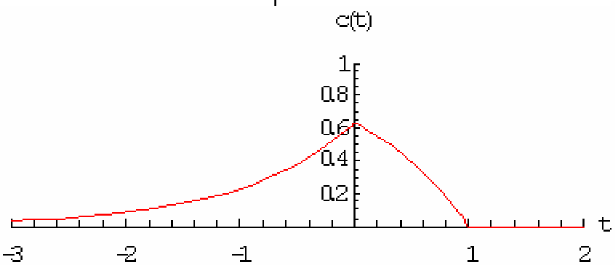
所以 $\mathcal{F}\{g(t)\circ f(t)\}=\mathcal{F}\{g(-t)*f(t)\}=G(-\omega)F(\omega)$

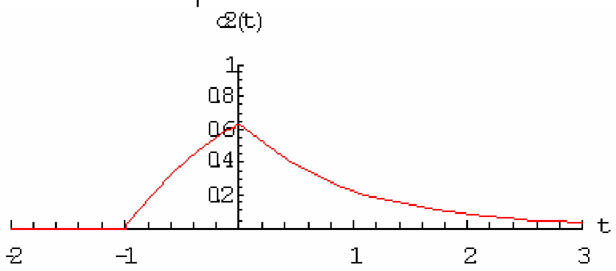
互為共軛，所以 $f(t)\circ g(t)$ 與 $g(t)\circ f(t)$ 為對 y 軸的鏡射。

$f(t)\circ g(t)$　與　$g(t)\circ f(t)$　分別對 t 作圖，互為對 y 軸的鏡射。



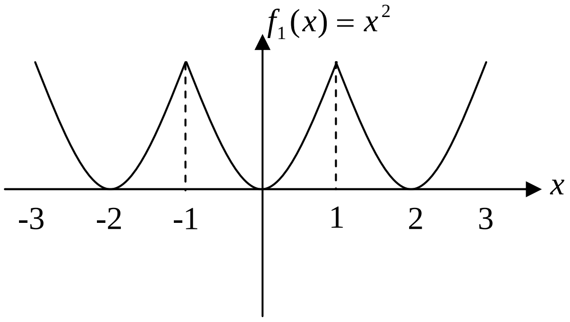


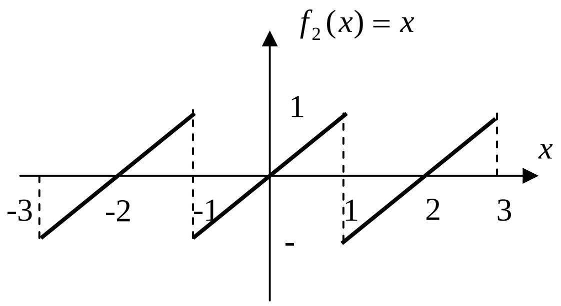




工數第二次大考(Fourierseries,integral,transform)09:20-12:00

日期：2008 年 5 月 22 日 姓名：___________ 學號：___________





1　Fourier series (real base) and summation application (period not 2π)

(1) Expand $f_1(x)$ into Fourier series. (10 %)

(2) Expand $f_2(x)$ into Fourier series. (10 %)

(3) Determine $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using (1) and (2), respectively. (10 %)

2. Consider $f(x) = x + \pi, \quad -\pi < x < \pi$ (period 2π)

(a) Determine whether the function f is even, odd, or neither (5 %)

(b) Find the Fourier series of f on the given interval $(-\pi, \pi)$ (10 %)

(c) Find the values that the series will converge at $x = -\pi,\ 0,\ \pi/2,\ \pi$ (10 %)　(converge in the mean)

(d) Use the result of (b) to find $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = ?$ (10 %)

3. Given $f(t) = \begin{cases} 1, & t \in (0, \pi) \\ 0, & t \in (\pi, 2\pi) \end{cases}$ and $f(t) = f(t + 2\pi)$.

(a) Find $f(t)$ by using the complex Fourier expansion. (10 %)

(b) Plot the frequency spectrum of $|c_n|$ versus *n*. (10 %)

(c) $c_n, n = -\infty, ..., 0, ...\infty$, real, imaginary or complex, why ? (5 %)

4. Given $g(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & otherwise \end{cases}$.

(a) Find the Fourier integral of *g(t)*, i.e, $G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$ (10 %)

(b) Plot the frequency spectrum of imaginary part of $G(\omega)$ versus ω (5 %)

(c) $G(\omega)$ is real, imaginary or complex, why ? (5 %)

5. Please point out the difference between Fourier series and Fourier integral. (10 %)

Mid2008-test2.doc Chen J T 製表

1

(1)

$$f_1(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\}$$

$$\begin{cases} a_0=\dfrac{1}{2p}\int_{-p}^{p}f(x)dx \\ a_n=\dfrac{1}{p}\int_{-p}^{p}f(x)\cos\dfrac{n\pi}{p}xdx \\ b_n=\dfrac{1}{p}\int_{-p}^{p}f(x)\sin\dfrac{n\pi}{p}xdx \end{cases}$$

偶函數 $b_n=0$

$$f_1(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\}$$

$$=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(-1)^n\cos(n\pi x)$$

(2)

$$f_2(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\}$$

$$\begin{cases} a_0=\dfrac{1}{2p}\int_{-p}^{p}f(x)dx \\ a_n=\dfrac{1}{p}\int_{-p}^{p}f(x)\cos\dfrac{n\pi}{p}xdx \\ b_n=\dfrac{1}{p}\int_{-p}^{p}f(x)\sin\dfrac{n\pi}{p}xdx \end{cases}$$

奇函數 $a_0=a_n=0$

$$f_2(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\}$$

$$=\sum_{n=1}^{\infty}\frac{-2}{n\pi}(-1)^n\sin(n\pi x)$$

(3)

$$f_1(1)=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(-1)^n\cos(n\pi)$$

$$=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

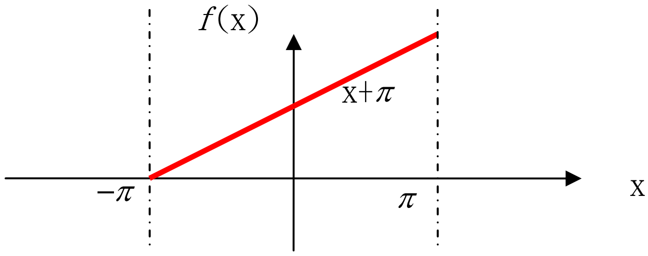
$$\int_{-p}^{p}[f_2(x)]^2=p[2(a_0^2)+\sum_{n=1}^{\infty}[(a_n^2)+(b_n^2)]$$

$$\Rightarrow\left[\frac{x^3}{3}\right]\Bigg|_{-1}^{1}=\sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

2.

(a)



→ 非奇非偶

(b)

$$f(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\}$$

$$\begin{cases} a_0=\dfrac{1}{2p}\displaystyle\int_{-p}^{p}f(x)dx \\ a_n=\dfrac{1}{p}\displaystyle\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx \\ b_n=\dfrac{1}{p}\displaystyle\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx \end{cases}$$

$$f(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\}$$

$$=\pi+\sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin(nx)$$

(c)

$$f(-\pi)=\frac{2\pi+0}{2}=\pi$$

$$f(0)=\pi$$

$$f(\frac{\pi}{2})=\frac{3\pi}{2}$$

$$f(\pi)=\frac{2\pi+0}{2}=\pi$$

(d)

$$f(\frac{\pi}{2})=\pi+\sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin(\frac{n\pi}{2})$$

$n=2m-1$

$$=\pi+\sum_{m=1}^{\infty}\frac{-2}{2m-1}(-1)^{2m-1}\sin(\frac{(2m-1)\pi}{2})$$ →僅奇數有值

$$\Rightarrow\frac{f(\frac{\pi}{2})-\pi}{2}=\sum_{m=1}^{\infty}\frac{1}{2m-1}(-1)^{m+1}\quad\Rightarrow\quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$$

3.

(a)

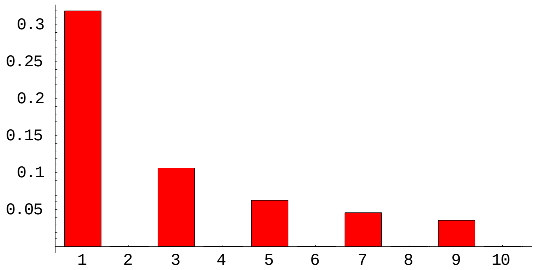
$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{p}t}$$

$$c_n=\frac{1}{2p}\int_{-p}^{p} f(t)e^{-i\frac{n\pi}{p}t}dt$$

$$c_0=\frac{1}{2p}\int_{-p}^{p} f(x)dt$$

$$f(t)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{-i(1-(-1)^n)}{2n\pi}e^{int}$$

(b)



(c)

Complex

4.

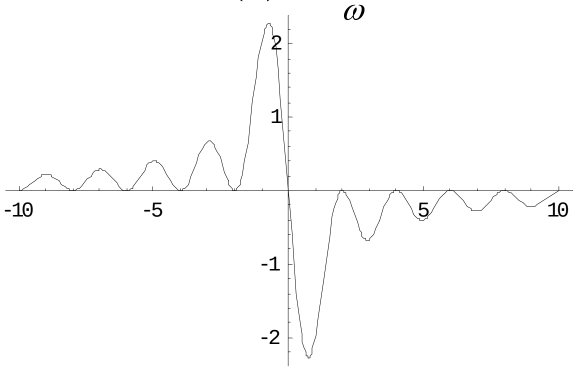
(a)

$$G(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt$$

$$=\int_{0}^{\pi} e^{-i\omega t}dt$$

$$=\frac{i}{\omega}(e^{-i\omega\pi}-1)$$

(b) $\operatorname{Im} G(\omega)=\dfrac{\sin\omega\pi}{\omega}$



(c)

Complex

5.

傅立葉積分其頻率爲連續的

傅立葉級數其頻率爲離散的

第四章 Laplace 轉換

4.1 拉氏轉換定義

拉氏轉換定義及基本函數之拉氏轉換----------------------------------216

一般函數的 Laplace transform--------------------------------------219

根號 t 與根號 t 分之一的 Laplace transform--------------------------220

Laplacee transform and Fourier transform 的一些性質-----------------222

Laplace transform 的庫存函數--------------------------------------224

4.2 用 Laplace transform 解 ODE

共振問題 Laplace 的 Ys 解法--225

激發、拍擊與共振問題的 Laplace transform----------------------------226

Laplace transform 解 ODE---229

Laplace Transform-1 ---232

補充性質---233

Bessel function 的 Laplace transform-------------------------------234

一些算例---235

Laplace Transform--2---236

4.3 摺積問題應用

重積分變換---237

褶積應用-(方形波)--238

摺積應用-(半正弦波)--239

兩函數的 convolution 與 correlation 在 Laplace 與 Fourier 轉換後的結果--242

4.4 電腦在工程數學上的應用

電腦在工程數學上的應用- 作業六------------------------------------244

電腦在工程數學上的應用- 作業七------------------------------------245

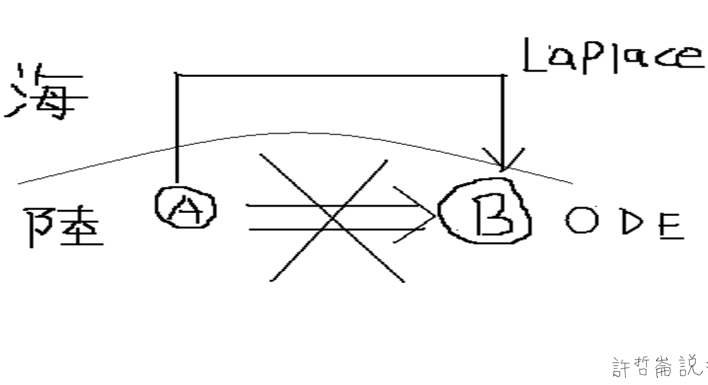
電腦在工程數學上的應用- 作業九------------------------------------246

4.5 歷屆考題

台大考題---247

小考與解答---248

大考與解答---253



許哲崙說: 本頁請勿塗改

Laplace transform

拉式轉換定義	Laplace transform 之 mathematica 語法(I)
一般函數的 Laplace transform Laplace transform 的庫存函數(影片)	Laplace transform 之 mathematica 語法(II)
根號 t 與根號 t 分之一的 Laplace transform	Laplace transform 之 mathematica 語法(III)
共振問題 Laplace 的 **Ys** 解法	台大考題
激發、拍擊與共振問題的 Laplace transform	摺積應用-(方形波)
Laplace transform 解 ODE(張毓玲)	Bessel function 的 Laplace transform
摺積應用-(半正弦波)	一些算例
Laplace Transform-1 (補充性質)	雙重積分轉換(Method1-2) (Method3)
Laplace Transform-2	重積分變換
兩函數的 convolution 與 correlation 在 Laplace 與 Fourier 轉換後的結果	
Laplacee transform and Fourier transform 的一些性質	

YouTube 學習影片

1.Laplace Transform

2.第一平移定理

3.第二平移定 理

4.Laplace transforms + ODEs

5.Laplace Transform－Convergence of Laplace transform (交通大學機械系白明憲教授)			
國內外好站連結			
取自 陳自雄博士教學公開網頁	拉氏轉換		
取自 陳自雄博士教學公開網頁	拉氏轉換之應用		
建議叢書區			
書名	作者	出版社	年代
Operational Mathematics (.jpg)	Churchill, Ruel Vance	McGraw-Hill	1980
Distribution Theory and Transform Analysis	A.H.Zemanian	DOVER PUBLICATIONS	1965
The Use of Integral Transforms	Ian N. Sneddon	McGraw-Hill	1972
Complex Variables and the Laplace Transform for Engineers	Wilbur R. LePage	DOVER PUBLICATIONS	1961

拉氏轉換定義

若函數 f(t)對所有 t≧0 皆有定義,則 f(t)的拉氏轉換記為$\mathcal{L}$[f(t)]。

$$\odot \mathcal{L}[\mathbf{f}(\mathbf{t})] = \mathbf{F}(\mathbf{s}) = \int_0^\infty \mathbf{f}(\mathbf{t})\ \mathbf{e}^{-\mathbf{st}}\mathbf{dt}$$

$$\mathcal{L}[f(t)] = F(s) \Leftrightarrow \underset{\text{反拉氏轉換}}{\mathcal{L}^{-1}}[F(s)] = f(t)$$

基本函數之拉氏轉換

◎指數函數:e^{at}

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}\ e^{-st}dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \frac{-1}{s-a}\ e^{-(s-a)t}\Big|_0^\infty$$

$$= \frac{1}{s-a}$$

◎三角函數:sin(bt)及 cos(bt)

(1)利用分部積分法

$$\mathcal{L}[\sin(bt)] = \int_0^\infty \sin(bt)\, e^{-st}dt$$

$$\xrightarrow{\text{by 分部積分}} = \frac{e^{-st}}{(-s)^2+b^2}\ [(-s)\sin(bt) - b\cos(bt)]\ \Big|_0^\infty$$

$$= -\frac{e^0}{(-s)^2+b^2}\ [-b\cos(0)]$$

$$\Rightarrow \mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$$

同理可得

$$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$$

(2)利用尤拉公式

$$e^{i\theta} = \cos\theta + i\sin\theta \cdots\cdots ①$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \cdots\cdots ②$$

$① + ②: \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$① - ②: \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\Rightarrow \mathcal{L}[\sin(bt)] = \mathcal{L}[\frac{e^{ibt} - e^{-ibt}}{2i}]$$
$$= \frac{1}{2i}\{\mathcal{L}[e^{ibt}] - \mathcal{L}[e^{-ibt}]$$
$$= \frac{1}{2i}[\frac{1}{s-ib} - \frac{1}{s+ib}]$$
$$= \frac{1}{2i}\frac{(s+ib)-(s-ib)}{(s-ib)(s+ib)}$$
$$= \frac{1}{2i}\frac{2ib}{s^2+b^2} = \frac{b}{s^2+b^2}$$

$\mathcal{L}\cos(bt)$交給同學幫忙~

◎雙曲線函數:sinh(at)及 cosh(at)

$$定義:\begin{cases} \sinh(at) = \dfrac{e^{at} - e^{-at}}{2} \\ \cosh(at) = \dfrac{e^{at} + e^{-at}}{2} \end{cases}$$

(1) $\mathcal{L}[\sin h(at)] = \mathcal{L}[\dfrac{e^{at} - e^{-at}}{2}]$

$$= \frac{1}{2}\{\mathcal{L}[e^{at}] - \mathcal{L}[e^{-at}]\}$$
$$= \frac{1}{2}[\frac{1}{s-a} - \frac{1}{s+a}]$$
$$= \frac{1}{2}[\frac{(s+a)-(s-a)}{(s-a)(s+a)} \Rightarrow \mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$$

(2)同理可得$\mathcal{L}[\cosh(at)] = \frac{s}{s^2-a^2}$

◎多項式函數:t^n

預備知識:Gamma 函數$\begin{cases} 定義: \Gamma(n) = \int_0^\infty t^{n-1} \cdot e^{-t}\,dt \\ 性質: \Gamma(n+1) = n \cdot \Gamma(n) \end{cases}$

$\mathcal{L}[t^n] = \int_0^\infty t^n\, e^{-st} dt$,令 $st = u$ $\therefore t = \frac{u}{s}$, $dt = \frac{du}{s}$

$$\Rightarrow \mathcal{L}[t^n] = \int_0^\infty (\frac{u}{s})^n\, e^{-u}\, \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty u^n\, e^{-u} du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty u^{(n+1)-1}\ e^{-u} du$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1)$$

$$\xRightarrow{\text{by Gamma 性質}} \mathcal{L}[t^n] = \frac{n\,\Gamma(n)}{s^{n+1}} = \frac{n \cdot \Gamma(n-1+1)}{s^{n+1}}$$

$$= \frac{n(n-1)\,\Gamma(n-1)}{s^{n+1}} = \frac{n(n-1)\Gamma(n-2+1)}{s^{n+1}}$$

$$= \frac{n(n-1)(n-2)\,\Gamma(n-2)}{s^{n+1}} = \frac{n(n-1)(n-2)\,\Gamma(n-3+1)}{s^{n+1}}$$

$$= \frac{\overbrace{n(n-1)(n-2)\cdots\cdots(2)(1)}^{n\,!}\ \Gamma(1)}{s^{n+1}}$$

$$\Rightarrow \Gamma(1) = \int_0^\infty t^0\ e^{-t} dt = -e^{-t}\big|_0^\infty = 1$$

$$\therefore \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

班級：____________學號：_____________姓名：_____________

國立台灣海洋大學河海工程學系工程數學(二) 2B 班第一次小考 參考解答

1. Find the Laplace transform of $y(t)$.

Table 1: Mapping for the Laplace transform

$f(t)$	$p_1(t)$	$p_2(t)$	$p_3(t)$	$p_4(t)$	$p_5(t)$	$p_6(t)$	$p_7(t)$	$p_8(t)$	$p_9(t)$	$p_{10}(t)$
$F(s)$	(1)	(2)	(4)	(7)	(9)	(6)	(3)	(5)	(10)	(12)

Time function, $f(t)$

$p_1(t)=1$
$p_2(t)=\cos(t)$
$p_3(t)=\sin(t)$
$p_4(t)=t\cos(t)$
$p_5(t)=t\sin(t)$
$p_6(t)=e^t$
$p_7(t)=\cosh(t)$
$p_8(t)=\sinh(t)$
$p_9(t)=t\cosh(t)$
$p_{10}(t)=t\sinh(t)$

s function, $F(s)$

(1) $\frac{1}{s}$	(7) $\frac{s^2-1}{(s^2+1)^2}$
(2) $\frac{s}{s^2+1}$	(8) $\frac{s}{(s^2+1)^2}$
(3) $\frac{s}{s^2-1}$	(9) $\frac{2s}{(s^2+1)^2}$
(4) $\frac{1}{s^2+1}$	(10) $\frac{s^2+1}{(s^2-1)^2}$
(5) $\frac{1}{s^2-1}$	(11) $\frac{s}{(s^2-1)^2}$
(6) $\frac{1}{s-1}$	(12) $\frac{2s}{(s^2-1)^2}$

1. Find the Laplace transform of $\sqrt{t}$ and $\frac{1}{\sqrt{t}}$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

(a). Method 1:
Set $\mathcal{L}\{\sqrt{t}\} = P(s), \mathcal{L}\{\frac{1}{\sqrt{t}}\} = Q(s)$,

$$\mathcal{L}\{p(t)\} = P(s) \Rightarrow \mathcal{L}\{p'(t)\} = s[P(s)] - p(0), p(t) = \sqrt{t}$$

$$\mathcal{L}\{q(t)\} = Q(s) \Rightarrow \mathcal{L}\{tq(t)\} = -(Q'(s)), q(t) = \frac{1}{\sqrt{t}}$$

we have

$$\mathcal{L}\{\sqrt{t}\} = P(s) \Rightarrow \mathcal{L}\{\frac{1}{2\sqrt{t}}\} = s[P(s)] - \sqrt{0} = \frac{1}{2}Q(s)$$

$$\mathcal{L}\{\frac{1}{\sqrt{t}}\} = Q(s) \Rightarrow \mathcal{L}\{t\frac{1}{\sqrt{t}} = \sqrt{t}\} = -(Q'(s)) = P(s)$$

$$\Rightarrow s[-Q'(s)] = \frac{1}{2}Q(s)$$

By solving the 1st order ODE for $Q(s)$, we have

$$\Rightarrow Q(s) = \frac{k}{\sqrt{s}}, P(s) = \frac{k}{2s^{3/2}}, \quad k \in \text{constant}$$

Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$$

if $s = 1$

$$\mathcal{L}\{t^{\frac{1}{2}}\} = \int_0^\infty t^{-\frac{1}{2}}e^{-st}dt = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt = \Gamma(\frac{1}{2})$$

set $t = v^2$, $dt = 2v\ \ dv$

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-v^2}dv$$

if

$$\int_0^\infty e^{-x^2}dx = C, \int_0^\infty e^{-y^2}dy = C$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = C^2$$

and $x = rcos\theta, y = rsin\theta$

$$\Rightarrow \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} rdrd\theta = -\frac{1}{2} e^{r^2}|_0^\infty = \frac{\pi}{4} = C^2$$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = C = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}, k = \sqrt{\pi}$$

$$\Rightarrow Q(s) = \sqrt{\frac{\pi}{s}}, P(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

(b). Method 2:

$$Hint : \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

Set $x^2 = ts$

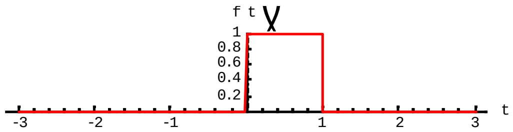
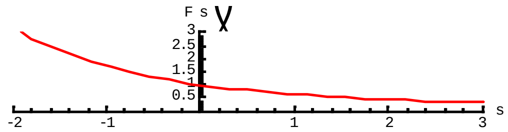
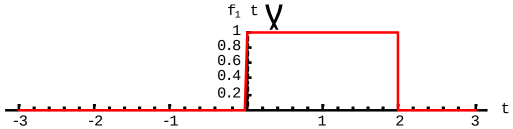
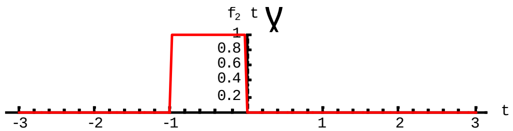
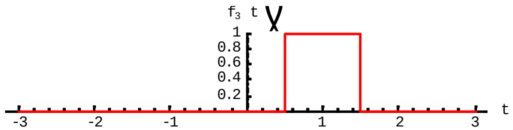
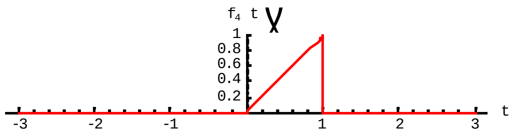
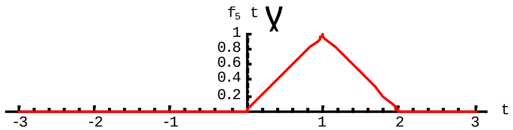
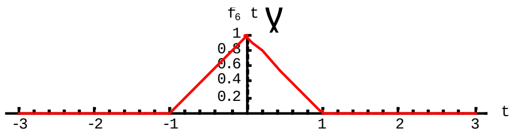
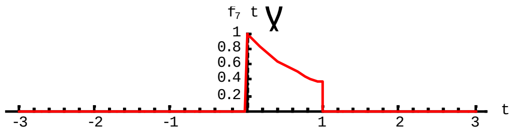
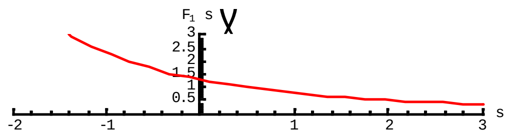
$$\Rightarrow 2x \ \ dx = s \ \ dt \Rightarrow dx = \frac{s}{2x} dt$$

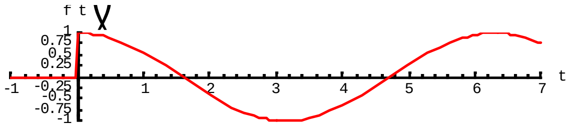
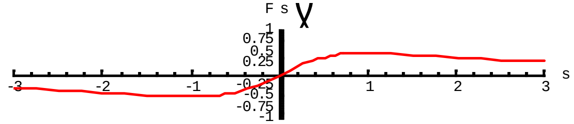
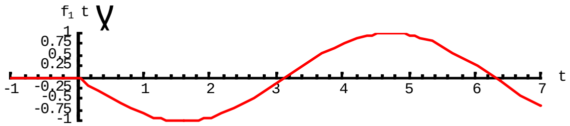
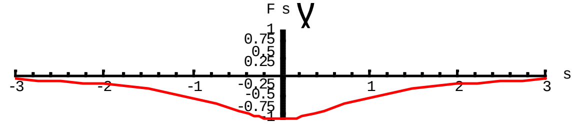
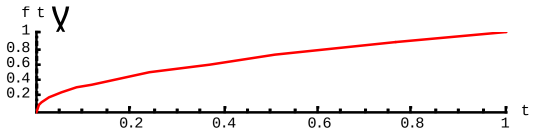
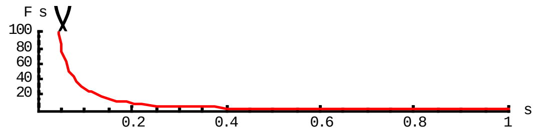
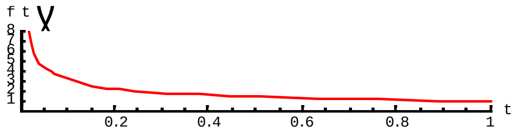
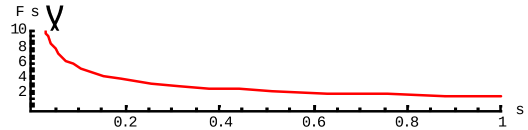
$$\Rightarrow \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-st} \frac{s}{2x} dt$$

$$= \int_0^\infty e^{-st} \frac{s}{2\sqrt{ts}} dt = \frac{\sqrt{s}}{2} \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} dt = \frac{1}{2}\sqrt{\pi}$$

$$\Rightarrow \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} dt = \sqrt{\frac{\pi}{s}}$$

$$\Rightarrow \int_0^\infty \sqrt{t} e^{-st} dt = \frac{\sqrt{\pi}}{2s^{3/2}}$$

function	figure	Fourier transform	Laplace transform $\hat{F}(s)$
$f(t)=\begin{cases}1, 0\le t\le 1\\ 0, otherwise\end{cases}$		$F(\omega)$	
$f(at)$		$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$	$\frac{1}{a}F\left(\frac{s}{a}\right), a>0$
$f(-t)$		$\frac{F(F(\omega))}{2\pi}$ or	N.A.
$f(t-a)$		$e^{-ia\omega}F(\omega)$	$e^{-sa}F(s), a>0$
$tf(t)$		?	$-\frac{dF(s)}{ds}$
$i\times tf(t)$		$-\frac{dF(\omega)}{d\omega}$	?.
$f(t)*f(t)$		$F(\omega)F(\omega)$	$F(s)F(s)$
$f(t)\circ f(t)$		$F(-\omega)F(\omega)$	N.A.
$e^{-\alpha t}f(t)$		$F(\omega+i\alpha)$	$F(s-\alpha)$ 

$f(t)=\begin{cases}\cos(t), t>0\\ 0, otherwise\end{cases}$		$F(\omega)$	$F(s)$ 
$f'(t)=\begin{cases}-\sin(t), t>0\\ 0, otherwise\end{cases}$		$i\omega F(\omega)$	$sF(s)-f(0)$ 
$f(t)=\sqrt{t}$		?	$F(s)=\dfrac{\sqrt{\pi}}{2s^{3/2}}$ 
$f(t)=\dfrac{1}{\sqrt{t}}$		?	$F(s)=\sqrt{\dfrac{\pi}{s}}$ 

Laplace transform 的庫存函數

	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^2}$
4	$t^n (n=0,1,2..)$	$\frac{n!}{s^{n+1}}$
5	e^{at}	$\frac{1}{s-a}$
6	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
7	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
8	$\cosh(at)$	$\frac{S}{s^2-a^2}$
9	$\sinh(at)$	$\frac{a}{s^2-a^2}$
10	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
11	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

班級：____________學號：_____________姓名：_____________

國立台灣海洋大學河海工程學系工程數學(二) 2B 班第二次小考 參考解答

1. **給定一常微分方程式,**

$$y''(t)+y(t)=\sin(t) \tag{1}$$

初始條件為 $y(0)=0, \dot{y}(0)=0$，**試用** Laplace **轉換求解** $y(t)$. (50%)

ANS

將(1)**式做拉式轉換**

$$s^2Y(s)-sy(0)-\dot{y}(0)+Y(s)=\frac{1}{s^2+1}$$

整理可得 $Y(s)=\frac{1}{(s^2+1)^2}=\frac{1}{2}[\frac{1}{s^2+1}-\frac{s^2-1}{(s^2+1)^2}]$

因此逆轉換可得 $y(t)=\frac{1}{2}[\sin(t)-t\cos(t)]$

2. **將一常微分方程式,**

$$t^2\ddot{y}(t)-4t\dot{y}(t)+6y(t)=0 \tag{2}$$

做 Laplace **轉換，則**

(1) $Y(s)$**需滿足何式？**(10%)

(2) **上式是多項式或微分方程式？**(10%)

(3) **求解** $Y(s)$。(15%)

(4) **求解** $y(t)$。(15%)

ANS

$L\{t^2\ddot{y}(t)\}=s^2Y''(s)+4sY'(s)+2Y(s)$

$L\{t\dot{y}(t)\}=-[sY'(s)+Y(s)]$

$L\{y(t)\}=Y(s)$

(1) $s^2Y''(s)+8sY'(s)+12Y(s)=0$

(2) **微分方程式**。

(3) $Y(s)=c_1s^{-3}+c_2s^{-4}$

(4) $y(t)=c_3t^2+c_4t^3$

(1)G.E

$$x''(t)+\omega^2 x(t)=\sin\varpi t$$

ω：natural frequency of system

ϖ：exciting frequency

I.C: $x(0)=0$ $x'(0)=0$

Taking Laplace transform, we have

$$X(s)=\frac{1}{s^2+\omega^2}\frac{\varpi}{s^2+\varpi^2}$$

$$X(s)=\frac{as+b}{s^2+\omega^2}+\frac{cs+d}{s^2+\varpi^2}$$

決定四個未定係數(a,b,c,d)

$$\begin{cases} a+c=0 \\ b+d=0 \\ a\omega^2+b\varpi^2=0 \\ b\omega^2+d\varpi^2=\varpi \end{cases}$$

$$\Rightarrow a=0\ ,\ c=0\ ,\ b=\frac{-\varpi}{\varpi^2-\omega^2}\ ,\ d=\frac{\varpi}{\varpi^2-\omega^2}$$

$$\therefore X(s)=\frac{\frac{-\varpi}{\varpi^2-\omega^2}}{s^2+\varpi^2}+\frac{\frac{\varpi}{\varpi^2-\omega^2}\frac{1}{\omega}\omega}{s^2+\omega^2}$$

(2)

$$x(t)=\frac{-1}{\varpi^2-\omega^2}\sin\varpi t+\frac{\frac{\varpi}{\omega}}{\varpi^2-\omega^2}\sin\omega t$$

$$=\frac{1}{\omega(\varpi^2-\omega^2)}[\varpi\sin\omega t-\omega\sin\varpi t]$$

(3)

$\omega\to\varpi$ set $\varpi=\omega+\varepsilon$

$$\therefore x(t)=\frac{1}{\omega}\cdot\frac{1}{(\varpi+\omega)(\varpi-\omega)}[(\omega+\varepsilon)\sin\omega t-\omega\sin(\omega t+\varepsilon t)]$$

$$=\frac{1}{\omega}\cdot\frac{1}{\varepsilon\cdot 2\omega}[(\omega+\varepsilon)\sin\omega t-\omega\sin\omega t\cdot\cos\varepsilon t-\omega\cos\omega t\cdot\sin\varepsilon t]$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\omega^2}\cdot\frac{1}{\varepsilon}[\varepsilon\sin\omega t-\omega\sin\omega t\cdot\sin\varepsilon t]$$

$$=\frac{1}{2\omega^2}\cdot[\sin\omega t-\omega t\cos\omega t]$$

(4)check $x(0)=0$ $x'(0)=0$

滿足初始條件，亦滿足非齊次微分方程式，即合外力項之解，故為全解。

(5)

Case1:

$$\varpi \neq \omega \to x(t) = \frac{1}{\omega} \cdot \frac{1}{(\varpi+\omega)(\varpi-\omega)}[\varpi \sin \omega t - \omega \sin \varpi t]$$

Case2:Beating

$$\varpi \approx \omega \to x(t) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \neq 0}} \frac{1}{2\omega^2} \cdot \frac{1}{\varepsilon}[\varepsilon \sin \omega t - \omega \sin \omega t \cdot \sin \varepsilon t]$$

$$= \frac{1}{2\omega^2} \cdot [\sin \omega t - \omega t \cos \omega t]$$

Case3:Resonance

$$\varpi = \omega \to x(t) = \frac{1}{2\omega^2} \cdot [\sin \omega t - \omega t \cos \omega t]$$

Case1: $\omega = 2\pi$, $\varpi = 4\pi$

Case2: $\omega = 2\pi$, $\varpi = 1.99\pi$

Case3: $\omega = 2\pi$, $\varpi = 2\pi$

(6)Set $\varpi = \omega$, Solve by Laplace transform

$$x(t) = \frac{\omega}{(s^2+\omega^2)^2} = \frac{s^2+\omega^2-s^2}{(s^2+\omega^2)^2} \cdot \frac{1}{\omega}$$

$$= \frac{1}{\omega}\left[\frac{1}{s^2+\omega^2} - \frac{s^2}{(s^2+\omega^2)^2}\right]$$

Using

$$L[\sin \omega t] = \frac{\omega}{s^2+\omega^2} \qquad L[\cos \omega t] = \frac{s}{s^2+\omega^2}$$

$$L[t \sin \omega t] = \frac{-\omega(2s)}{(s^2+\omega^2)^2} \cdot (-1) \qquad L[t \cos \omega t] = -\left[\frac{1}{(s^2+\omega^2)} - \frac{2s^2}{(s^2+\omega^2)^2}\right]$$

⇨ $L^{-1}\left[\frac{s^2}{(s^2+\omega^2)^2}\right] = \frac{1}{2} t \cos \omega t + \frac{1}{\omega} \sin \omega t$

⇨

$$x(t)=\{-\frac{1}{2}[t\cos\omega t+\frac{1}{\omega}\sin\omega t]+\frac{1}{\omega}\sin\omega t\}\cdot\frac{1}{\omega}$$

$$=\frac{1}{2\omega^2}[\sin\omega t-\omega t\cos\omega t]$$

和前面假設不同，做出後 *x(t)*，再以ϖ→ω逼近，所得結果相同。

另以迴旋積分(Convolution)求解

$$X(s)=\omega\cdot(\underbrace{\frac{1}{s^2+\omega^2}}_{L[\sin\omega t]})\cdot(\underbrace{\frac{1}{s^2+\omega^2}}_{L[\sin\omega t]})$$

$$x(t)=\frac{1}{\omega}\int_0^t\sin\omega u\cdot\sin\omega(t-u)\,du=\frac{-1}{2\omega}\int_0^t-2\sin\omega u\cdot\sin\omega(t-u)\,du$$

$$-2\sin\alpha\cdot\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$$

$$\therefore\ -2\sin\omega u\cdot\sin\omega(t-u)=\cos\omega t-\cos(2\omega u-\omega t)$$

⇨

$$x(t)=\frac{-1}{2\omega}\int_0^t[\cos\omega t-\cos(2\omega u-\omega t)]\,du$$

$$=\frac{-1}{2\omega}[t\cos\omega t-\underbrace{\int_0^t\cos(2\omega u-\omega t)\,du}_{set\,u=\bar{u}+\frac{t}{2}}]$$

in which

$$-\int_0^t\cos(2\omega u-\omega t)\,du=-\int_{-\frac{t}{2}}^{\frac{t}{2}}\cos 2\omega\bar{u}\,d\bar{u}$$

$$=-\frac{1}{2\omega}\sin 2\omega\bar{u}\Big|_{-\frac{t}{2}}^{\frac{t}{2}}=-\frac{1}{\omega}\sin\omega t$$

⇨

$$x(t)=\frac{-1}{2\omega}[t\cos\omega t-\frac{1}{\omega}\sin\omega t]$$

$$=\frac{1}{2\omega^2}[\sin\omega t-\omega t\cos\omega t]$$

用Laplace transform 解ODE

指導學長:林羿州

報告者:張毓玲

日期:2008/06/16

求常微分方程式

$y''(t)+y(t)=\sin(t)$

$y(0)=0\ ,\ y'(0)=0$ 試用 Laplace 轉換求解 $y(t)$.

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{1}{s^2+1}$$

$$Y(s)=\frac{1}{(s^2+1)^2}=\frac{1}{(s^2+1)}\frac{1}{(s^2+1)}$$

$$=L\{\sin(t)\}L\{\sin(t)\}$$

$$=L\{\sin(t)*\sin(t)\}$$

$$=L\left\{\int_{-\infty}^{\infty}\sin(u)\times\sin(t-u)du\right\}$$

$$\sin m\sin n=-\frac{1}{2}\left[\cos(m+n)-\cos(m-n)\right]$$

$$\sin(u)\times\sin(t-u)$$

$$=-\frac{1}{2}\left[\cos(t)-\cos(2u-t)\right]$$

$$=\frac{1}{2}\left[\cos(2u-t)-\cos(t)\right]$$

$$\frac{1}{2}\int_0^t\cos(2u-t)-\cos(t)du$$

$$=\frac{1}{2}\left(\frac{\sin(2u-t)}{2}-u\cos(t)\right)\Bigg|_0^t$$

$$=\frac{1}{2}\left(\frac{\sin(u)}{2}-t\cos(t)-\frac{\sin(-u)}{2}\right)$$

$$=\frac{1}{2}\left(\sin(t)-t\cos(t)\right)$$

用Laplace transform 解ODE.doc 張毓玲 製表

求常微分方程式

$$t^2 y''(t) - 4ty'(t) + 6y(t) = 0$$

做 Laplace 轉換，則

(1) $Y(s)$需滿足何式？

(2) 上式是多項式或微分方程式？

(3) 求解 $Y(s)$。

(4) 求解 $y(t)$。

求常微分方程式

$$L\{t^2 y''(t)\} = s^2 Y''(s) + 4sY'(s) + 2Y(s)$$

$$t \to -\frac{d}{ds}\left(s^2 Y(s) - sy(0) - y'(0)\right)$$

$$= -2sY(s) - s^2 Y'(s) + y(0)$$

$$t^2 \to -\frac{d}{ds}\left(-2sY(s) - s^2 Y'(s) + y(0)\right)$$

$$= 2sY'(s) + 2Y(s) + 2sY'(s) + s^2 Y''(s)$$

$$= s^2 Y''(s) + 4sY'(s) + 2Y(s)$$

$$L\{ty'(t)\} = -\frac{d}{ds}\left(sY(s) - y(0)\right)$$

$$= -\left(Y(s) + sY'(s)\right)$$

$$L\{y(t)\} = Y(s)$$

$$t^2 y''(t) - 4ty'(t) + 6y(t) = 0$$

$$\begin{cases} L\{t^2 y''(t)\} = s^2 Y''(s) + 4sY'(s) + 2Y(s) \\ L\{ty'(t)\} = -\left(Y(s) + sY'(s)\right) \\ L\{y(t)\} = Y(s) \end{cases}$$

$$s^2 Y''(s) + 8sY'(s) + 12Y(s) = 0$$

求常微分方程式

(3) 求解 $Y(s)$。

$$m(m-1)s^m + 8ms^m + 12s^m = 0$$
$$m^2 - m + 8m + 12 = 0$$
$$m^2 + 7m + 12 = 0$$
$$m = -4, -3$$
$$Y(s) = c_1 s^{-3} + c_2 s^{-4}$$

(4) 求解 $y(t)$。

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{c_1 s^{-3} + c_2 s^{-4}\}$$
$$= d_1 t^2 + d_2 t^3 \qquad f(t) \to F(s)$$
$$t^n \to \frac{n!}{s^{n+1}}$$

驗証：

$$m(m-1)t^m - 4mt^m + 6t^m = 0$$
$$m^2 - m - 4m + 6 = 0$$
$$m^2 - 5m + 6 = 0$$
$$m = 2, 3$$
$$y(s) = d_1 t^2 + d_2 t^3$$

用Laplace transform 解ODE.doc 張毓玲 製表

Laplace transform

t	s
Real world	Laplace domain
$y''(t) + y(t) = \cos(t)$ $y(0) = y'(0) = 0$	$s^2 Y(s) + Y(s) = \frac{s}{s^2+1}$
$y(t)$	$Y(s) = \frac{s}{(s^2+1)^2}$
$y(t)$	$Y(s)$
$y'(t)$	$s\,Y(s)$
$y''(t)$	$s^2\,Y(s)$
$\cos(t)$	$\frac{s}{s^2+1}$
$\frac{t}{2}\sin(t)$	$Y(s) = \frac{s}{(s^2+1)^2}$

$$f(t) \longrightarrow F(s)$$

$$(t)f(t) \longrightarrow -F'(s)$$

$$\frac{f(t)}{t} \longrightarrow \int_s^\infty F(\tau)d\tau$$

p.f $L[f(t)]=F(s)$

$$-F'(s)=-\frac{dF(s)}{ds}=-\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt$$

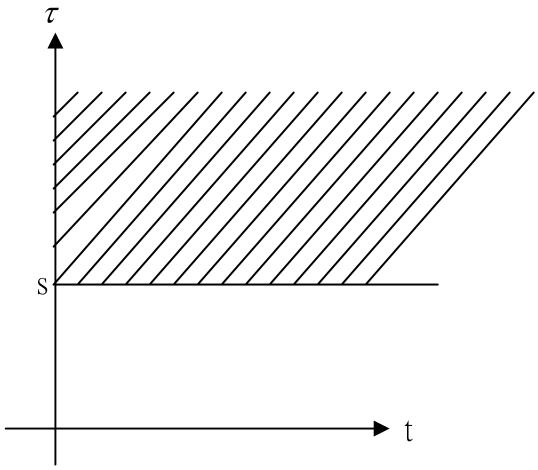
$$=\int_0^\infty -f(t)\frac{de^{-st}}{ds}dt$$

$$=\int_0^\infty -f(t)(-t)e^{-st}dt$$

$$=\int_0^\infty (t)f(t)e^{-st}dt$$

$$=L[(t)f(t)]$$

$$\Rightarrow L[(t)f(t)]=-F'(s)$$



$$\int_s^\infty F(\tau)d\tau = \int_s^\infty\int_0^\infty f(t)e^{-\tau t}dt d\tau$$

$$= \int_0^\infty\int_s^\infty f(t)e^{-\tau t}d\tau dt$$

$$= \int_0^\infty f(t)\int_s^\infty e^{-\tau t}d\tau dt$$

$$= \int_0^\infty f(t)\frac{1}{t}e^{-st}dt$$

$$=\int_0^\infty \frac{f(t)}{t}e^{-st}dt$$

$$\Rightarrow L[\frac{f(t)}{t}]=\int_s^\infty F(\tau)d\tau$$

班級：＿＿＿＿＿＿學號：＿＿＿＿＿＿＿姓名：＿＿＿＿＿＿＿

國立台灣海洋大學河海工程學系工程數學(二) 2B 班第四次小考 參考解答

1. In the course, we determine $\mathcal{L}\{J_0(t)\}$ by considering

$$t^2 J_0''(t) + tJ_0'(t) + t^2 J_0(t) = 0 \tag{1}$$

If we change Eq.(1) into

$$tJ_0''(t) + J_0'(t) + tJ_0(t) = 0 \tag{2}$$

Repeat the process and find the ODE for $\mathcal{L}\{J_0(t)\}$, and $\mathcal{L}\{J_0(t)\}$.

解 **令** $y(t) = J_0(t)$, $\mathcal{L}\{J_0(t)\} = Y(s)$ **則可得**

$$\mathcal{L}\{tJ_0''(t)\} = -\frac{d}{ds}Y(S)$$

$$\mathcal{L}\{J_0'(t)\} = sY(s) - 1$$

$$\mathcal{L}\{tJ_0''\} = -\frac{d}{ds}[s^2Y(s) - s] = -s^2Y'(s) - 2sY(s) + 1$$

因此， Eq.(2) **可變為**

$$(s^2 + 1)Y'(s) + sY(s) = 0 \Rightarrow \qquad Y(s) = \frac{c}{\sqrt{s^2+1}}$$

利用起始值定理可決定常數 c ($y(0)=1$)

$$\lim_{s\to\infty} \frac{c}{\sqrt{s^2+1}} = c = 1 \Rightarrow \qquad J_0(x) = \frac{1}{\sqrt{s^2+1}}$$

範圍:拉氏轉換　班級：____________學號：____________姓名：____________

國立台灣海洋大學河海工程學系工程數學(二) 2B 班第一次大考解答

1. Define zero order Bessel function $J_0(t)$ which is one of the solution for

$$t\ddot{y}(t) + \dot{y}(t) + ty(t) = 0 \,, \qquad (1)$$

Solve $\mathcal{L}\{J_0(t)\}$. (10%) **解 1.** $1/\sqrt{s^2+1}$

2. $J_0(t)$ can be expressed by

$$J_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m (m!)^2} t^{2m} \,, \qquad (2)$$

解 2. $\sum_{m=0}^{\infty} \frac{(-1)^m (2m)!}{2^m (m!)^2 s^{2m+1}}$

Determine the series form for $\mathcal{L}\{J_0(t)\}$. (10%)

3. Compare the solutions 1. with 2., and prove that they are the same. (10%) **解 3.略**

4. By using the Laplace transform, solve the integral equation for $y(t)$

$$y(t) = t^3 + \int_0^t \sin(t-\tau)y(\tau)d\tau \,. \text{ (10\%)} \qquad (3)$$

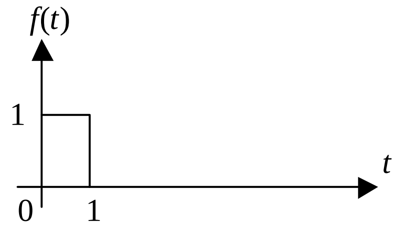
解 4. $y(t) = t^3 + \frac{1}{20}t^5$

5. Given a function $f(t)$,

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

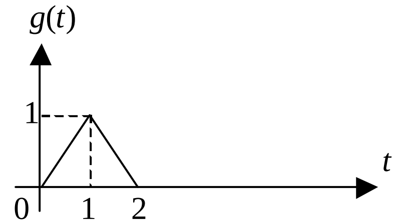
determine $f(t) * f(t)$. (10%)

解 5. $f(t) * f(t) = g(t)$



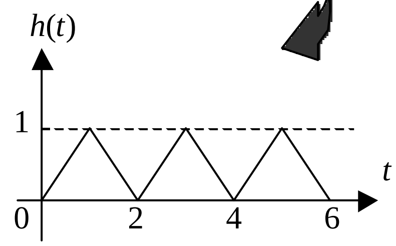
6. Given a function $g(t)$,

$$g(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$



determine $\mathcal{L}\{g(t)\}$. (10%) **解 6.** $(1-e^{-s})^2 / s^2$

7. Given a function $h(t)$, determine $\mathcal{L}\{h(t)\}$. (10%)



解 7. $\frac{1-e^{-s}}{s^2(1+e^{-s})}$

8. Explain the initial and final theorems in Laplace transform. (10%) **解 8. 略**

9. Determine $\mathcal{L}\{\delta(t-3)\}$ and $\mathcal{L}\{H(t-3)\}$, where δ and H are the Dirac-Delta function and Heaviside function, respectively. (10%) **解 9.** $e^{-3s}, (1/s)e^{-3s}$

10. Determine $\mathcal{L}^{-1}\left\{\ell n \frac{s+a}{s-a}\right\}$. (10%) **解 10.** $\frac{1}{t}(e^{at} - e^{-at})$

【工數二第一次大考解答，考試日期：2000/3/31，存檔：e:/jtchen' s/engmath/2000ex1a.doc】

Laplace transform

t	s
Real world	Laplace domain
$f(t)$	$F(s)$
$f_1(t)$	$F_1(s)$
$f_2(t)$	$F_2(s)$
$c_1 f_1(t)+c_2 f_2(t)$	$c_1 F_1(s)+c_2 F_2(s)$
$f(t-c)u(t-c)$	$e^{-cs} F(s)$
$e^{bt} f(t)$	$F(s-b)$
$f'(t)$	$sF(s)-f(0)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f(t)*g(t)$	$F(s)G(s)$
$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$

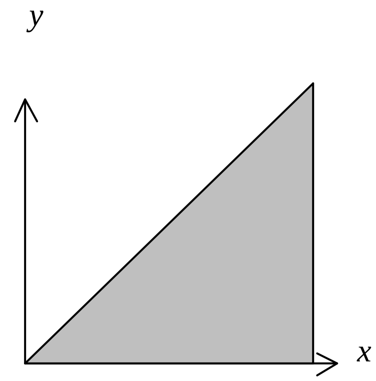
班級：_____________學號：______________姓名：______________

國立台灣海洋大學河海工程學系工程數學(二) 2B 班第三次小考 參考解答

1. $\int_0^\infty \int_y^\infty f(x, y)dxdy = \int_a^d \int_b^c f(x, y)dydx$ (1)

Please determine *a*, *b*, *c* and *d* and plot the domain of integration on *x*-*y* plane. (50%)

a=0,
b=0,
c=x,
d=∞



2. Determine $\sin(t) * \sin(t)$, where "＊" means convolution. (50%)

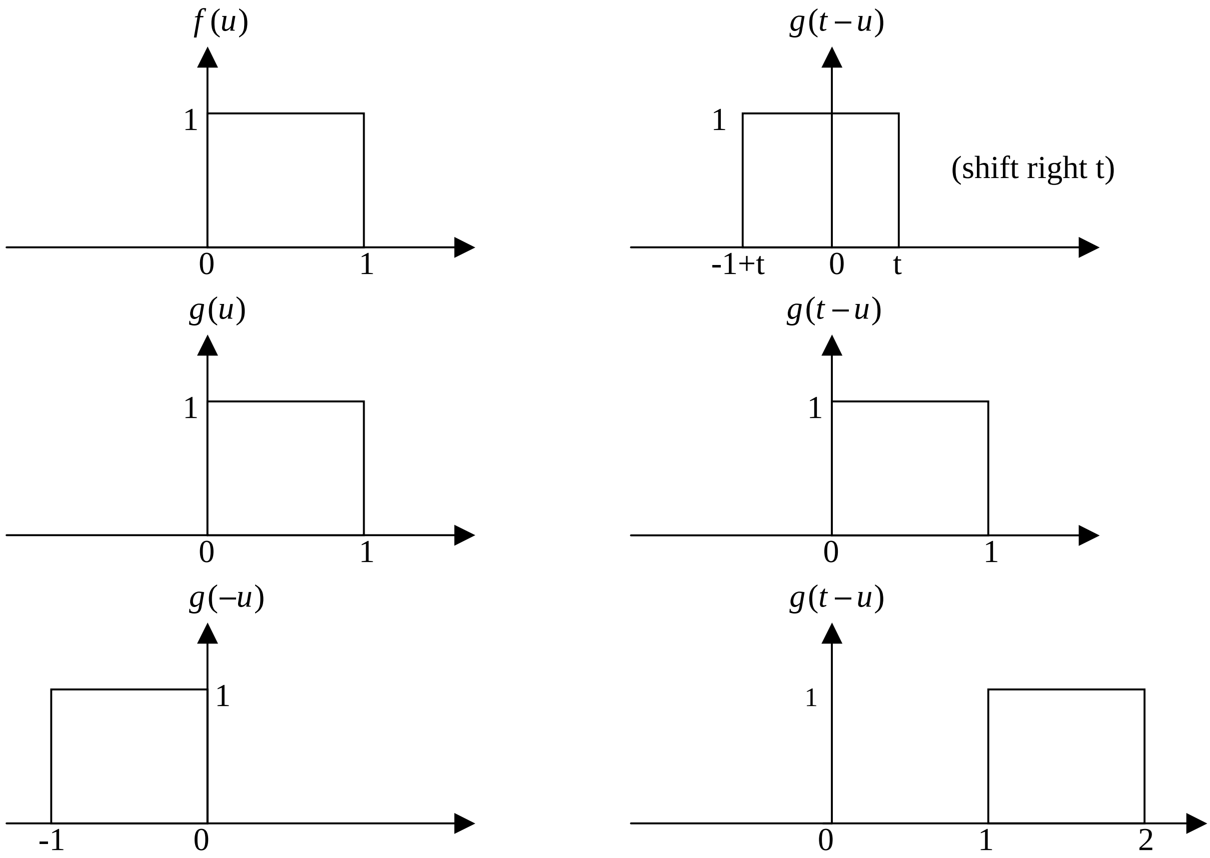
解 $\sin(t) * \sin(t)$

$$= \int_0^t \sin(\tau)\sin(t-\tau)d\tau$$

$$= \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos(t)]d\tau$$

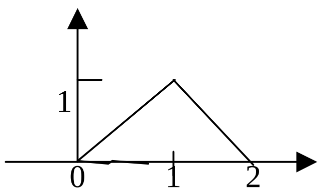
$$= \frac{1}{2}\left[\frac{\sin(2\tau - t)}{2} - \tau\cos(t)\right]\Bigg|_0^t$$

$$= \frac{1}{2}[\sin(t) - t\cos(t)]$$



$0<t<1 \quad \Rightarrow \quad \int_0^t 1\times 1 dt = t$

$1<t<2 \quad \Rightarrow \quad \int_{-1+t}^{1} 1\times 1 d\,t = 1-\left(-1-t\right) = 2-t$



(1)

$$\int_0^t f\,(t)\,e^{-st}\,d\,t = \int_0^1 1e^{-st}\,d\,t = \left(\frac{e^{-st}}{-s}\right)\Big|_{t=0}^{t=1} = \frac{-e^{-s}}{s} + \frac{1}{s} = \frac{1-e^{-s}}{s}$$

(2)

$$\int_0^1 t\,e^{-st}\,d\,t = \left(\frac{t\,e^{-st}}{-s}\right)\Bigg|_{t=0}^{t=1} - \int_0^1 \frac{e^{-st}}{-s}\,d\,t = \left(\frac{e^{-s}}{-s}\right) + \int_0^1 \frac{1}{s} e^{-st} d\,t = \frac{e^{-s}}{-s} + \frac{e^{-st}}{(s)(-s)}\Bigg|_0^1 = \frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

(3)

$$\int_1^2 (2-t)e^{-st} d\,t = \frac{(2-t)}{-s} e^{-st}\Big|_1^2 + \int_1^2 \frac{-1}{s} e^{-st} d\,t = \frac{1}{s} e^{-s} + \frac{-1e^{-st}}{s(-s)}\Big|_{t=1}^{t=2} = \frac{1}{s} e^{-s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$

$$f(t)*g(t) \quad \rightarrow \quad F(s)G(s) \qquad \int_0^1 t\,e^{-st} d\,t + \int_1^2 (2-t)\,e^{-st} d\,t = \left(\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right) + \left(\frac{1}{s} e^{-s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}\right) = \left(\frac{1-e^{-s}}{s}\right)^2$$

If the function $f(t)=\sin t$, then try to draw that

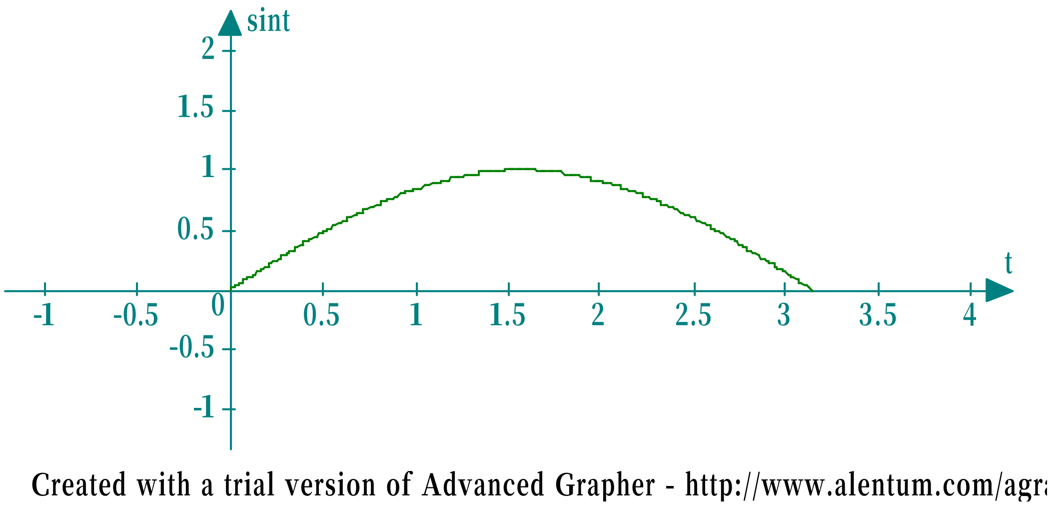
$f(t)\left[u(t-0)-u(t-p)\right]*f(t)\left[u(t-0)-u(t-p)\right]$

$$
\begin{aligned}
\int \sin(t-u)\sin u\,du &= \int(\sin t\cos u-\cos t\sin u)\sin u\,du \\
&=\frac{1}{2}\int\left[\sin t\sin(2u)-\cos t\left(1-\cos(2u)\right)\right]du \\
&=\frac{1}{2}\int\left[\left(\sin t\sin(2u)+\cos t\cos(2u)\right)-\cos t\right]du \\
&=\frac{1}{2}\int\left(\cos(t-2u)-\cos t\right)du \\
&=\frac{1}{2}\left(-\frac{1}{2}\sin(t-2u)-u\cos t\right) \\
&=-\frac{1}{4}\sin(t-2u)-\frac{1}{2}u\cos t
\end{aligned}
$$

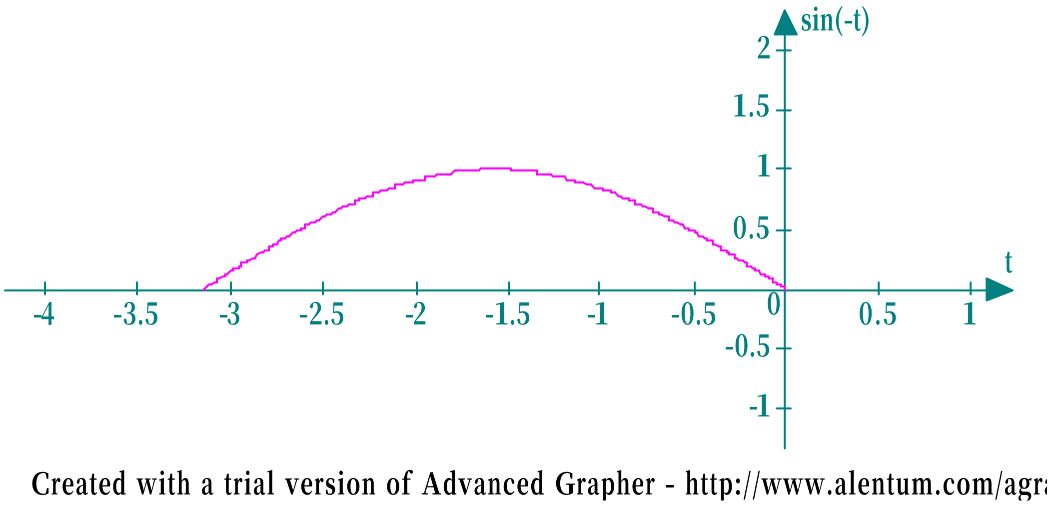
If $0\le t\le p$, then
$$
\begin{aligned}
\int \sin(t-u)\sin u\,du &= \left(-\frac{1}{4}\sin(t-2u)-\frac{1}{2}u\cos t\right)\Bigg|_{0}^{t} \\
&=\frac{1}{2}\sin t-\frac{1}{2}t\cos t
\end{aligned}
$$

If $p\le t\le 2p$, then
$$
\begin{aligned}
\int \sin(t-u)\sin u\,du &= \left(-\frac{1}{4}\sin(t-2u)-\frac{1}{2}u\cos t\right)\Bigg|_{t-p}^{p} \\
&=-\frac{1}{2}\sin t-\frac{1}{2}(2p-t)\cos t
\end{aligned}
$$

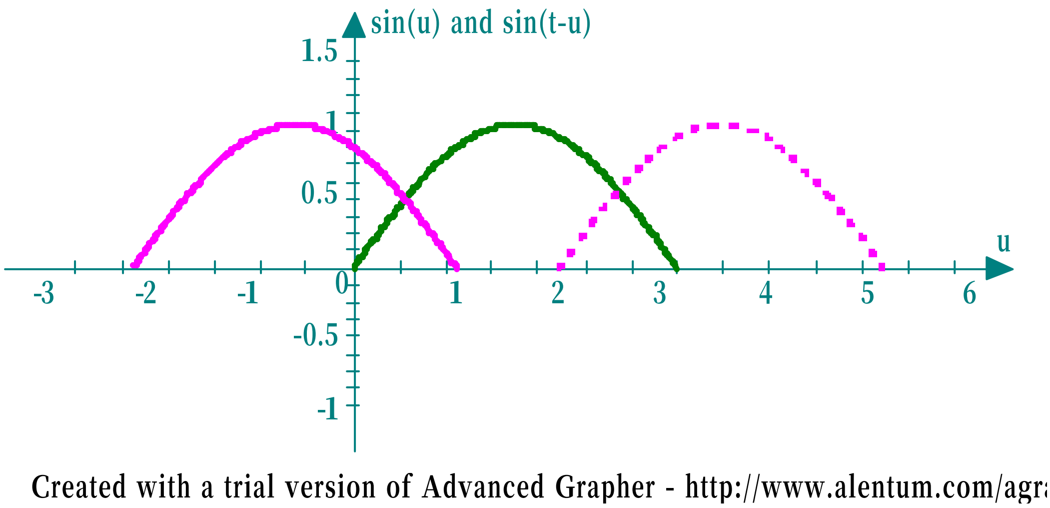
sint



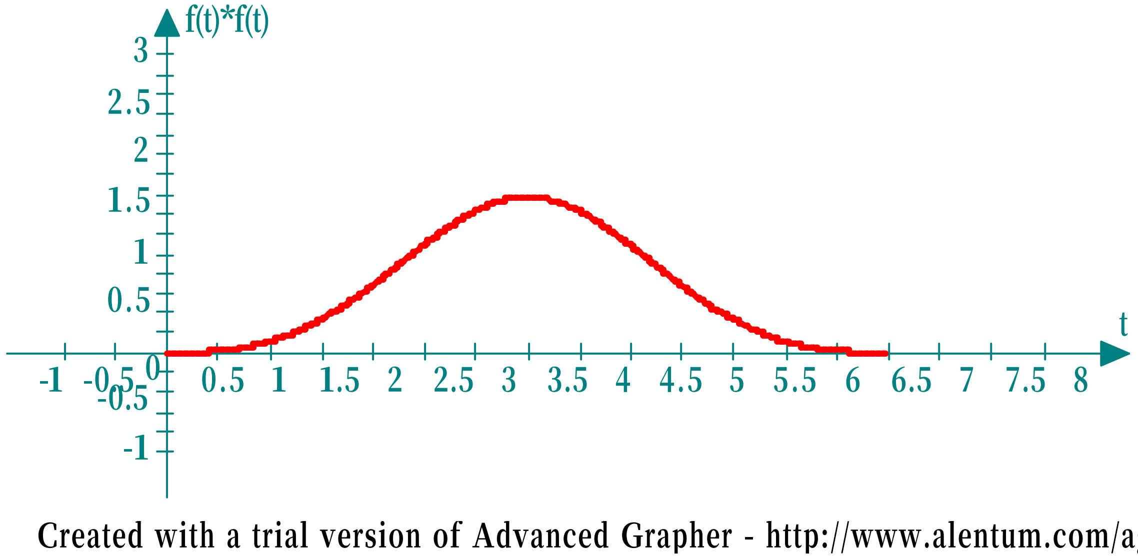
sin(-t)



sin(u) and sin(t-u)



$$f(t)\left[u(t-0)-u(t-p)\right]*f(t)\left[u(t-0)-u(t-p)\right]$$



兩函數的convolution與correlation

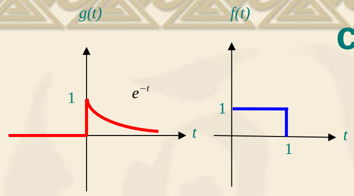
在Laplace與Fourier轉換後的結果

雅馨
指導學長：班主任&魯蛋
指導教授：陳正宗終身特聘教授
時間：2008/06/16

兩函數的convolution與correlation 在Laplace與Fourier轉換後的結果

	Laplace	Fourier
convolution	$h(t)=\int_{-\infty}^{\infty} f(u)g(t-u)du$ $H(s)=F(s)G(s)$	$h(t)=\int_{-\infty}^{\infty} f(u)g(t-u)du$ $H(\omega)=F(\omega)G(\omega)$
correlation	$c(t)=\int_{-\infty}^{\infty} f(u)g(t+u)du$ $C(s)\neq F(-s)G(s)$	$c(t)=\int_{-\infty}^{\infty} f(u)g(t+u)du$ $C(\omega)=F(-\omega)G(\omega)$ $=\overline{F}(\omega)G(\omega)$
	$F(s)=\int_{0}^{\infty} f(t)\, e^{-st} dt$	$F(\omega)=\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$

convolution



$$g(t)=\begin{cases} e^{-t}\ ,\ t\ge 0 \\ 0\ ,\ otherwise \end{cases}\qquad f(t)=\begin{cases} 1\ ,\ 0\le t\le 1 \\ 0\ ,\ otherwise \end{cases}$$

$$h(t)=\begin{cases} 0 & ,\ t<0 \\ 1-e^{-t} & ,\ 0\le t\le 1 \\ e^{-t}(e-1) & ,\ t>1 \end{cases}$$

$$F(\omega)=\frac{1}{-i\omega}(e^{-i\omega}-1)\qquad F(s)=\frac{1}{-s}(e^{-s}-1)$$

$$G(\omega)=\frac{1}{1+i\omega}\qquad G(s)=\frac{1}{1+s}$$

OK! OK!

$$H(\omega)=\frac{(e^{-i\omega}-1)}{-i\omega(1+i\omega)}=F(\omega)G(\omega)\qquad H(s)=\frac{(e^{-s}-1)}{-s(1+s)}=F(s)G(s)$$

correlation

$$c(t)=\begin{cases} 0 & ,\ t<-1 \\ 1-e^{-t-1} & ,\ -1\le t\le 0 \\ e^{-t}-e^{-t-1} & ,\ t>0 \end{cases}$$

$$F(\omega)=\frac{1}{-i\omega}(e^{-i\omega}-1)$$

$$G(\omega)=\frac{1}{1+i\omega}$$

$$C(\omega)=\frac{1}{\omega^2-i\omega}(1-e^{i\omega})$$

$$=F(-\omega)G(\omega)$$

OK!

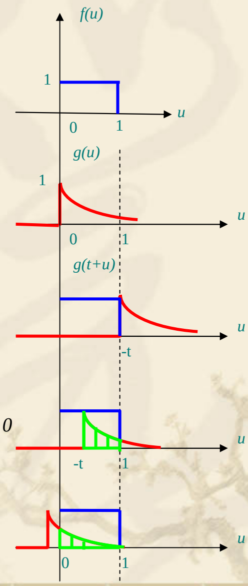
$$F(s)=-\frac{1}{s}\ (e^{-s}-1)$$

$$G(s)=\frac{1}{1+s}$$

$$C(s)=\frac{-1}{1+s}(1-e^{-1})$$

$$\ne\ F(-s)\ G(s)$$

NG!



電腦在工程數學應用-作業六

HW1: Nonlinear ODE

$$\frac{dy}{dx} = y^2, y(0) = 1$$

Solve $y(x)$ and plot $y(x)$.

HW2: Stability of ODE

$$\frac{dy}{dx} - 2y = -3e^{-x}, y(0) = y_0$$

Solve $y(x)$ and plot $y(x)$ for $y_0 = 0.97, 1.0, 1.03$.

HW3: Qualitative approach

$$\frac{dx}{dt} = e^{-t} - 2x$$

```
<< Graphics ‘PlotField‘
PlotVectorField[{1, Exp[-t]-2x}, {t,-2,3}, {x,-1,2}, ScaleFunction->(1&),
Axes->True, Ticks->None, Frame->True,AspectRatio->1]
```

HW4: Find $y(x), x = 1.1, 1.2...2.0$ by Euler Method for the ODE

$$\frac{dy}{dx} = x + 2y, y(1) = 0.5$$

HW5: Find Wronskian of $e^x, e^{-x}, cosh(x)$.

HW6: Find Wronskian of $e^x, e^{-x}, cosh(x), sinh(x)$.

HW7: If $x_p(t) = Asin(\bar{\omega}t + \phi)$ is a particular solution for

$$\ddot{x}(t) + 2\xi\omega\dot{x}(t) + \omega^2 x(t) = sin(\bar{\omega}t)$$

Find A, ϕ in terms of $(\xi, \omega, \bar{\omega})$.

HW8: Find Laplace transform of $1, t, t^2$.

HW9: Find Laplace transform of $1/t, 1/t^2$.

HW10: Solve the Riccati ODE of

$$u' = (1 - 2x^2) + xu + u^2$$

One solution is x, find the other one by Mathematica.

電腦在工程數學應用-作業七 (Laplace Transform)

```
<< Calculus`LaplaceTransform`
```

HW1: Find Laplace transform of 1.

```
<< Integrate[Exp[-s t], {t,0, Infinity}]
<< LaplaceTransform[1,t,s]
```

```
<< Integrate[Exp[a t] Exp[-s t], {t,0, Infinity}]
<< LaplaceTransform[Exp[a t], t, s]
```

HW2: Find Laplace transform of t.

```
<< Integrate[t Exp[-s t], {t,0, Infinity}
<< LaplaceTransform[t, t, s]
```

HW3: Solve the following ODE by Laplace transform:

$$\ddot{x}(t)+\omega^2 x(t) = cos(\bar{\omega}t), if\ \bar{\omega} \neq \omega,\ or\ \bar{\omega} = \omega$$

HW4: Solve the following ODE by Laplace transform:

$$\ddot{x}(t)+\omega^2 x(t) = sin(\bar{\omega}t), if\ \bar{\omega} \neq \omega,\ or\ \bar{\omega} = \omega$$

HW5: Find Inverse Laplace Transform of $s/(s^2 + 64)$.

```
<< InverseLaplaceTransform[s/(s^2+64), s, t]
```

HW6: Find Inverse Laplace Transform of $s/(s^4 - 64)$.

```
<< InverseLaplaceTransform[s/(s^4 - 64), s, t]
```

HW7: Find $\mathcal{L}^{-1}\{dF(s)/ds\}$.

```
<< InverseLaplaceTransform[D[LaplaceTransform[f[t]],t,s],s],s,t]
```

HW8: Find Laplace transform for Heaviside function $\mathcal{L}\{H(t-a)\}$.

```
<< Transform[UnitStep[t-a],t,s]
```

HW9: Find the solution by Laplace transform

$$y''(t) - 5y'(t) + 6y(t) = e^{2t}$$

subjected to

$$y(0) = 1, y'(0) = -2$$

電腦在工程數學應用-作業九 (Laplace Transform)

```
<< Calculus‘LaplaceTransform‘

<< LPT.m (* Load LPT.m *)
```

HW1: Define unit step function for $H(t)$.

```
<< UnitStep[x_]:=If[TrueQ[x>0],1,0]/;NumberQ[N[x]]
```

HW2: Plot $Sin[2t]UnitStep[Sin[2t]]$ versus t from 0 to 4π.

HW3: Take Laplace transform of $Sin[2t]UnitStep[Sin[2t]]$.

HW4: Find Laplace transform for Heaviside function $\mathcal{L}\{H(t-a)\}$.

```
<< LaplaceTransform[UnitStep[t-a],t,s]
```

HW5: Find Laplace transform for Dirac Delta function $\mathcal{L}\{\delta(t-a)\}$.

HW6: Find Laplace transform for $\sqrt{t}$.

HW7: Find Laplace transform for $y(t)$ where

$$ty''(t)+y'(t)+ty(t)=0$$

and $y(0)=0, \dot{y}(0)=1$.

HW8: Find the convolution of $sin(t)$ and $sin(t)$.
1. 期末報告, 分書面與口頭報告, 兩人一組, 自行選定題目。
2. 書面報告, 以 10 頁爲原則, 請打字。
3. 口頭報告, 以 20 分鐘爲原則,5 分鐘討論, 請使用投影機與 PC Notebook。
4. 口頭報告, 在星期一早上三、四堂, 於本系會議室舉行。
5. 投影機、麥克風與 PC Notebook, 由下一組同學幫忙準備。

89 下 工數(二) 第一次期中考 90/3/29

1. Find $L^{-1}\left\{\dfrac{s+3}{s(s^2+1)}e^{-3s}\right\}$ (12%)

   Hint : $L[\delta(t-t_0)]=e^{-st_0}$, $L[H(t-t_0)]=\dfrac{1}{s}e^{-st_0}$

2. For the following 1st order O.D.E

$$\frac{dy}{dx}=\frac{2x+3y-4}{-3x+2y+3} \quad , \qquad (2)$$

use the method specified below to solve the general solution.
(No credit for other methods.)

(a) Use a transformation, $(x,y)\rightarrow(X,Y)$, so that equation (2) becomes a homogeneous equation. Then solve this homogeneous equation with $Y=vX$ (9%)

(b) Solve equation (2) as an exact equation (if not exact, find the integrating factor). If the solution passes $(x, y) = (1, 1)$, write down the specific solution. (12%)

3. Find the general solutions for the following ODEs :

(a) $\dfrac{d^2y}{dx^2}+9y = x\cos x$

(16%)

(b) $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=\sin 2x+\cos 3x$ (17%)

4. For the boundary value problem (20%)

$$\frac{d^2y}{dx^2}=2x, \quad y(0)=2, \quad y(1)=0,$$

(a) Formulate the Green's function $G(x, z)$:

(a1) governing equation (a2) boundary condition

(a3) jump condition (a4) continuity condition

(b) Find $G(x, z)$

(c) Write the solution $y(x)$ in terms of the Green's function $G(x, z)$.

5. Solve the initial value problem (14%)

$$y\frac{d^3y}{dt^3}+\left(3\frac{dy}{dt}+y\right)\frac{d^2y}{dt^2}+\left(\frac{dy}{dt}\right)^2=e^{-t},$$

$$y(0)=1, \quad \frac{dy}{dt}(0)=\frac{d^2y}{dt^2}(0)=0,$$

for $y(t)$.

1. $y'' - 5y' + 6y = 2e^{t}$，$y(0) = 0, y'(0) = 0$

求 $y(t) = ?$ 請用上學期學過的方法，以及 Laplace transform。

2. $y'' - y = 2\sin t$，$y(0) = 0, y'(0) = 0$

求 $y(t) = ?$ 請用上學期學過的方法，以及 Laplace transform。

1. $y'' - 5y' + 6y = 2e^t$，$y(0) = 0, y'(0) = 0$

求 $y(t) = ?$ 請用上學期學過的方法，以及 Laplace transform。

(1) 令

$$y_h = c_1 e^{-2t} + c_2 e^{3t}$$

$$y_p = e^t$$

$$\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{3t} + e^t \quad \begin{cases} c_1 = -2 \\ c_2 = 1 \end{cases}$$

(2)

$$L[y''] - 5L[y'] + 6L[y] = 2e^t$$

$$s^2 Y(s) - sy(0) - y'(0) - 5[sY(s) - y(0)] + 6Y(s) = \frac{2}{s-1}$$

$$s^2 Y(s) - 5sY(s) + 6Y(s) = \frac{2}{s-1}$$

$$(s^2 - 5s + 6)Y(s) = \frac{2}{s-1}$$

$$Y(s) = \frac{\frac{2}{s-1}}{(s^2 - 5s + 6)}$$

$$= \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-1} \begin{cases} A = -2 \\ B = 1 \\ C = 1 \end{cases}$$

$$y(t) = L^{-1}[Y(s)] = -2e^{2t} + e^{3t} + e^t$$

2. $y'' - y = 2\sin t$, $y(0) = 0, y'(0) = 0$

求 $y(t) = ?$請用上學期學過的方法，以及 Laplace transform。

(1) 令 $y_h = c_1 e^{-t} + c_2 e^{t}, y_p = A\sin t + B\cos t$

$$=> y(t) = y_h + y_p = c_1 e^{-t} + c_2 e^{t} - \sin t \qquad \begin{cases} c_1 = -\dfrac{1}{2} \\ c_2 = \dfrac{1}{2} \end{cases}$$

(2)

$$L[y''] - L[y] = \frac{2}{s^2+1}$$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{2}{s^2+1}$$

$$Y(s)(s^2-1) = \frac{2}{s^2+1}$$

$$Y(s) = \frac{2}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-1}$$

$$A = 0, B = -1, C = -\frac{1}{2}, D = \frac{1}{2}$$

$$y(t) = L^{-1}[y(s)] = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - \sin t$$

第十三次小考

1. Solve particular solution (use Laplace transform)

(1)$y' - y = e^t$, no initial condition

(2) Solve the total solution

$y' - y = e^t$ $\quad y(0) = 0$ $\quad y'(0) = 1$

$y' - y = e^t$ $\quad y(0) = 1$ $\quad y'(0) = 0$

第十三次小考

1. Solve particular solution (use Laplace transform)

(1)$y' - y = e^t$, no initial condition

$\Rightarrow \mathcal{L}[y'] - \mathcal{L}[y] = \mathcal{L}[e^t]$

$sY(s) - y(0) - Y(s) = \frac{1}{s-1} \qquad y(0) = k \ \ (\text{constant})$

$(s-1)Y(s) = \frac{1}{s-1} + k$

$Y(s) = \frac{1}{(s-1)^2} + \frac{k}{s-1}$

$= -\frac{d}{ds}\left(\frac{1}{s-1}\right) + \frac{k}{s-1}$

$\therefore y(t) = \mathcal{L}^{-1}[Y(s)] = te^k + ke^t$

(2) Solve the total solution

$y' - y = e^t \qquad y(0) = 0 \qquad y'(0) = 1$

$y' - y = e^t \qquad y(0) = 1 \qquad y'(0) = 0$

(1)I.C.S. $y(0) = 0$ (1)

$\Rightarrow Y(s) = \frac{1}{(s-1)^2}$

$\Rightarrow y(t) = te^t$

(2) I.C.S. $y(0) = 1$ (2)

$\Rightarrow Y(s) = \frac{1}{(s-1)^2} + \frac{1}{(s-1)}$

$\Rightarrow y(t) = te^t + e^t$

工數第三次大考(Fourier and Laplace transform) 09:20-12:00

1. 拉氏轉換: $F(s) \equiv L(f) = \int_0^\infty e^{-st} f(t)dt$　　拉氏反轉換: $f(t) = L^{-1}(F(s)) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds$

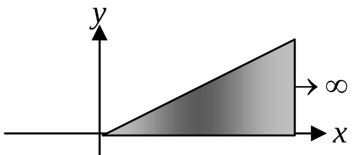
2. Laplace transform table:

	f(t)	F(s)		f(t)	F(s)
1	1	$\frac{1}{s}$	7	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
2	t	$\frac{1}{s^2}$	8	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$\frac{2!}{s^3}$	9	$\cosh(at)$	$\frac{s}{s^2-a^2}$
4	$t^n(n=0,1,2,...)$	$\frac{n!}{s^{n+1}}$	10	$\sinh(at)$	$\frac{a}{s^2-a^2}$
5	$t^a(a>0)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

3. 第一平移定理，s-平移→ $L(e^{at}f(t)) = F(s-a)$
4. 微分的拉氏轉換: $L(f') = sL(f) - f(0), L(f'') = s^2L(f) - sf(0) - f'(0)$
5. 積分的拉氏轉換: $L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}F(s)$
6. 階梯函數的拉氏轉換: $L(u(t-a)) = \frac{e^{-as}}{s}$
7. 第二平移定理，t-平移 → $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$
8. Dirac 衝擊函數的拉氏轉換: $L(\delta(t-a)) = e^{-as}$
9. 摺積定義: $(f*g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$
10. correlation 定義: $(f\circ g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t+\tau)d\tau$
11. 摺積的拉氏轉換: $L((f*g)(t)) = F(s)G(s)$
12. 摺積的傅立葉轉換: $F((f*g)(t)) = F(\omega)G(\omega)$, $\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega$
13. correlation 的傅立葉轉換: $F((f\circ g)(t)) = F(-\omega)G(\omega) = \bar{F}(\omega)G(\omega)$
14. 拉氏轉換的微分: $L(tf(t)) = -F'(s)$，拉氏轉換的積分 $L(\frac{f(t)}{t}) = \int_s^\infty F(\tilde{s})d\tilde{s}$

日期：2008 年 6 月 19 日 姓名：__________ 學號：__________ Mid2008-test3.doc Chen J T 製表

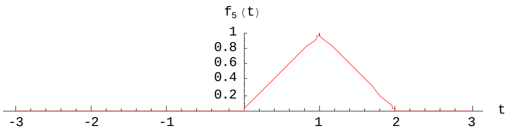
1 $\int_0^\infty\int_y^\infty f(x,y)dxdy=\int_a^c\int_b^d f(x,y)dydx$



(a) Please determine a, b, c and d, where the domain of integration is shown. (10 %)

(b) $\int_0^X\int_0^{X_1} f(\tau)d\tau dx_1 \Rightarrow \int_e^f \Box f(\xi)d\xi$ Please determine e, f and $\Box$. (10 %)

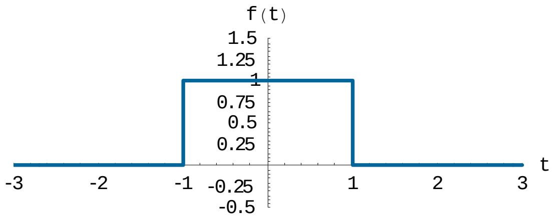
2. IF Laplace transform of f(t) is $F_L(s)$, Fourier transform is $F_f(\omega)$ where f(t) is shown as



Please plot the inverse transform of $\frac{1}{(1-e^{-2s})}F_L(s)$. (10 %) Please find $\int_{-\infty}^{\infty}\left|F_f(\omega)\right|^2 d\omega$ (10 %)

3. Given f(t)=g(t)=1, -1<t<1, otherwise zero, please find $(f*g)=\int_{-\infty}^{\infty} f(u)g(t-u)du=h(t)$ (10 %)

4. Given f(t)=g(t)=1, -1<t<1, otherwise zero, please find $(f\circ g)=\int_{-\infty}^{\infty} f(u)g(t+u)du=c(t)$ (10 %)



5. Find $H(\omega)=\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$ (5%), $C(\omega)=\int_{-\infty}^{\infty} c(t)e^{-i\omega t}dt$ (5%), is it right that $H(\omega)=C(\omega)$? Why ? (5%)

6. If $f(t)=\sqrt{t}$ and $F(s)=\frac{\sqrt{\pi}}{2s^{3/2}}$, find the Laplace transform of $f(t)=\frac{1}{\sqrt{t}}$ (10 %)

7. Find the inverse Laplace transform: (20%)

(a) $\frac{se^{-s}}{s^2+\omega^2}$ (b) $\frac{1}{(s^2+0.5s)}$ (c) $\frac{1}{(s-3)(s+5)}$ (d) $\frac{s}{\left(s^2+16\right)^2}$

8. Solve the ODE using Laplace transform (10 %)

$$\dot{y}+y=e^{-t},\quad y(0)=0,\quad \dot{y}(0)=1$$

$$\dot{y}+y=e^{-t},\quad y(0)=1,\quad \dot{y}(0)=0$$

9. Transform the following ODE once and twice (10 %)

$$t^2y''(t)-4ty'(t)+6y(t)=0$$

10. Solve the ODE using the Laplace transform. (10 %)

$$y''(t)+y(t)=\cos(t)\qquad y'(0)=y(0)=0$$

日期：2008 年 6 月 19 日 姓名：__________ 學號：__________ Mid2008-test3.doc Chen J T 製表

國立台灣海洋大學　河海工程學系 2B　　工程數學（二）陳正宗終身特聘教授

工數第三次大考(Fourier and Laplace transform) 09:20-12:00

1. 拉氏轉換: $F(s)\equiv L(f)=\int_0^{\infty} e^{-st} f(t)dt$　　拉氏反轉換: $f(t)=L^{-1}(F(s))=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds$

2. Laplace transform table:

	f(t)	F(s)		f(t)	F(s)
1	1	$\frac{1}{s}$	7	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
2	t	$\frac{1}{s^2}$	8	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$\frac{2!}{s^3}$	9	$\cosh(at)$	$\frac{s}{s^2-a^2}$
4	$t^n (n=0,1,2,...)$	$\frac{n!}{s^{n+1}}$	10	$\sinh(at)$	$\frac{a}{s^2-a^2}$
5	$t^a (a>0)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

3. 第一平移定理，s-平移→ $L(e^{at}f(t))=F(s-a)$

4. 微分的拉氏轉換: $L(f')=sL(f)-f(0), L(f'')=s^2L(f)-sf(0)-f'(0)$

5. 積分的拉氏轉換: $L\left(\int_0^t f(\tau)d\tau\right)=\frac{1}{s}F(s)$

6. 階梯函數的拉氏轉換: $L(u(t-a))=\frac{e^{-as}}{s}$

7. 第二平移定理，t-平移 → $L\{f(t-a)u(t-a)\}=e^{-as}F(s)$

8. Dirac 衝擊函數的拉氏轉換: $L(\delta(t-a))=e^{-as}$

9. 摺積定義: $(f*g)(t)\equiv\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

10. correlation 定義: $(f\circ g)(t)\equiv\int_{-\infty}^{\infty} f(\tau)g(t+\tau)d\tau$

11. 摺積的拉氏轉換: $L((f*g)(t))=F(s)G(s)$

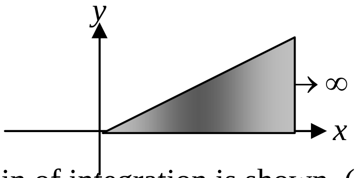
12. 摺積的傅立葉轉換: $F((f*g)(t))=F(\omega)G(\omega)$, $\int_{-\infty}^{\infty} f^2(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega$

13. correlation 的傅立葉轉換: $F((f\circ g)(t))=F(-\omega)G(\omega)=\bar{F}(\omega)G(\omega)$

14. 拉氏轉換的微分: $L(tf(t))=-F'(s)$，拉氏轉換的積分 $L(\frac{f(t)}{t})=\int_s^{\infty} F(\tilde{s})d\tilde{s}$

日期：2008 年 6 月 19 日　姓名：____________　學號：___________Mid2008-test3.doc Chen J T 製表

1 $\int_0^{\infty}\int_y^{\infty} f(x,y)dxdy = \int_a^c\int_b^d f(x,y)dydx$



(a) Please determine a, b, c and d, where the domain of integration is shown. (10 %)

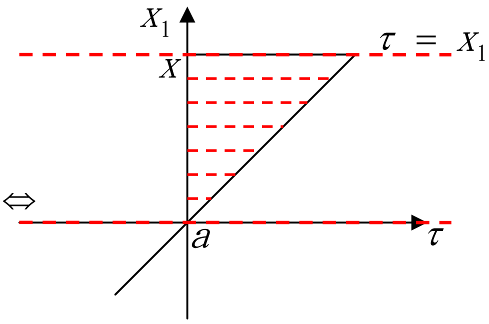
Sol.

$a=0,\quad b=0,\quad c=\infty,\quad d=x$

(b) $\int_0^X\int_0^{X_1} f(\tau)d\tau dx_1 \Rightarrow \int_e^f \square f(\xi)d\xi$ Please determine e, f and $\square$. (10 %)

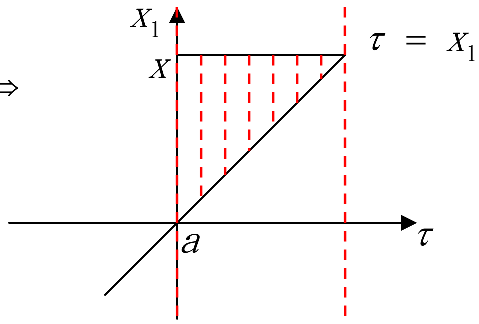
Sol.

$\int_0^X\int_0^{X_1} f(\tau)d\tau dx_1$ ⇔



⇕　　⇕

$\int_0^X\int_\tau^X f(\tau)dx_1 d\tau$ ⇔



$= \int_0^X (x-\tau)f(\tau)d\tau$

$= \int_0^X (x-\xi)f(\xi)d\xi$

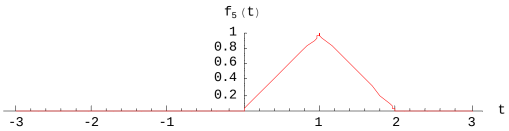
$e=0$

$f=x$

$\square = (x-\xi)$

日期：2008 年 6 月 19 日　姓名：__________　學號：__________ Mid2008-test3.doc Chen J T 製表

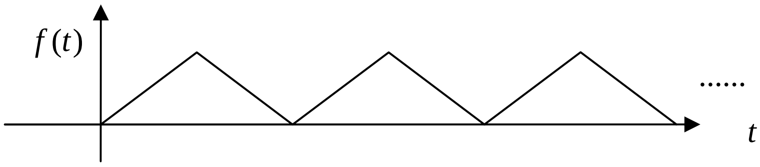
2. IF Laplace transform of f(t) is $F_L(s)$, Fourier transform is $F_f(\omega)$ where f(t) is shown as



Please plot the inverse transform of $\frac{1}{(1-e^{-2s})}F_L(s)$. (10 %) Please find $\int_{-\infty}^{\infty}|F_f(\omega)|^2\,d\omega$ (10 %)

Sol

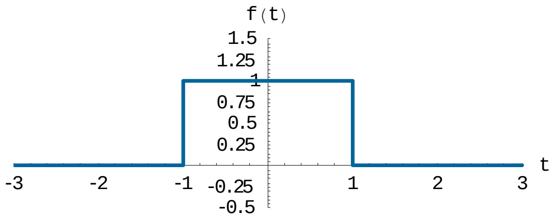
(a).



(b). $\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega = 2\pi\int_{-\infty}^{\infty} f^2(t)dt = 2\pi(\int_0^1 t^2dt + \int_1^2 (2-t)^2 dt) = \frac{4}{3}\pi$

3. Given f(t)=g(t)=1, -1<t<1, otherwise zero, please find $(f*g)=\int_{-\infty}^{\infty} f(u)g(t-u)du = h(t)$ (10 %)

4. Given f(t)=g(t)=1, -1<t<1, otherwise zero, please find $(f\circ g)=\int_{-\infty}^{\infty} f(u)g(t+u)du = c(t)$ (10 %)



Sol.

3.

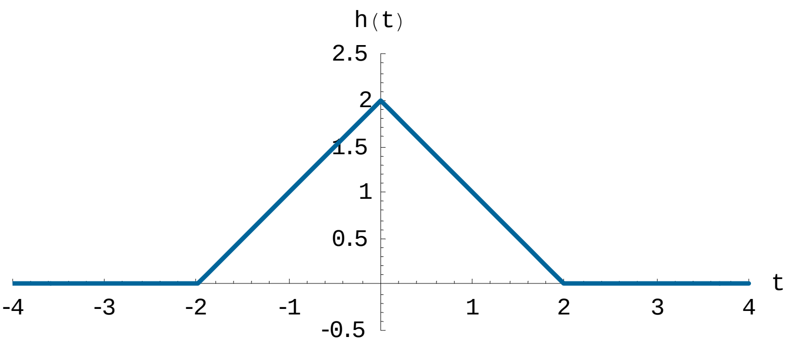
If $-2\le t\le 0$, $h(t)=\int_{-1}^{1+t} 1du = (1+t)-(-1) = 2+t$

If $0\le t\le 2$,

$h(t)=\int_{t-1}^{1} 1du = 1-(t-1) = 2-t$

If $t<-2$ or $t>2$, $h(t)=0$

$$h(t)=\begin{cases}2+t, & -2\le t\le 0\\ 2-t, & 0\le t\le 2\\ 0, & \text{otherwise}\end{cases}$$



4.

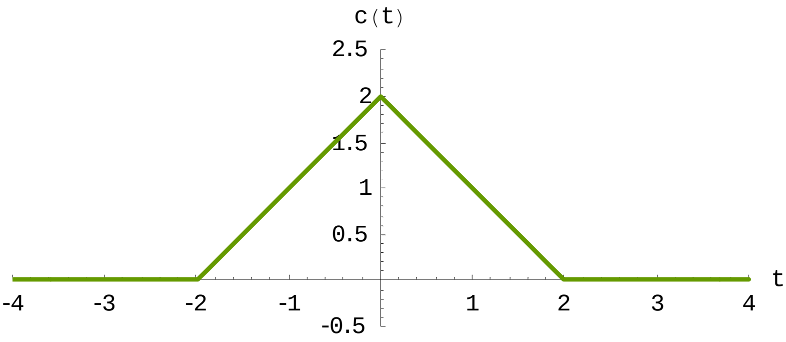
If $-2 \le t \le 0$, $c(t) = \int_{-1-t}^{1} 1du = 1-(-1-t) = 2+t$

If $0 \le t \le 2$,

$c(t) = \int_{-1}^{1-t} 1du = (1-t)-(-1) = 2-t$

If $t < -2$ or $t > 2$, $c(t) = 0$

$$c(t) = \begin{cases} 2+t, -2 \le t \le 0 \\ 2-t, 0 \le t \le 2 \\ 0, \quad \text{otherwise} \end{cases}$$



5. Find $H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$ (5%), $C(\omega) = \int_{-\infty}^{\infty} c(t) e^{-i\omega t} dt$ (5%), is it right that $H(\omega) = C(\omega)$? Why ?(5%)

Sol:

方法一:

a. $h(t) = \begin{cases} 2+t, -2 \le t \le 0 \\ 2-t, 0 \le t \le 2 \\ 0, \quad \text{otherwise} \end{cases}$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_{-2}^{0} (2+t) e^{-i\omega t} dt + \int_{0}^{2} (2-t) e^{-i\omega t} dt$$

$$= \left[\frac{1}{\omega^2}(1-e^{2i\omega}) - \frac{2}{i\omega} \right] + \left[\frac{1}{\omega^2}(1-e^{-2i\omega}) + \frac{2}{i\omega} \right]$$

$$= \frac{1}{\omega^2}(1-e^{2i\omega}) + \frac{1}{\omega^2}(1-e^{-2i\omega})$$

$$= \frac{2-(e^{2i\omega}+e^{-2i\omega})}{\omega^2}$$

b. $c(t) = \begin{cases} 2+t, -2 \le t \le 0 \\ 2-t, 0 \le t \le 2 \\ 0, \quad \text{otherwise} \end{cases}$

$$\therefore C(\omega) = \int_{-\infty}^{\infty} c(t) e^{-i\omega t} dt$$

$$= \int_{-2}^{0} (2+t) e^{-i\omega t} dt + \int_{0}^{2} (2-t) e^{-i\omega t} dt$$

$$= \left[\frac{1}{\omega^2}(1-e^{2i\omega}) - \frac{2}{i\omega} \right] + \left[\frac{1}{\omega^2}(1-e^{-2i\omega}) + \frac{2}{i\omega} \right]$$

$$= \frac{1}{\omega^2}(1-e^{2i\omega}) + \frac{1}{\omega^2}(1-e^{-2i\omega})$$

$$= \frac{2-(e^{2i\omega}+e^{-2i\omega})}{\omega^2}$$

方法二:

$a.\ H(\omega) = F(\omega) G(\omega)$

$$F(\omega) = \int_{-1}^{1} 1 \times e^{-i\omega t} dt = \frac{-(e^{-i\omega t} - e^{i\omega t})}{i\omega}$$

$$G(\omega)=\int_{-1}^{1}1\times e^{-i\omega t}dt=\frac{-(e^{-i\omega t}-e^{i\omega t})}{i\omega}$$

$$\therefore H(\omega)=F(\omega)G(\omega)=\frac{2-(e^{2i\omega}+e^{-2i\omega})}{\omega^2}$$

$$b.\ C(\omega)=F(-\omega)G(\omega)$$

$$F(-\omega)=\int_{-1}^{1}1\times e^{i\omega t}dt=\frac{e^{i\omega t}-e^{-i\omega t}}{i\omega}$$

$$G(\omega)=\int_{-1}^{1}1\times e^{-i\omega t}dt=\frac{-(e^{-i\omega t}-e^{i\omega t})}{i\omega}$$

$$\therefore C(\omega)=F(-\omega)G(\omega)=\frac{2-(e^{2i\omega}+e^{-2i\omega})}{\omega^2}$$

$$\rightarrow H(\omega)=C(\omega)=\frac{2-(e^{2i\omega}+e^{-2i\omega})}{\omega^2}$$

$\because f(t)$ 為偶函數

$\therefore F(\omega)=F(-\omega)$，故可得證

6. If $f(t)=\sqrt{t}$ and $F(s)=\frac{\sqrt{\pi}}{2s^{3/2}}$, find the Laplace transform of $g(t)=\frac{1}{\sqrt{t}}$ (10 %)

Sol.

方法一:

$$g(t)=\frac{1}{\sqrt{t}}\rightarrow G(s)\quad ;tg(t)=\frac{t}{\sqrt{t}}=\sqrt{t}\rightarrow -G'(s)=F(s)$$

$$\rightarrow G'(s)=-\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}=-\int\frac{s^{-\frac{3}{2}}\sqrt{\pi}}{2}$$

$$=-\frac{(-2)\sqrt{\pi}}{2}s^{-\frac{1}{2}}=s^{-\frac{1}{2}}\sqrt{\pi}=\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}=\sqrt{\frac{\pi}{s}}$$

方法二:

$$f(t)=\sqrt{t}\qquad ;f'(t)=-\frac{1}{2\sqrt{t}}$$

$$-2f'(t)=\frac{1}{\sqrt{t}}\ and\ L\{f'(t)\}=sF(s)-f(0)$$

$$\rightarrow G(s)=L\{-2f'(t)\}=-2L\{f'(t)\}$$

$$=-2\frac{\sqrt{\pi}}{2s^{\frac{1}{2}}}=\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}=\sqrt{\frac{\pi}{s}}$$

7. Find the inverse Laplace transform: (20%)

(a) $\frac{se^{-s}}{s^2+\omega^2}$ (b) $\frac{1}{(s^2+0.5s)}$ (c) $\frac{1}{(s-3)(s+5)}$ (d) $\frac{s}{\left(s^2+16\right)^2}$

Sol.

(a)

$$F(s)=\frac{se^{-s}}{s^2+\omega^2}=e^{-s}\times\frac{s}{s^2+\omega^2}$$

$$\Rightarrow f(t)=u(t-1)\cos[\omega(t-1)]$$

(b)

$$F(s)=\frac{1}{s^2+0.5s}=\frac{1}{s(s+0.5)}=\frac{2}{s}-\frac{2}{(s+0.5)}$$

$$\Rightarrow f(t)=2-2e^{-0.5t}$$

(c)

$$F(s)=\frac{1}{(s-3)(s+5)}=\frac{1}{8}\times\frac{1}{s-3}-\frac{1}{8}\times\frac{1}{s+5}$$

$$\Rightarrow f(t)=\frac{1}{8}e^{3t}-\frac{1}{8}e^{-5t}$$

(d)

$$F(s)=\frac{s}{\left(s^2+16\right)^2}=s(s^2+4^2)^{-2}=-\frac{1}{2}\frac{d[(s^2+4^2)^{-1}]}{ds}=-\frac{1}{8}\frac{d[\frac{4}{s^2+4^2}]}{ds}$$

$$\Rightarrow f(t)=\frac{1}{8}t\sin(4t)$$

8. Solve the ODE using Laplace transform (10 %)

$$(a).\ \dot{y}+y=e^{-t},\quad y(0)=0,\quad \dot{y}(0)=1$$

$$(b).\ \dot{y}+y=e^{-t},\quad y(0)=1,\quad \dot{y}(0)=0$$

Sol.

(a).

$$\Rightarrow L[\dot{y}]+L[y]=L\left[e^{-t}\right]$$

$$sY(s)-y(0)+Y(s)=\frac{1}{s+1},$$

$$(s+1)Y(s)=\frac{1}{s+1}$$

$$Y(s)=\frac{1}{(s+1)^2}$$

$$=-\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

$$\therefore\ y(t)=L^{-1}\left[Y(s)\right]=te^{-t}$$

(b).

$$\Rightarrow L[\dot{y}]+L[y]=L\left[e^{-t}\right]$$

$$sY(s)-y(0)+Y(s)=\frac{1}{s+1}$$

$$(s+1)Y(s)=\frac{1}{s+1}+1$$

$$Y(s)=\frac{1}{(s+1)^2}+\frac{1}{s+1}$$

$$=-\frac{d}{ds}\left(\frac{1}{s+1}\right)+\frac{1}{s+1}$$

$$\therefore\ y(t)=L^{-1}\left[Y(s)\right]=te^{-t}+e^{-t}$$

9. Transform the following ODE, $t^2y''(t)-4ty'(t)+6y(t)=0$, once and twice (10 %)

Sol.

$$\boldsymbol{L}\left\{\boldsymbol{y}(\boldsymbol{t})\right\}=\boldsymbol{Y}(\boldsymbol{s})$$

$$\boldsymbol{L}\left\{\boldsymbol{t}^2\boldsymbol{y}''(\boldsymbol{t})\right\}=\boldsymbol{s}^2\boldsymbol{Y}''(\boldsymbol{s})+4\boldsymbol{s}\boldsymbol{Y}'(\boldsymbol{s})+2\boldsymbol{Y}(\boldsymbol{s})$$

$$\boldsymbol{L}\left\{\boldsymbol{t}\boldsymbol{y}'(\boldsymbol{t})\right\}=-\frac{\boldsymbol{d}}{\boldsymbol{ds}}\left(\boldsymbol{s}\boldsymbol{Y}(\boldsymbol{s})-\boldsymbol{y}(0)\right)=-\left(\boldsymbol{Y}(\boldsymbol{s})+\boldsymbol{s}\boldsymbol{Y}'(\boldsymbol{s})\right)$$

$$\boldsymbol{t}^2\boldsymbol{y}''(\boldsymbol{t})-4\boldsymbol{t}\boldsymbol{y}'(\boldsymbol{t})+6\boldsymbol{y}(\boldsymbol{t})=0\rightarrow\boldsymbol{s}^2\boldsymbol{Y}''(\boldsymbol{s})+8\boldsymbol{s}\boldsymbol{Y}'(\boldsymbol{s})+12\boldsymbol{Y}(\boldsymbol{s})=0$$

Transform twice:

由於 Euler-Cauchy Equation 拉氏轉換兩次可以得到原來的微分方程，故可得

$$\boldsymbol{p}^2\overline{\boldsymbol{Y}}''(\boldsymbol{p})-4\boldsymbol{p}\overline{\boldsymbol{Y}}'(\boldsymbol{p})+6\overline{\boldsymbol{Y}}(\boldsymbol{p})=0$$

10. Solve the ODE using the Laplace transform. (10 %)

$$y''(t)+y(t)=\cos(t)\quad y'(0)=y(0)=0$$

Sol:

$$L\left[y''(t)+y(t)=cos(t)\right]$$

$$\Rightarrow S^2Y(s)-Sy(0)-y'(0)+Y(s)=\frac{S}{S^2+1}$$

$$\Rightarrow S^2Y(s)+Y(s)=\frac{S}{S^2+1}$$

$$\Rightarrow (S^2+1)Y(s)=\frac{S}{S^2+1}$$

$$\Rightarrow Y(s)=\frac{S}{(S^2+1)^2}$$

$$L^{-1}\left[Y(s)=\frac{S}{(S^2+1)^2}\right]=\frac{1}{2}t\,sin\,t$$

陳正宗終身特聘教授簡介

陳正宗　終身特聘教授，生於 1962 年，分別於 1984 年、1986 年、1994 年取得台灣大學土木工程學系學士學位、台灣大學應力所碩士學位及台灣大學土木工程研究所博士學位。1986 至 1990 年間，於中山科學研究院火箭飛彈系統結構部門從事結構力學計算。1994 年至海洋大學河海工程學系擔任副教授一職，1998 年晉升為教授。2001 與 2004 年分別獲聘海洋大學第一屆優良教師與特聘教授。2005 年獲選台大傑出校友(土木)。2007 年獲聘海洋大學終身特聘教授。2011 年獲第 55 屆教育部學術獎(工科)與華人計算力學會士獎。主要研究領域為計算力學，曾與洪宏基教授推導出對偶積分方程再以對偶邊界元素法求解含退化邊界的邊界值問題作出貢獻。陳正宗教授帶領海大 NTOU/MSV 研究團隊發展出四套對偶邊界元素法程式，Laplace 方程，Helmholtz 方程，修正 Helmholtz 方程與 Navier 方程，並撰寫了兩本有關邊界元素法和有限元素法的中文書籍，也曾受邀到保加利亞（Colloquium of Numerical Analysis, 1996,1997）、阿根廷（WCCM 1998）、奧地利(WCCM 2002)、聖彼得堡（BEM-FEM 2003）、日本京都(ICAM 2007)、中國合肥(ICOME 2009)、香港(ICIP 2010)發表論文演說(Plenary lecture)。連續三次獲得國科會傑出研究獎(1999-2012)及第一屆吳大猷先生紀念獎(2002-2005)並獲聘 A 級計畫主持人(2005-2007)與國科會傑出學者計劃，發表逾百篇(172) SCI 論文分佈於 61 種 SCI 期刊並被超過九百餘篇(911)論文引用過。研究論文入榜 ESI 高引用率資料庫。兩篇論文(ASCE, ASME)分別在 SCOPUS 與 WOS 被引超過百次。陳正宗終身特聘教授曾任中國土木水利學刊常務編輯、中國工程學刊土木編委(SCI)、海洋學刊執行編輯與國際計算機材料與連體期刊(Computers, Materials and Continua)編委，國際 Beteq、WCCM、IABEM、Betech、BEM、ICCES、ICCESMM、ICCM、ECOMAS、ICOME、APCOM、ISCM-III 執行委員與 MFS-Trefftz 國際會議共同主席。現為海洋學刊編委(SCI)，亞太工程學報編委，國際計算方法期刊(Int. J. Comp. Meth.)、國際邊界元素法通訊編委(Boundary Element Communications)、工程科技計算模擬期刊(CMES, SCI)與國際邊界元素法電子期刊(Electronics Journal on BEM)編委，現任工程中邊界元素法期刊編輯(EABE, SCI)與力學期刊(JoM, SCI)副編輯、並審過 77 種期刊論文。由北京清華工程力學系根據最新 ISI Web of Science 資訊統計查得陳正宗終身特聘教授為二十一世紀世界邊界元素法研究學者 Top 10 。

2012 年元月 08 日 NTOU/MSV 室友回娘家　【Author12-01.doc/chen 製表】

About the author: Prof. J T Chen

Jeng-Tzong Chen, born in 1962, received a BS degree in Civil Engineering, an M.S. in Applied Mechanics, and a Ph.D. in Civil Engineering, respectively, in 1984, 1986 and 1994, from National Taiwan University, Taipei, Taiwan, R.O.C. He had worked as a research assistant in the Structural Division of the Department of Rocket and Missile System, Chung Shan Institute of Science and Technology, from 1986 to 1990. In 1994, he was invited to be an Associate Professor in the Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan, R.O.C. He was promoted to a full professor in 1998. Later in 2004, he was selected to be the Distinguished Professor. In 2007, he was selected as the Life-time Distinguished Professor. He is also the Professor of Department of Mechanical and Mechatronic Engineering of Taiwan Ocean University. In 2011, he won the MOE academic award and the ICACM Fellow Award. His major interest is computational mechanics. He had derived the theory of dual integral equations for boundary value problems with degenerate boundary. Prof. Chen also developed four dual BEM programs for the BVPs of Laplace equation, Helmholtz equation, bi-Helmholtz and modified Helmhotlz equation and Navier equation. Recently, he also employed the null field integral equations to solve BVPs with circular and/or elliptical boundaries including holes and inclusions. He wrote two books in Chinese on dual BEM and FEM using MSC/NASTRAN, respectively. He was ever invited to give plenary and keynote lectures, e.g., twice in World Congress on Computational Mechanics (WCCM4 (1998) in Buenos Aries and WCCM5 (2002) in Vienna), twice in ICOME 2006 and 2009, FEM/BEM 2003 in St. Petersburg, Russia and ICIP 2010 in Hong Kong. Also, he is now the editor of Engineering Analysis with Boundary Elements and the associate editor of Journal of Mechanics. He has been the editor of Journal of Marine Science and Technology and the guest editor of J. Chinese Institute of Engineers. He won three times of Outstanding Research Awards from National Science Council, Taiwan. He also won the first Wu, Ta-You Memorial Award in 2002. In 2011, he also won the HiWin Award. He is currently the member of editorial board of six international SCI journals. Until now, he has published more than 172 SCI papers on BEM and FEM in technical Journals. More than 1458 citings from 911 papers are found to cite Chen's work.. Two papers (ASCE and ASME) were both cited more than 100 times form SCOPUS and WOS systems. Boundary element method is one focus of Professor Chen's research interests. Others may be categorized into two areas. One is vibration and acoustics, and the other is computational mechanics.

E-mail: jtchen@mail.ntou.edu.tw
URL: http://ind.ntou.edu.tw/~msvlab

Autobiography2012-Feb.doc

jtchen@mail.ntou.edu.tw
http://ind.ntou.edu.tw/~msvlab/
886-24622192 ext 6177or6140
2012.02~2012.06