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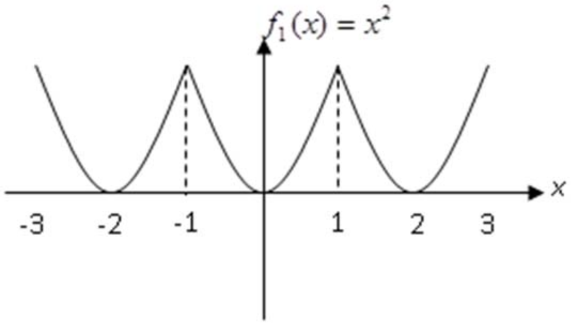
系級:__________ 學號:________ 姓名:________

如何用以下 Fourier 展開函數方式去求$\sum \frac{1}{n^4}$



$$a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$$

$$a_n = \int_{-1}^{1} x^2 \cos(n\pi x) dx = (\frac{1}{n\pi})^2 4\cos(n\pi)$$

$$b_n = \int_{-1}^{1} x^2 \sin(n\pi x) = 0$$

$$f_1(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x)$$

方法一：

用 $f_1(1)$ 算 $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$

方法二：透過 Parseval's theorem

算 $\frac{1}{2}\int_{-1}^{1} x^4 dx = \frac{1}{9} + \sum_{n=1}^{\infty} \frac{8}{n^4\pi^4}$， $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.08232$

方法三：那我們可否靠積分兩次來算$\sum \frac{1}{n^4} = ?$

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x)$$

$$\frac{1}{3}x^3 = \frac{1}{3}x + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \frac{1}{n\pi}\sin(n\pi x)$$

$$\frac{1}{12}x^4 = \frac{1}{6}x^2 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \frac{1}{n\pi}(-\frac{1}{n\pi})\cos(n\pi x) + C$$

$$\frac{1}{12}x^4 - \frac{1}{6}x^2 = C + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \frac{1}{n\pi}(-\frac{1}{n\pi})\cos(n\pi x)$$ (好比偶函數 $a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$)

上式可看成$\frac{1}{12}x^4-\frac{1}{6}x^2$的傅立葉展開

所以常數C可看成傅立葉的常數項

$$C=\frac{1}{2}\int_{-1}^{1}(\frac{1}{12}x^4-\frac{1}{6}x^2)dx=\frac{-7}{180}$$

$$\frac{1}{12}x^4=\frac{1}{6}x^2+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(-1)^n\frac{1}{n\pi}(-\frac{1}{n\pi})\cos(n\pi x)+\frac{-7}{180}$$

令$x=1$

$$\frac{1}{12}=\frac{1}{6}+\sum_{n=1}^{\infty}\frac{-4}{\pi^4}\frac{1}{n^4}-\frac{7}{180}$$

$$\frac{1}{12}-\frac{1}{6}+\frac{7}{180}=\frac{-4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

方法四：陳聖匡想要在積分一次可行嗎?