Gibbs phenomenon by Fourier transform

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Given a function $f(t)$, we have

$$f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dte^{i\omega u}d\omega \ \rightarrow \ f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-u)}dtd\omega$$

$$f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-u)}d\omega dt \ \rightarrow \ f_{\omega_0}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\frac{e^{-i\omega(t-u)}}{-i(t-u)}\Big|_{-\omega_0}^{\omega_0} dt$$

$$f_{\omega_0}(u) = \int_{-\infty}^{\infty}\frac{sin(\omega_0(t-u))}{(t-u)}f(t)dt \ \rightarrow \ f_{\omega_0}(u) = \int_{-\infty}^{\infty} R_{\omega_0}(u,t)f(t)dt$$

where reproducing kernel $R_{\omega_0}(u,t) = \frac{sin(\omega_0(t-u))}{(t-u)}$

After substituting the Heaviside function into $f(t)$, we have

$$f_{\omega_0}(u) = \int_{0}^{\infty}\frac{sin(\omega_0(t-u))}{(t-u)}dt$$

By changing the variable, we have

$$f_{\omega_0}(u) = \int_{-\infty}^{\omega_0 u}\frac{1}{\pi}\frac{sin(x)}{(x)}dx = \int_{-\infty}^{0}\frac{1}{\pi}\frac{sin(x)}{(x)}dx + \int_{0}^{\omega_0 t}\frac{1}{\pi}\frac{sin(x)}{(x)}dx$$

$$f_{\omega_0}(u) = 0.5 + \int_{0}^{\omega_0 u}\frac{1}{\pi}\frac{sin(x)}{(x)}dx$$

$$f_{\omega_0}(u) = 0.5 + \frac{1}{\pi}Si(\omega_0 u)$$

where Si function is defined by

$$Si(z) = \int_{0}^{z}\frac{sin(x)}{x}dx$$

The maximum value occurs as the derivative is zero, *i.e.*,

$$\frac{sin(z)}{z} = 0$$

The maximum may occur as $z = 0, \pi, 2\pi, \cdots$.

The maximum of 1.09 can be obtained by substituting $u = \pi/\omega_0$