

Frenet formula

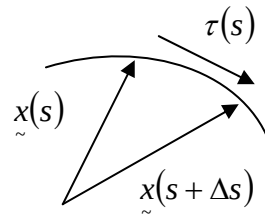
Oct,13,2004

Step 1. Give a space curve of $(x(t), y(t), z(t))$

Step 2. Path coordinate $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

Step 3. $((x(t), y(t), z(t))) \rightarrow X(s) = ((\tilde{x}(s), \tilde{y}(s), \tilde{z}(s)))$ transform t to s

Step 4. Tangent vector $\tau(s) = \frac{\dot{X}(s)}{\|\dot{X}(s)\|}$



Step 5. $\tau \cdot \tau = 1 \Rightarrow \tau \cdot \dot{\tau} = 0 \quad \dot{\tau} \perp \tau$ (Note that $|\tau| = |\nu| = |\beta| = 1$)

Choose $\nu = \rho \dot{\tau}$ (for simplicity)

Step 6. Why ρ is the radius of curvature

$$\tau(s) \cdot \tau(s + \Delta s) = |\tau(s)| |\tau(s + \Delta s)| \cos(d\theta) = \cos(d\theta)$$

$$\begin{aligned} \tau(s) \cdot \tau(s + \Delta s) &= \tau(s) \cdot \left[\tau(s) + \frac{1}{1!} \dot{\tau}(s)(ds) + \frac{1}{2!} \ddot{\tau}(s)(ds)^2 + \frac{1}{3!} \ddot{\tau}(s)(ds)^3 + \dots \right] \\ &= 1 + \tau(s) \cdot \dot{\tau}(s)(ds) + \frac{1}{2} \tau(s) \cdot \ddot{\tau}(s)(ds)^2 + \frac{1}{6} \tau(s) \cdot \ddot{\tau}(s)(ds)^3 + \dots \\ &= 1 + \frac{1}{2} \tau(s) \cdot \ddot{\tau}(s)(ds)^2 + \dots \end{aligned}$$

$$\cos(d\theta) = 1 - \frac{1}{2!}(d\theta)^2 + \frac{1}{4!}(d\theta)^4 - \frac{1}{6!}(d\theta)^6 + \dots = 1 - \frac{1}{2!}(d\theta)^2 + \dots$$

$$1 + \frac{1}{2} \tau(s) \cdot \ddot{\tau}(s)(ds)^2 = 1 - \frac{1}{2!}(d\theta)^2 \Rightarrow \tau(s) \cdot \ddot{\tau}(s)(ds)^2 = -(d\theta)^2 \quad (1)$$

$$\tau \cdot \dot{\tau} = 0 \Rightarrow \dot{\tau} \cdot \dot{\tau} + \tau \cdot \ddot{\tau} = 0 \quad \therefore \tau \cdot \ddot{\tau} = -\dot{\tau} \cdot \dot{\tau} \quad \text{代入(1)}$$

$$-\dot{\tau} \cdot \dot{\tau}(ds)^2 = -(d\theta)^2 \Rightarrow |\dot{\tau}|^2 = \left(\frac{d\theta}{ds} \right)^2 = \frac{1}{\left(\frac{ds}{d\theta} \right)^2} = \frac{1}{\left(\frac{\rho d\theta}{d\theta} \right)^2} = \frac{1}{\rho^2} \quad \therefore \rho d\theta = ds$$

$$\therefore |\dot{\tau}| = \frac{1}{\rho}, \quad \rho = \frac{ds}{d\theta}$$

Step 7. $\beta \cdot \beta = 1 \Rightarrow \dot{\beta} \cdot \beta = 0$

$$\therefore \dot{\beta} = p \tau + q \nu \Rightarrow$$

高聖凱提出

$$\tau \cdot \beta = 0$$

$$\tau \cdot \dot{\beta} + \dot{\tau} \cdot \beta = 0$$

$$\tau \cdot \dot{\beta} + \frac{1}{\rho} v \cdot \beta = 0$$

$$\therefore \tau \cdot \dot{\beta} = 0$$

$$\dot{\beta} \cdot \tau = p \tau \cdot \tau + q v \cdot \tau$$

$$0 = p$$

$$\therefore \text{set } \dot{\beta} = -\frac{1}{\sigma} v \quad (\text{for simplicity})$$

Step 8. $v = \beta \times \tau$

$$\dot{v} = \dot{\beta} \times \tau + \beta \times \dot{\tau}$$

$$= -\frac{1}{\sigma} v \times \tau + \beta \times \frac{1}{\rho} v$$

$$= \frac{1}{\sigma} \beta - \frac{1}{\rho} \tau$$

$\therefore$ we have

$$\begin{Bmatrix} \dot{\tau} \\ \dot{v} \\ \dot{\beta} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & -\frac{1}{\sigma} & 0 \end{bmatrix} \begin{Bmatrix} \tau \\ v \\ \beta \end{Bmatrix} \rightarrow \dot{P} = AP$$

$$\text{where } A = \begin{bmatrix} 0 & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & -\frac{1}{\sigma} & 0 \end{bmatrix} \quad (\text{反對稱}), \quad P = \begin{Bmatrix} \tau \\ v \\ \beta \end{Bmatrix}$$