1. 已知$\underset{\sim}{\tau}(s)=\left(\dot{X}(s),\dot{Y}(s)\right)$

$$\dot{\underset{\sim}{\tau}}(s)=\left(\ddot{X}(s),\ddot{Y}(s)\right)$$

$\underset{\sim}{\tau}\cdot\underset{\sim}{\tau}=1\Rightarrow\underset{\sim}{\tau}\cdot\dot{\underset{\sim}{\tau}}=0$

$\therefore\underset{\sim}{\tau}\perp\dot{\underset{\sim}{\tau}}$

$\underset{\sim}{\tau}\times\dot{\underset{\sim}{\tau}}=\left(\dot{X}\ddot{Y}-\dot{Y}\ddot{X}\right)\vec{k}\Rightarrow\left|\underset{\sim}{\tau}\times\dot{\underset{\sim}{\tau}}\right|=\sqrt{\left(\dot{X}\ddot{Y}-\dot{Y}\ddot{X}\right)^2}=\left|\left(\dot{X}\ddot{Y}-\dot{Y}\ddot{X}\right)\right|$ (外積向量長度)

$\left|\underset{\sim}{\tau}\times\dot{\underset{\sim}{\tau}}\right|=\sqrt{\left(\dot{X}\right)^2+\left(\dot{Y}\right)^2}\sqrt{\left(\ddot{X}\right)^2+\left(\ddot{Y}\right)^2}\cdot 1$ (外積向量長度)

$\Rightarrow$ 故得證：$\sqrt{\left(\ddot{X}\right)^2+\left(\ddot{Y}\right)^2}=\left|\left(\dot{X}\ddot{Y}-\dot{Y}\ddot{X}\right)\right|$

2.

$$\begin{bmatrix}\tau\\ \upsilon\\ \beta\end{bmatrix}=e^{As}\begin{bmatrix}\tau(0)\\ \upsilon(0)\\ \beta(0)\end{bmatrix},A=\begin{bmatrix}0&\frac{1}{2}&0\\ -\frac{1}{2}&0&\frac{1}{2}\\ 0&-\frac{1}{2}&0\end{bmatrix}$$

$$Av=\lambda v\Rightarrow\lambda(\lambda^2+\frac{1}{2})=0\Rightarrow\begin{cases}\lambda_1=0\\ \lambda_2=\frac{1}{\sqrt{2}i}\\ \lambda_3=-\frac{1}{\sqrt{2}i}\end{cases}\Rightarrow\begin{cases}v_1=(1,0,1)^T\\ v_2=\left(1,\sqrt{2}i,-1\right)^T\\ v_3=\left(1,-\sqrt{2}i,-1\right)^T\end{cases}$$

$$set\ C=\begin{bmatrix}0&0&0\\ 0&\frac{1}{\sqrt{2}i}&0\\ 0&0&-\frac{1}{\sqrt{2}i}\end{bmatrix},D=\begin{bmatrix}1&1&1\\ 0&\sqrt{2}i&-\sqrt{2}i\\ 1&-1&-1\end{bmatrix}$$

$$AD=DC\Rightarrow\begin{bmatrix}0&\frac{1}{2}&0\\ -\frac{1}{2}&0&\frac{1}{2}\\ 0&-\frac{1}{2}&0\end{bmatrix}\begin{bmatrix}1&1&1\\ 0&\sqrt{2}i&-\sqrt{2}i\\ 1&-1&-1\end{bmatrix}=\begin{bmatrix}1&1&1\\ 0&\sqrt{2}i&-\sqrt{2}i\\ 1&-1&-1\end{bmatrix}\begin{bmatrix}0&0&0\\ 0&\frac{1}{\sqrt{2}i}&0\\ 0&0&-\frac{1}{\sqrt{2}i}\end{bmatrix}\Rightarrow A=DCD^{-1}$$

$$e^{As}=\sum_{n=0}^{\infty}\frac{A^n}{n!}=\sum_{n=0}^{\infty}\frac{DC^nD^{-1}}{n!}=D\sum_{n=0}^{\infty}\frac{C^n}{n!}D^{-1}$$

$$=D\begin{bmatrix} e^{0s} & 0 & 0\\ 0 & e^{\frac{1}{\sqrt{2}i}s} & 0\\ 0 & 0 & e^{-\frac{1}{\sqrt{2}i}s}\end{bmatrix}D^{-1}$$

$$=\begin{bmatrix} 1 & 1 & 1\\ 0 & \sqrt{2}i & -\sqrt{2}i\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix} e^{0s} & 0 & 0\\ 0 & e^{\frac{1}{\sqrt{2}i}s} & 0\\ 0 & 0 & e^{-\frac{1}{\sqrt{2}i}s}\end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2\sqrt{2}i} & -\frac{1}{4}\\ \frac{1}{4} & -\frac{1}{2\sqrt{2}i} & -\frac{1}{4}\end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{2}+\frac{1}{4}e^{\frac{-i}{\sqrt{2}}s}+\frac{1}{4}e^{\frac{i}{\sqrt{2}}s} & -\frac{ie^{\frac{-i}{\sqrt{2}}s}}{2\sqrt{2}}+\frac{ie^{\frac{i}{\sqrt{2}}s}}{2\sqrt{2}} & \frac{1}{2}-\frac{1}{4}e^{\frac{-i}{\sqrt{2}}s}-\frac{1}{4}e^{\frac{i}{\sqrt{2}}s}\\ \frac{ie^{\frac{-i}{\sqrt{2}}s}}{2\sqrt{2}}-\frac{ie^{\frac{i}{\sqrt{2}}s}}{2\sqrt{2}} & \frac{1}{2}e^{\frac{-i}{\sqrt{2}}s}+\frac{1}{2}e^{\frac{i}{\sqrt{2}}s} & -\frac{ie^{\frac{-i}{\sqrt{2}}s}}{2\sqrt{2}}+\frac{ie^{\frac{i}{\sqrt{2}}s}}{2\sqrt{2}}\\ \frac{1}{2}-\frac{1}{4}e^{\frac{-i}{\sqrt{2}}s}-\frac{1}{4}e^{\frac{i}{\sqrt{2}}s} & \frac{ie^{\frac{-i}{\sqrt{2}}s}}{2\sqrt{2}}-\frac{ie^{\frac{i}{\sqrt{2}}s}}{2\sqrt{2}} & \frac{1}{2}+\frac{1}{4}e^{\frac{-i}{\sqrt{2}}s}+\frac{1}{4}e^{\frac{i}{\sqrt{2}}s}\end{bmatrix}$$

$$=\begin{bmatrix} \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2} & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2}\\ -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \cos(\frac{\sqrt{2}}{2}s) & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2}\\ \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2} & -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2}\end{bmatrix}$$

$$\begin{bmatrix}\tau(s)\\ \upsilon(s)\\ \beta(s)\end{bmatrix}=\begin{bmatrix} \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2} & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2}\\ -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \cos(\frac{\sqrt{2}}{2}s) & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2}\\ \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2} & -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2}\end{bmatrix}\begin{bmatrix}\tau(0)\\ \upsilon(0)\\ \beta(0)\end{bmatrix}$$

$$=\begin{bmatrix} \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2} & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2}\\ -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \cos(\frac{\sqrt{2}}{2}s) & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2}\\ \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2} & -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2}\end{bmatrix}\begin{bmatrix}\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\\ (-1,0,0)\\ \left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\end{bmatrix}=\begin{bmatrix}\left(-\frac{\sin(\frac{\sqrt{2}}{2}s)}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\left(\cos^2\left(\frac{1}{2\sqrt{2}}s\right)-\sin^2\left(\frac{1}{2\sqrt{2}}s\right)\right) \quad \frac{1}{\sqrt{2}}\right)\\ \left(-\cos(\frac{\sqrt{2}}{2}s) \quad -\sin(\frac{\sqrt{2}}{2}s) \quad 0\right)\\ \left(\frac{\sin(\frac{\sqrt{2}}{2}s)}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\left(\sin^2\left(\frac{1}{2\sqrt{2}}s\right)-\cos^2\left(\frac{1}{2\sqrt{2}}s\right)\right) \quad 1\right)\end{bmatrix}$$

$$\tau(s)=\left(-\frac{\sin(\frac{\sqrt{2}}{2}s)}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\left(\cos^2\left(\frac{1}{2\sqrt{2}}s\right)-\sin^2\left(\frac{1}{2\sqrt{2}}s\right)\right) \quad \frac{1}{\sqrt{2}}\right)\Rightarrow\begin{cases}X(s)=\cos(\frac{\sqrt{2}}{2}s)+c_1\\ Y(s)=\sin\left(\frac{\sqrt{2}}{2}s\right)+c_2\\ Z(s)=\frac{\sqrt{2}}{2}s+c_3\end{cases}\Rightarrow\begin{cases}X(0)=1\\ Y(0)=0\\ Z(0)=0\end{cases}\Rightarrow\begin{cases}X(s)=\cos(\frac{\sqrt{2}}{2}s)\\ Y(s)=\sin\left(\frac{\sqrt{2}}{2}s\right)\\ Z(s)=\frac{\sqrt{2}}{2}s\end{cases}$$

∴正算反算的弧長參數表示式一樣

方法二

$$\begin{bmatrix}\tau\\ \upsilon\\ \beta\end{bmatrix}=e^{As}\begin{bmatrix}\tau(0)\\ \upsilon(0)\\ \beta(0)\end{bmatrix}, As=\bar{A}=\begin{bmatrix}0 & \frac{1}{2}s & 0\\ -\frac{1}{2}s & 0 & \frac{1}{2}s\\ 0 & -\frac{1}{2}s & 0\end{bmatrix}$$

$$\bar{A}v=\lambda v\Rightarrow\lambda(\lambda^2+\frac{1}{2}s^2)=0\Rightarrow\begin{cases}\lambda_1=0\\ \lambda_2=\frac{1}{\sqrt{2}i}s\\ \lambda_3=-\frac{1}{\sqrt{2}i}s\end{cases}$$

$$e^x=\left(x(x^2+\frac{1}{2})\right)Q(x)+ax^2+bx+c$$

$$\Rightarrow\begin{cases}e^{0s}=a(0s)^2+b(0s)+c\\ e^{\frac{1}{\sqrt{2}i}s}=a\left(\frac{1}{\sqrt{2}i}s\right)^2+b\left(\frac{1}{\sqrt{2}i}s\right)+c\\ e^{\frac{-1}{\sqrt{2}i}s}=a\left(-\frac{1}{\sqrt{2}i}s\right)^2+b\left(-\frac{1}{\sqrt{2}i}s\right)+c\end{cases}\Rightarrow\begin{cases}a=\frac{1}{s^2}\left(2-e^{\frac{-\sqrt{2}}{2}is}-e^{\frac{\sqrt{2}}{2}is}\right)\\ b=\frac{1}{s}\frac{-\sqrt{2}}{2}\left(e^{\frac{\sqrt{2}}{2}is}-e^{\frac{-\sqrt{2}}{2}is}\right)\\ c=1\end{cases}$$

$$\Rightarrow e^x=\left(x(x^2+\frac{1}{2})\right)Q(x)+\frac{1}{s^2}\left(2-e^{\frac{-\sqrt{2}}{2}is}-e^{\frac{\sqrt{2}}{2}is}\right)x^2+\left(\frac{1}{s}\frac{-\sqrt{2}}{2}\left(e^{\frac{\sqrt{2}}{2}is}-e^{\frac{-\sqrt{2}}{2}is}\right)\right)x+1$$

$$\Rightarrow e^{\bar{A}}=\left(\bar{A}(\bar{A}^2+\frac{1}{2}I)\right)Q(\bar{A})+\frac{1}{s^2}\left(2-e^{\frac{-\sqrt{2}}{2}is}-e^{\frac{\sqrt{2}}{2}is}\right)\bar{A}^2+\left(\frac{1}{s}\frac{-\sqrt{2}}{2}\left(e^{\frac{\sqrt{2}}{2}is}-e^{\frac{-\sqrt{2}}{2}is}\right)\right)\bar{A}+I$$

$$e^{As}=\begin{bmatrix}\frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2} & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2}\\ -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \cos(\frac{\sqrt{2}}{2}s) & \frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2}\\ \frac{1-\cos(\frac{\sqrt{2}}{2}s)}{2} & -\frac{\sqrt{2}\sin(\frac{\sqrt{2}}{2}s)}{2} & \frac{1+\cos(\frac{\sqrt{2}}{2}s)}{2}\end{bmatrix}$$
與前一題的e^{As}展開一樣，後續方法相同