河工系 工數二 B 第四次作業

已知 $\begin{cases} \underset{\sim}{\tau}(0)=\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) \\ \underset{\sim}{\nu}(0)=(-1,0,0) \\ \underset{\sim}{\beta}(0)=\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) \end{cases}$ 滿足 $\begin{Bmatrix} \dot{\underset{\sim}{\tau}} \\ \dot{\underset{\sim}{\nu}} \\ \dot{\underset{\sim}{\beta}} \end{Bmatrix}=\begin{bmatrix} 0 & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & -\frac{1}{\sigma} & 0 \end{bmatrix}\begin{Bmatrix} \underset{\sim}{\tau} \\ \underset{\sim}{\nu} \\ \underset{\sim}{\beta} \end{Bmatrix}$，其中$\rho$=2，$\sigma$=2

初位置在$\left(X(0),Y(0),Z(0)\right)=(1,0,0)$出發 ，在上述條件($\rho$=2，$\sigma$=2)的控制下，

請問此路線之 $\begin{cases} X(s) \\ Y(s) \\ Z(s) \end{cases}$ 為何?

ans

$$\begin{Bmatrix} \tau(\mathrm{s}) \\ \nu(\mathrm{s}) \\ \beta(\mathrm{s}) \end{Bmatrix}=e^{As}\begin{Bmatrix} \tau(0) \\ \nu(0) \\ \beta(0) \end{Bmatrix},$$

$$A\bar{x}=\lambda\bar{x}$$

$$\Rightarrow(\mathrm{A}-\lambda I)\,\bar{\mathrm{x}}=0I$$

$$\begin{vmatrix} -\lambda & \frac{1}{2} & 0 \\ \frac{-1}{2} & -\lambda & \frac{1}{2} \\ 0 & \frac{-1}{2} & -\lambda \end{vmatrix}=0$$

其特徵方程式為$-\lambda^3-\frac{\lambda}{2}=\lambda^3+\frac{\lambda}{2}=0$,所以 $A^3+\frac{A}{2}=0$

而 $f(\mathrm{A})=(A^3+\frac{A}{2})Q(\mathrm{A})+a(\mathrm{A}^2+\frac{1}{2}\mathrm{I})+b(\mathrm{A}-\frac{i}{\sqrt{2}}\mathrm{I})+\mathrm{cI}$

經由類比(實數$\Rightarrow$矩陣)可得

$$f(\mathrm{x})=(\mathrm{x}^3+\frac{x}{2})Q(\mathrm{x})+a(\mathrm{x}^2+\frac{1}{2})+b(\mathrm{x}-\frac{i}{\sqrt{2}})+\mathrm{c}$$

$$e^{xs}=(\mathrm{x}^3+\frac{x}{2})Q(\mathrm{x})+a(\mathrm{x}^2+\frac{1}{2})+b(\mathrm{x}-\frac{i}{\sqrt{2}})+\mathrm{c}$$

$$e^{\frac{is}{\sqrt{2}}}=c$$

$$e^{\frac{-is}{\sqrt{2}}}=b(-\frac{i}{\sqrt{2}}-\frac{i}{\sqrt{2}})+\mathrm{c}=-\sqrt{2}bi+e^{\frac{is}{\sqrt{2}}}$$

$$\Rightarrow b=\frac{e^{\frac{is}{\sqrt{2}}}-e^{\frac{-is}{\sqrt{2}}}}{\sqrt{2}i}=\frac{2i\sin\frac{s}{\sqrt{2}}}{\sqrt{2}i}=\sqrt{2}\sin\frac{s}{\sqrt{2}}$$

$$e^{0s}=\frac{1}{2}a-\frac{bi}{\sqrt{2}}+c=\frac{1}{2}a-i\sin\frac{s}{\sqrt{2}}+e^{\frac{is}{\sqrt{2}}}=\frac{1}{2}a+\cos\frac{s}{\sqrt{2}}$$

$$\Rightarrow a=2(1-\cos\frac{s}{\sqrt{2}})$$

將 a,b,c 代回得

$$e^{As}=(A^3+\frac{A}{2})Q(s)+a(s)(\mathrm{A}^2+\frac{1}{2}\mathrm{I})+b(s)(\mathrm{A}-\frac{i}{\sqrt{2}}\mathrm{I})+\mathrm{c}(s)\,\mathrm{I}$$

$$=a(\mathrm{A}^2+\frac{1}{2}\mathrm{I})+b(\mathrm{A}-\frac{i}{\sqrt{2}})+\mathrm{I}\qquad(\because A^3+\frac{A}{2}=0$$

$$a(\mathrm{A}^2+\frac{1}{2}\mathrm{I})=a\begin{bmatrix}0&\frac{1}{2}&0\\\frac{-1}{2}&0&\frac{1}{2}\\0&\frac{-1}{2}&0\end{bmatrix}\begin{bmatrix}0&\frac{1}{2}&0\\\frac{-1}{2}&0&\frac{1}{2}\\0&\frac{-1}{2}&0\end{bmatrix}+\frac{a}{2}I=\begin{bmatrix}\frac{-a}{4}&0&\frac{a}{4}\\0&\frac{-a}{2}&0\\\frac{a}{4}&0&\frac{-a}{4}\end{bmatrix}+\frac{a}{2}I=\begin{bmatrix}\frac{a}{4}&0&\frac{a}{4}\\0&0&0\\\frac{a}{4}&0&\frac{a}{4}\end{bmatrix}$$

$$b(\mathrm{A}-\frac{i}{\sqrt{2}}\mathrm{I})=\begin{bmatrix}0&\frac{b}{2}&0\\\frac{-b}{2}&0&\frac{b}{2}\\0&\frac{-b}{2}&0\end{bmatrix}-\frac{bi}{\sqrt{2}}\mathrm{I}=\begin{bmatrix}-\frac{bi}{\sqrt{2}}&\frac{b}{2}&0\\\frac{-b}{2}&-\frac{bi}{\sqrt{2}}&\frac{b}{2}\\0&\frac{-b}{2}&-\frac{bi}{\sqrt{2}}\end{bmatrix}$$

$$e^{As} = a(\mathrm{A}^2+\frac{1}{2}\mathrm{I})+b(\mathrm{A}-\frac{i}{\sqrt{2}}\mathrm{I})+c\mathrm{I} = \begin{bmatrix} \frac{a}{4} & 0 & \frac{a}{4} \\ 0 & 0 & 0 \\ \frac{a}{4} & 0 & \frac{a}{4} \end{bmatrix} + \begin{bmatrix} -\frac{bi}{\sqrt{2}} & \frac{b}{2} & 0 \\ \frac{-b}{2} & -\frac{bi}{\sqrt{2}} & \frac{b}{2} \\ 0 & \frac{-b}{2} & -\frac{bi}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} e^{\frac{is}{\sqrt{2}}} & 0 & 0 \\ 0 & e^{\frac{is}{\sqrt{2}}} & 0 \\ 0 & 0 & e^{\frac{is}{\sqrt{2}}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\cos\frac{s}{\sqrt{2}}}{2} & 0 & \frac{1-\cos\frac{s}{\sqrt{2}}}{2} \\ 0 & 0 & 0 \\ \frac{1-\cos\frac{s}{\sqrt{2}}}{2} & 0 & \frac{1-\cos\frac{s}{\sqrt{2}}}{2} \end{bmatrix} + \begin{bmatrix} -i\sin\frac{s}{\sqrt{2}} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & 0 \\ \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & -i\sin\frac{s}{\sqrt{2}} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} \\ 0 & \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & -i\sin\frac{s}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \cos\frac{s}{\sqrt{2}}+i\sin\frac{s}{\sqrt{2}} & 0 & 0 \\ 0 & \cos\frac{s}{\sqrt{2}}+i\sin\frac{s}{\sqrt{2}} & 0 \\ 0 & 0 & \cos\frac{s}{\sqrt{2}}+i\sin\frac{s}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-\cos\frac{s}{\sqrt{2}}}{2} & 0 & \frac{1-\cos\frac{s}{\sqrt{2}}}{2} \\ 0 & 0 & 0 \\ \frac{1-\cos\frac{s}{\sqrt{2}}}{2} & 0 & \frac{1-\cos\frac{s}{\sqrt{2}}}{2} \end{bmatrix} + \begin{bmatrix} \cos\frac{s}{\sqrt{2}} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & 0 \\ \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \cos\frac{s}{\sqrt{2}} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} \\ 0 & \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \cos\frac{s}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos\frac{s}{\sqrt{2}}}{2} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \frac{1-\cos\frac{s}{\sqrt{2}}}{2} \\ \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \cos\frac{s}{\sqrt{2}} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} \\ \frac{1-\cos\frac{s}{\sqrt{2}}}{2} & \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \frac{1+\cos\frac{s}{\sqrt{2}}}{2} \end{bmatrix}$$

$$\begin{Bmatrix} \tau(\mathrm{s}) \\ \nu(\mathrm{s}) \\ \beta(\mathrm{s}) \end{Bmatrix} = e^{As} \begin{Bmatrix} \tau(0) \\ \nu(0) \\ \beta(0) \end{Bmatrix} = \begin{bmatrix} \frac{1+\cos\frac{s}{\sqrt{2}}}{2} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \frac{1-\cos\frac{s}{\sqrt{2}}}{2} \\ \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \cos\frac{s}{\sqrt{2}} & \frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} \\ \frac{1-\cos\frac{s}{\sqrt{2}}}{2} & \frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}} & \frac{1+\cos\frac{s}{\sqrt{2}}}{2} \end{bmatrix} \begin{Bmatrix} (0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) \\ (-1,0,0) \\ (0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}) \end{Bmatrix}$$

$$= \begin{Bmatrix} (0,\frac{1+\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}},\frac{1+\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}})+(\frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}},0,0)+(0,\frac{-1+\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}},\frac{1-\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}}) \\ (0,\frac{-\sin\frac{s}{\sqrt{2}}}{2},\frac{-\sin\frac{s}{\sqrt{2}}}{2})+(-\cos\frac{s}{\sqrt{2}},0,0)+(0,\frac{-\sin\frac{s}{\sqrt{2}}}{2},\frac{\sin\frac{s}{\sqrt{2}}}{2}) \\ (0,\frac{1-\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}},\frac{1-\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}})+(\frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}},0,0)+(0,\frac{-1-\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}},\frac{1+\cos\frac{s}{\sqrt{2}}}{2\sqrt{2}}) \end{Bmatrix} = \begin{Bmatrix} (\frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}},\frac{\cos\frac{s}{\sqrt{2}}}{\sqrt{2}},\frac{1}{\sqrt{2}}) \\ (-\cos\frac{s}{\sqrt{2}},-\sin\frac{s}{\sqrt{2}},0) \\ (\frac{\sin\frac{s}{\sqrt{2}}}{\sqrt{2}},\frac{-\cos\frac{s}{\sqrt{2}}}{\sqrt{2}},\frac{1}{\sqrt{2}}) \end{Bmatrix}$$

$$\tau(\mathrm{s}) = (\frac{-\sin\frac{s}{\sqrt{2}}}{\sqrt{2}},\frac{\cos\frac{s}{\sqrt{2}}}{\sqrt{2}},\frac{1}{\sqrt{2}}) \Rightarrow \begin{cases} X(s) = \cos\frac{s}{\sqrt{2}} + c_1 \\ Y(s) = \sin\frac{s}{\sqrt{2}} + c_2 \\ Z(s) = \frac{s}{\sqrt{2}} + c_3 \end{cases} \Rightarrow \left(X(0),Y(0),Z(0)\right) = (1,0,0) \Rightarrow \begin{cases} X(s) = \cos\frac{s}{\sqrt{2}} \\ Y(s) = \sin\frac{s}{\sqrt{2}} \\ Z(s) = \frac{s}{\sqrt{2}} \end{cases}$$