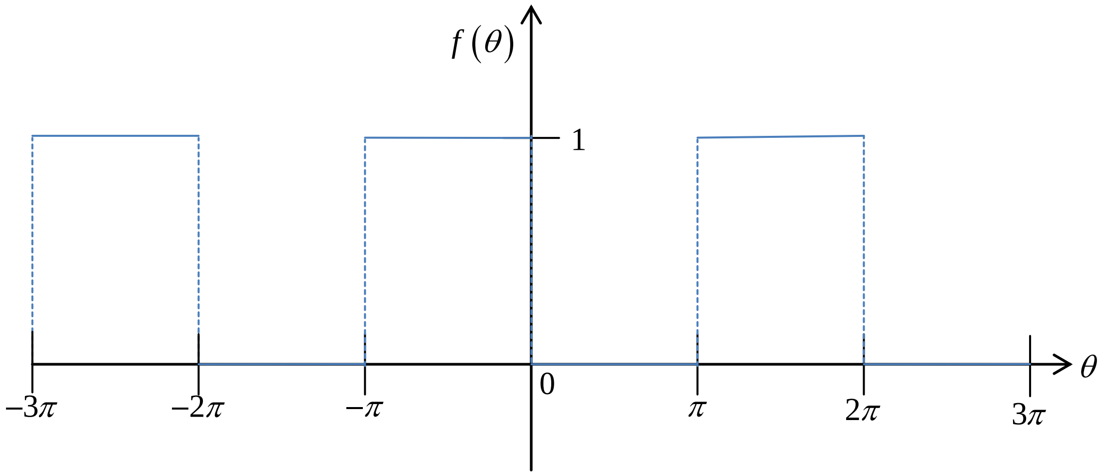
河工系 工數二B 第五次作業

1.

若一週期函數 $f(\theta+2\pi)=f(\theta)$, $f(\theta)=\begin{cases}0, & 0\le\theta\le\pi\\ 1, & \pi\le\theta\le 2\pi\end{cases}$ 如圖，試以複數型傅立葉級數展開求其係數，並用複數型傅立葉級數計算 Parseval's 定理，計算 $\int_0^{2\pi} f^2(\theta)\,d\theta=?$ 此題老師演練給同學參考



$$f(\theta)=\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$c_0=\frac{1}{2\pi}\int_{\pi}^{2\pi} 1\,d\theta=\frac{1}{2}$$

$$c_n=\frac{1}{2\pi}\int_{\pi}^{2\pi} e^{-in\theta}\,d\theta=\frac{1}{2\pi}\times\frac{-1}{in}e^{-in\theta}\Big|_{\pi}^{2\pi}=\frac{i}{2\pi n}\left(e^{-in2\pi}-e^{-in\pi}\right)$$

$$=\frac{i}{2\pi n}\left(\cos(-2n\pi)+i\sin(-2n\pi)-\cos(-n\pi)-i\sin(-n\pi)\right)=\frac{i}{2\pi n}\left(1-(-1)^n\right)$$

$$f(\theta)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{i}{2\pi n}\left(1-(-1)^n\right)e^{in\theta}$$

$$\int_0^{2\pi} f^2(\theta)\,d\theta=\int_{\pi}^{2\pi}1\,d\theta=\int_{\pi}^{2\pi}1\,d\theta=\pi$$

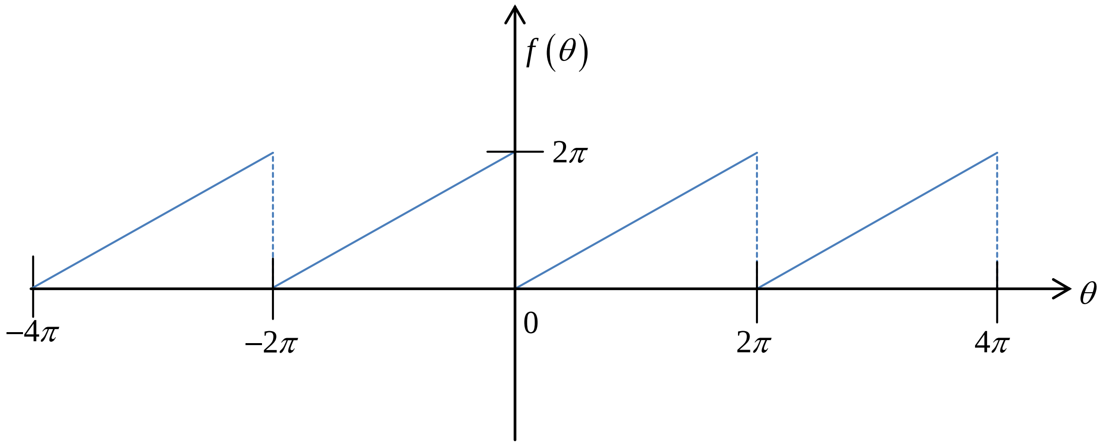
$$\int_0^{2\pi} f^2(\theta)\,d\theta=2\pi\sum_{n=-\infty}^{-1}c_nc_{-n}+2\pi c_0c_0+2\pi\sum_{n=1}^{\infty}c_nc_{-n}$$

$$=2\pi\sum_{n=-\infty}^{-1}\frac{i}{2\pi n}\left(1-(-1)^n\right)\frac{-i}{2\pi n}\left(1-(-1)^n\right)+2\pi\times\frac{1}{2}\times\frac{1}{2}+2\pi\sum_{n=1}^{\infty}\frac{i}{2\pi n}\left(1-(-1)^n\right)\frac{-i}{2\pi n}\left(1-(-1)^n\right)$$

$$=\sum_{n=-\infty}^{-1}\frac{1}{2\pi n^2}\left(1-(-1)^n\right)^2+\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{1}{2\pi n^2}\left(1-(-1)^n\right)^2=\pi$$

已知 $f(\theta+2\pi)=f(\theta)$， $f(\theta)=\theta$，$0\le\theta\le 2\pi$，如圖，試以複數型傅立葉級數展開求其係數，

並用複數型傅立葉級數計算 Parseval's 定理，計算 $\int_0^{2\pi} f^2(\theta)\,d\theta=?$



$$f(\theta)=\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$c_0=\frac{1}{2\pi}\int_0^{2\pi}\theta\,d\theta=\pi$$

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}\theta e^{-in\theta}\,d\theta=\frac{1}{2\pi}\left(\frac{-\theta}{in}e^{-in\theta}\Big|_0^{2\pi}+\frac{1}{n^2}e^{-in\theta}\Big|_0^{2\pi}\right)=\frac{1}{2\pi}\left(\frac{2\pi i}{n}e^{-in2\pi}+\frac{1}{n^2}e^{-in2\pi}-\frac{1}{n^2}\right)$$

$$=\frac{1}{2\pi}\left(\frac{2\pi i}{n}\left(\cos(-2n\pi)+i\sin(-2n\pi)\right)+\frac{1}{n^2}\left(\cos(-2n\pi)+i\sin(-2n\pi)\right)-\frac{1}{n^2}\right)=\frac{1}{2\pi}\left(\frac{2\pi i}{n}\right)=\frac{i}{n}$$

$$f(\theta)=\pi+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{i}{n}e^{in\theta}$$

$$\int_0^{2\pi} f^2(\theta)\,d\theta=\int_0^{2\pi}\theta^2\,d\theta=\int_0^{2\pi}\frac{\theta^3}{3}\,d\theta=\frac{8\pi^3}{3}$$

$$\int_0^{2\pi} f^2(\theta)\,d\theta=2\pi\sum_{n=-\infty}^{-1}c_n c_{-n}+2\pi c_0 c_0+2\pi\sum_{n=1}^{\infty}c_n c_{-n}=2\pi\sum_{n=-\infty}^{-1}\frac{i}{n}\frac{-i}{n}+2\pi^3+2\pi\sum_{n=1}^{\infty}\frac{i}{n}\frac{-i}{n}$$

$$=2\pi\sum_{n=-\infty}^{-1}\frac{1}{n^2}+2\pi^3+2\pi\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{8\pi^3}{3}$$