Acoustic eigenfrequencies in a spheroidal cavity with a concentric penetrable sphere

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The acoustic eigenfrequencies f_{nsm} in a spheroidal cavity containing a concentric penetrable sphere are determined analytically, for both Dirichlet and Neumann conditions in the spheroidal boundary. Two different methods are used for the evaluation. In the first, the pressure field is expressed in terms of both spherical and spheroidal wave functions, connected with one another by well-known expansion formulas. In the second, a shape perturbation method, this field is expressed in terms of spherical wave functions only, while the equation of the spheroidal boundary is given in spherical coordinates. The analytical determination of the eigenfrequencies is possible when the solution is specialized to small values of $h=d/(2R_2)$, $(h\ll 1)$, with d the interfocal distance of the spheroidal boundary and $2R_2$ the length of its rotation axis. In this case exact, closed-form expressions are obtained for the expansion coefficients $g_{nsm}^{(2)}$ and $g_{nsm}^{(4)}$ in the resulting relation $f_{nsm}(h)=f_{ns}(0)[1+h^2g_{nsm}^{(2)}+h^4g_{nsm}^{(4)}+O(h^6)]$. Analogous expressions are obtained with the use of the parameter $v=1-(R_2/R_2')^2$, $(|v|\ll 1)$, with $2R_2'$ the length of the other axis of the spheroidal boundary. Numerical results are given for various values of the parameters. [S0001-4966(99)05803-8]

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INTRODUCTION

Calculation of eigenfrequencies in acoustic cavities of various shapes is an important problem with many applications in room acoustics,[1] acoustic levitation[2,3] and high accuracy measurements of sound speed in gases.[4] The shape of the boundaries severely limits the possibility for analytical solution of such problems. For complicated geometries numerical techniques are used. Analytical, perturbational methods were used elsewhere, in order to obtain the acoustic eigenfrequencies in a spherical cavity with an eccentric inner sphere, for both Dirichlet and Neumann boundary conditions, in the case of small eccentricity between the two spheres,[5] or for a small inner sphere.[6,7] In spheroidal cavities calculation is more complex, due to the complexity of spheroidal functions. In Refs. 8 and 9 the eigenfrequencies of a prolate spheroidal cavity were calculated, for Dirichlet and Neumann boundary conditions, too. The same is valid also in Refs. 10 and 11 for concentric spheroidal–spherical cavities, by analytical, perturbational methods. In this last case not only the prolate but also the oblate spheroidal boundaries are examined.

In the present paper the acoustic cavity, shown in Fig. 1, is examined also for both Dirichlet and Neumann conditions in its spheroidal boundary, which has major and minor semiaxes R_2 and R_2', respectively, and interfocal distance d. It contains a concentric penetrable sphere with radius R_1. This cavity is a perturbation of the concentric spherical one with radii R_1 and R_2. Only the prolate spheroidal boundary is shown, but corresponding formulas for the oblate one are obtained immediately. The length of the rotation axis in each case is $2R_2$, while that of the other axis is $2R_2'$.

The acoustic eigenfrequencies in the former cavity are determined by two different methods. In the first of them the pressure field is expressed in terms of both spherical and spheroidal wave functions, while use is made of the well-known expansion formulas connecting these functions.[12] In the second method we use shape perturbation. In this case the pressure field is expressed in terms of spherical wave functions only, while the equation of the spheroidal boundary is given in spherical coordinates r and θ. In both cases, after the satisfaction of the boundary conditions, we obtain an infinite determinantal equation for the evaluation of the eigenfrequencies. In the special case of small $h=d/(2R_2)$, $(h\ll 1)$ we are led to an exact evaluation, up to the order h^4, for the elements of the infinite determinant and, finally, for the determinant itself. It is then possible to obtain the eigenfrequencies in the form $f_{nsm}(h)=f_{ns}(0)[1+h^2g_{nsm}^{(2)}+h^4g_{nsm}^{(4)}+O(h^6)]$. The expansion coefficients $g_{nsm}^{(2)}$ and $g_{nsm}^{(4)}$ are independent of h and are given by exact, closed-form expressions, while $f_{ns}(0)$ are the eigenfrequencies of the corresponding spherical cavity with $h=0$.

The main advantage of such an analytical solution lies in its general validity for each small value of h and for all modes, while numerical techniques require repetition of the evaluation for each different h, with accuracy deteriorating quickly for higher order modes.

Analogous expansions are obtained by using the parameter $v=1-(R_2/R_2')^2$, $(|v|\ll 1)$.

Our method can be applied also in the corresponding exterior (scattering) problem.

The cases of the Dirichlet and Neumann conditions in the spheroidal boundary are examined in Secs. I and II, re-

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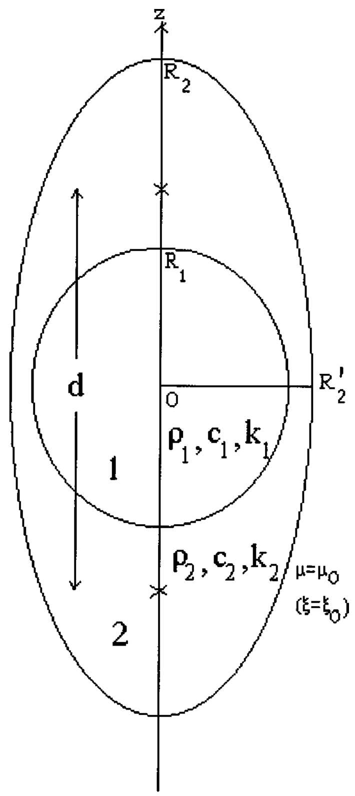

FIG. 1. Geometry of the cavity.

spectively. Finally, Sec. III includes numerical results and discussion.

I. DIRICHLET BOUNDARY CONDITIONS

As shown in Fig. 1, the density, the sound speed and the wave number are ρ_1, c_1, k_1 and ρ_2, c_2, k_2 inside the penetrable sphere (region 1) and between it and the spheroidal boundary (region 2), respectively. The materials of both regions are considered as fluids or fluidlike, i.e., they do not support shear waves.

Let p_1 and p_2 be the acoustic pressure fields in regions 1 and 2, respectively. These fields, which satisfy the scalar Helmholtz equation, have the following expressions:

$$p_1=\sum_{n=0}^{\infty}\sum_{m=0}^{n} j_n(k_1 r)P_n^m(\cos\theta)[C_{nm}\cos m\varphi + D_{nm}\sin m\varphi], \quad (1)$$

$$p_2=\sum_{n=0}^{\infty}\sum_{m=0}^{n} [j_n(k_2 r)-E_n n_n(k_2 r)]P_n^m(\cos\theta) \times[A_{nm}\cos m\varphi+B_{nm}\sin m\varphi]. \quad (2)$$

In Eqs. (1), (2) r, θ, φ are the spherical coordinates with respect to O, j_n and n_n are the spherical Bessel functions of the first and second kind, respectively, and P_n^m is the associated Legendre function of the first kind.

By satisfying the boundary conditions at $r=R_1$

$$p_1=p_2, \quad \frac{1}{\rho_1 c_1}\frac{\partial p_1}{\partial(k_1 r)}=\frac{1}{\rho_2 c_2}\frac{\partial p_2}{\partial(k_2 r)}, \quad (3)$$

and using the orthogonal relations for the associated Legendre[13] and the trigonometric functions we obtain the following expression for E_n

$$E_n=\frac{j_n(x_1)j_n'(w_1)-q j_n(w_1)j_n'(x_1)}{n_n(x_1)j_n'(w_1)-q j_n(w_1)n_n'(x_1)}, \quad (4)$$

where

$$w_1=k_1R_1, \quad x_1=k_2R_1, \quad q=\frac{\rho_1 c_1}{\rho_2 c_2}, \quad (5)$$

and the primes denote derivatives with respect to the argument.

In order to satisfy the remaining boundary condition $p_2=0$ at the spheroidal boundary, denoted by $\mu=\mu_0$, we follow two different methods. In the first of them we expand the spherical wave functions, appearing in Eq. (2), into concentric spheroidal ones by the formula[12]

$$z_n^{(\sigma)}(k_2 r)P_n^m(\cos\theta) =\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\sum_{\ell=m,m+1}^{\infty}{}' \frac{i^{\ell-n}}{N_{m\ell}} \times d_{n-m}^{m\ell}S_{m\ell}(c,\eta)R_{m\ell}^{(\sigma)}(c,\xi), \quad c=k_2 d/2. \quad (6)$$

In Eq. (6) $\xi=\cosh\mu$, η are the spheroidal coordinates (φ is common in both systems), $z_n^{(\sigma)}$ $(\sigma=1-4)$ is the spherical Bessel function of any kind, $R_{m\ell}^{(\sigma)}$ is the corresponding radial spheroidal function of the same kind, $S_{m\ell}$ and $d_{n-m}^{m\ell}$ are the angular spheroidal function of the first kind and its expansion coefficients, while

$$N_{mn}=2\sum_{r=0,1}^{\infty}{}'\frac{(d_r^{mn})^2(r+2m)!}{(2r+2m+1)r!}. \quad (7)$$

The prime over the summation symbols in Eqs. (6) and (7) indicates that when $n-m$ is even/odd these summations start with the first/second value of their summation index and continue only with values of the same parity with it.

We substitute from Eq. (6) into Eq. (2) satisfying the boundary condition $p_2=0$ at $\mu=\mu_0(\xi=\xi_0)$ and we next use the orthogonal properties of the angular spheroidal[12] and the trigonometric functions, to obtain finally the following infinite set of linear homogeneous equations for the expansion coefficients A_{nm} (or B_{nm})

$$\sum_{n=m,m+1}^{\infty}{}' \alpha_{\ell nm}A_{nm}=0, \quad \ell\geqslant m,m+1, \quad (8)$$

where

$$\alpha_{\ell nm}=\frac{2i^{-n}(n+m)!}{(2n+1)(n-m)!} \times d_{n-m}^{m\ell}[R_{m\ell}^{(1)}(c,\cosh\mu_0)-E_nR_{m\ell}^{(2)}(c,\cosh\mu_0)]. \quad (9)$$

In Eqs. (8) and (9) ℓ and n are both even or odd, starting with that value of m or $m+1$, which has the same parity with them. So, Eq. (8) separates into two distinct subsets, one with ℓ, n even and the other with ℓ, n odd.

We next substitute $R_{m\ell}^{(1)}$ and $R_{m\ell}^{(2)}$ from Eq. (7) of Ref. 10 into Eq. (9) and set each one of the two determinants

$\Delta(\alpha_{\ell nm})$ (one with ℓ, n even and the other with ℓ, n odd) of the coefficients $\alpha_{\ell nm}$ in Eq. (8) equal to 0. So, we obtain two determinantal equations of the same form for the evaluation of the eigenfrequencies, which are treated simultaneously under the symbol $\Delta(\alpha_{\ell nm})$. By dividing $\alpha_{\ell nm}$ by the product $2i^{-n}\tanh^m\mu_0(d_{\ell-m}^{m\ell})^2(n+m)/[(2n+1)(n-m)!]$, as in Ref. 10, we do not change the roots of the determinantal equation. We next use the symbol $\alpha_{\ell n}$ for the resulting coefficient, deleting the subscript m for simplicity, and replace $c=k_2d/2$ by its equal one $c=x_2h$, where $x_2=c\cosh\mu_0=k_2R_2$ and $h=d/(2R_2)$. For large values of h the determinantal equation can be solved only numerically, but for small h ($h\ll 1$) an analytical solution is possible. In this last case we can set up to the order h^4,

$$\alpha_{nn}=D_{nn}^{(0)}+h^2D_{nn}^{(2)}+h^4D_{nn}^{(4)}+O(h^6),$$
$$\alpha_{n\pm 2,n}=h^2D_{n\pm 2,n}^{(2)}+h^4D_{n\pm 2,n}^{(4)}+O(h^6), \qquad (10)$$
$$\alpha_{n\pm 4,n}=h^4D_{n\pm 4,n}^{(4)}+O(h^6).$$

Exact expressions for the various D's used in our calculations are given in Eqs. (A1)–(A5) of the Appendix.

Relations (10) allow a closed-form evaluation of the determinant $\Delta(\alpha_{\ell n})=\Delta(\alpha_{\ell nm})$, up to the order h^4, in steps exactly the same with those in Ref. 10, which will not be repeated here.

The resonant wave numbers $k_2=k_2(h)$, as well as $x_2=x_2(h)=k_2(h)R_2$ have also expansions of the form

$$k_2(h)=k_2^{(0)}+h^2k_2^{(2)}+h^4k_2^{(4)}+O(h^6), \qquad (11)$$
$$x_2(h)=x_2^{(0)}+h^2x_2^{(2)}+h^4x_2^{(4)}+O(h^6),$$
$$x_2^{(\rho)}=k_2^{(\rho)}R_2, \quad \rho=0,2,4, \qquad (12)$$

where $k_2^{(0)}\equiv k_2^0$ and $x_2^{(0)}\equiv x_2^0$ correspond to the concentric spherical cavity with radii R_1 and R_2 (h=0).

The expressions of $x_2^{(2)}$ and $x_2^{(4)}$ in terms of D's are exactly the same as in Ref. 10 and are given by the formulas

$$x_2^{(2)}=-\left[\frac{dD_{nn}^0(x_2^0)}{dx_2}\right]^{-1}D_{nn}^{(2)}(x_2^0), \qquad (13)$$

$$x_2^{(4)}=-\left[\frac{dD_{nn}^0(x_2^0)}{dx_2}\right]^{-1}\left[\frac{(x_2^{(2)})^2}{2}\frac{d^2D_{nn}^0(x_2^0)}{dx_2^2}+x_2^{(2)}\frac{dD_{nn}^{(2)}(x_2^0)}{dx_2}+D_{nn}^{(4)}(x_2^0)-\frac{D_{n+2,n}^{(2)}(x_2^0)D_{n,n+2}^{(2)}(x_2^0)}{D_{n+2,n+2}^0(x_2^0)}-\frac{D_{n,n-2}^{(2)}(x_2^0)D_{n-2,n}^{(2)}(x_2^0)}{D_{n-2,n-2}^0(x_2^0)}\right], \qquad (14)$$

where $D_{nn}^{(0)}\equiv D_{nn}^0$. As it is evident from Eq. (8), the various subscripts in Eq. (10) and so also in Eqs. (13), (14) should be equal or greater than $m\geqslant 0$. In the opposite case the corresponding α's and D's are equal to zero and so disappear.

In Eqs. (13) and (14) we have used the relations $x_1=\tau x_2$, $w_1=\tau x_2c_2/c_1$, where $\tau=R_1/R_2=$constant, so x_2 is the only variable.

Formulas (13) and (14) are also valid for the oblate cavity, with the only difference that $D^{(2)}$'s change their signs and R_2 is the minor semiaxis of the oblate boundary. So, $x_2^{(2)}$ changes its sign, while $x_2^{(4)}$ remains the same.

The eigenfrequencies for the problem of two concentric spheres with radii R_1 and R_2, used in Eqs. (13), (14), are given by the equation [Eq. (A1) in the Appendix] $D_{nn}^0=0$, or

$$\frac{j_n(x_2^0)}{n_n(x_2^0)}=E_n(x_2^0), \quad x_1^0=\tau x_2^0, \quad w_1^0=\tau x_2^0\frac{c_2}{c_1}, \quad \tau=\frac{R_1}{R_2}. \qquad (15)$$

By using Eqs. (15), (A1), (A22) from the Appendix and the Wronskian[13] $j_n(x_2^0)n_n'(x_2^0)-j_n'(x_2^0)n_n(x_2^0)=1/(x_2^0)^2$, we obtain

$$\frac{dD_{nn}^0(x_2^0)}{dx_2}=-\frac{1}{(x_2^0)^2n_n(x_2^0)}-n_n(x_2^0)\frac{dE_n(w_1^0,x_1^0)}{dx_2}. \qquad (16)$$

Equations (16) and (A2) substituted in (13) give $x_2^{(2)}$. The expression for $x_2^{(4)}$ is much more lengthy, but is obtained immediately from Eqs. (13), (14), (16), (A1)–(A5) and (A22)–(A24).

By setting Eq. (12) in the form $x_2(h)=x_2^0[1+h^2g^{(2)}+h^4g^{(4)}+O(h^6)]$ we obtain the eigenfrequencies in the cavity of Fig. 1 by the expression

$$f_{\text{nsm}}(h)=f_{ns}(0)[1+h^2g_{\text{nsm}}^{(2)}+h^4g_{\text{nsm}}^{(4)}+O(h^6)],$$
$$n=0,1,2,\ldots, \quad s=1,2,3,\ldots, \quad m=0,1,2,\ldots,n, \qquad (17)$$

where $f_{ns}(0)=c_2(x_2^0)_{ns}/(2\pi R_2)$ are the eigenfrequencies of the concentric spherical cavity, $x_2^0=(x_2^0)_{ns}$ are the successive positive roots of Eqs. (15) and $g^{(2),(4)}=x_2^{(2),(4)}/x_2^0[g_{\text{nsm}}^{(2),(4)}=(x_2^{(2),(4)})_{\text{nsm}}/(x_2^0)_{ns}]$.

We next apply the second method for the determination of the eigenfrequencies. This is a shape perturbation method with no use of spheroidal wave functions. Equations (1)–(5) are also valid in this case. In order to satisfy the remaining boundary condition $p_2=0$, at the spheroidal surface, we express the equation of this surface in terms of r and θ, as in Ref. 14

$$r=\frac{R_2}{\sqrt{1-v\sin^2\theta}}, \qquad (18)$$

where[11]

$$v=1-\left(\frac{R_2}{R_2'}\right)^2=\mp h^2-h^4+O(h^6). \qquad (19)$$

The upper/lower sign in Eq. (19) corresponds to the prolate ($v<0$)/oblate ($v>0$) spheroidal boundary.

We expand Eq. (18) into power series in h, thus obtaining up to the order h^4

$$r=R_2\left[1\mp\frac{h^2}{2}\sin^2\theta-\frac{h^4}{2}\sin^2\theta\left(1-\frac{3}{4}\sin^2\theta\right)+O(h^6)\right]. \qquad (20)$$

By using Eq. (20) we get the following expansion[11] ($x_2=k_2R_2$):

$$j_n(k_2r)=j_n(x_2)\mp\frac{h^2}{2}x_2 j_n'(x_2)\sin^2\theta-\frac{h^4}{2}x_2\sin^2\theta \cdot\{j_n'(x_2)-\tfrac{1}{4}[3j_n'(x_2)+x_2 j_n''(x_2)]\sin^2\theta\} +O(h^6), \tag{21}$$

and a similar one for $n_n(k_2r)$.

We next substitute the former expansions into Eq. (2) satisfying the boundary condition at the spheroidal surface and we use the orthogonal properties of the associated Legendre[13] and the trigonometric functions, concluding finally to the following infinite set of linear homogeneous equations for the expansion coefficients A_{nm} (or B_{nm}), up to the order h^4:

$$\alpha_{n-4,n}A_{n-4,m}+\alpha_{n-2,n}A_{n-2,m}+\alpha_{nn}A_{nm}+\alpha_{n+2,n}A_{n+2,m} +\alpha_{n+4,n}A_{n+4,m}=0,\quad n\geqslant m. \tag{22}$$

The third subscript m is omitted from the various α's in Eq. (22), for simplicity. Their expressions are also given by the general expansions (10), but with different D's, which are given in Eqs. (A6)–(A9) of the Appendix. As it is evident from Eq. (2), the first subscripts of A's (and B's) should be always equal or greater than $m\geqslant 0$. In the opposite case A's (and B's) are equal to zero and disappear. The same is valid also for the corresponding α's and D's.

If m has the same/opposite parity with n, i.e., $n-m$ is even/odd, the first subscript of the α's in Eq. (22) starts from the minimum value $m/m+1$ and continues with the values $m+2/m+3$, $m+4/m+5$, etc. So, Eq. (22) separates into two distinct subsets, one with n even and the other with n odd. Setting each one of the determinants of the coefficients α, in these subsets, equal to zero, we obtain two determinantal equations of the same form for the evaluation of the eigenfrequencies, which are treated simultaneously. The rest steps are exactly the same as with the first method. So, Eqs. (11)–(17) are also valid here with identical final results as in that method [$x_2^{(2)}$ is obtained from Eqs. (13), (16) and (A7), while $x_2^{(4)}$ from Eqs. (13), (14), (16), (A6)–(A9) and (A22)–(A24)], as it is expected for the same problem. This consists a very good check for their correctness.

The problem can be also solved, from the beginning, by using the eccentricity parameter v instead of h. In this case the expansion of the general quantity y with respect to v is

$$y=y(v)=y^0+vy_v^{(1)}+v^2y_v^{(2)}+O(v^3), \tag{23}$$

while its expansion with respect to h is

$$y=y(h)=y^0+h^2y_h^{(2)}+h^4y_h^{(4)}+O(h^6). \tag{24}$$

By using Eq. (19) into (23), as well as the relation

$$v^2=h^4+O(h^6) \tag{25}$$

we finally obtain[11]

$$y_v^{(1)}=\mp y_h^{(2)},\; y_v^{(2)}=\mp y_h^{(2)}+y_h^{(4)}. \tag{26}$$

These last expressions are unique for both the prolate and the oblate cavity (v includes the sign), because $y_h^{(2)}$ simply changes its sign in these two cases.

By using the limiting value $\rho_1\to 0$ $(q\to 0)$, with c_1 finite, in Eq. (4), we obtain $E_n=j_n(x_1)/n_n(x_1)$, corresponding to a soft inner sphere. In the special case with $\rho_1=\rho_2$ and $c_1=c_2$, $q=1$, $w_1=x_1$ and $E_n=0$. Use of the small argument formulas for the various Bessel functions[13] in Eq. (4) as $R_1\to 0$, gives also $E_n=0$. The last two cases correspond to a simple spheroidal cavity, i.e., in the absence of the inner sphere. In all three cases the various results become identical with the corresponding ones in Refs. 10 and 11. For $E_n=0$, Eq. (16) is replaced by $dD_{nn}^0(x_2^0)/dx_2=j_n'(x_2^0)$.

II. NEUMANN BOUNDARY CONDITIONS

Equations (1)–(5) are also valid in this case. In order to satisfy the boundary condition $\partial p_2/\partial\mu=0$ $(\partial p_2/\partial\xi=0)$ at $\mu=\mu_0$ $(\xi=\xi_0)$, according to the first method, we follow steps identical to those for the Dirichlet case. So, we use again formulas (6) and (7) and conclude finally to the infinite set (8), with the difference that $\alpha_{\ell nm}$ is now given by the expression

$$\alpha_{\ell nm}=\frac{2i^{-n}(n+m)!}{(2n+1)(n-m)!} \times d_{n-m}^{m\ell}\left[\frac{\partial R_{m\ell}^{(1)}(c,\cosh\mu_0)}{\partial\mu}-E_n\frac{\partial R_{m\ell}^{(2)}(c,\cosh\mu_0)}{\partial\mu}\right]. \tag{27}$$

The remarks after Eq. (9) are again valid in this case. We next substitute $\partial R_{m\ell}^{(1)}/\partial\mu$ and $\partial R_{m\ell}^{(2)}/\partial\mu$ from Eq. (33) of Ref. 10 into Eq. (27) and follow the same procedure as in the Dirichlet case. So, we obtain again Eqs. (10)–(14) and (17) but with different expressions for the various expansion coefficients, which are given in Eqs. (A12)–(A15) of the Appendix. In place of Eq. (15) we now have

$$\frac{j_n'(x_2^0)}{n_n'(x_2^0)}=E_n(x_2^0),\quad x_1^0=\tau x_2^0,\quad w_1^0=\tau x_2^0\frac{c_2}{c_1}, \tag{28}$$

while, by using Eqs. (28), (A12), (A22) and the Wronskian $j_n'(x_2^0)n_n''(x_2^0)-j_n''(x_2^0)n_n'(x_2^0)=[(x_2^0)^2-n(n+1)]/(x_2^0)^4$, we obtain in place of Eq. (16)

$$\frac{dD_{nn}^0(x_2^0)}{dx_2}=-\frac{(x_2^0)^2-n(n+1)}{(x_2^0)^3n_n'(x_2^0)}-x_2^0n_n'(x_2^0)\frac{dE_n(w_1^0,x_1^0)}{dx_2}. \tag{29}$$

Equations (29) and (A13) substituted in (13) give $x_2^{(2)}$. The expression for $x_2^{(4)}$ is much more lengthy, but is obtained immediately from Eqs. (13), (14), (29), (A12)–(A15) and (A22)–(A24).

According to the second method, the boundary condition at the spheroidal surface is expressed as $\hat{u}\cdot\nabla p_2=0$, with $\hat{u}$ the normal unit vector there, where[11]

TABLE I. Dirichlet conditions, $\tau = R_1/R_2 = 0.2(0.5)$, $\rho_2/\rho_1 = 0.820$, $c_2/c_1 = 0.787$.

			s			
	n	m	1	2	3	4
$(x_2^0)_{ns}$	0		3.177 90(3.559 71)	6.501 60(7.210 62)	9.936 19(10.366 79)	13.365 58(14.325 24)
	1		4.486 10(4.711 36)	7.745 53(8.862 57)	11.066 88(12.032 23)	14.494 62(15.773 77)
	2		5.761 16(5.865 33)	9.088 70(10.073 77)	12.345 25(13.870 28)	15.656 05(17.058 48)
	3		6.987 53(7.031 11)	10.414 10(11.145 51)	13.694 39(15.437 97)	16.949 25(18.601 91)
$g_{nsm}^{(2)}$	0	0	0.344 06(0.428 86)	0.360 30(0.313 11)	0.366 44(0.409 89)	0.358 75(0.364 37)
	1	0	0.199 71(0.244 14)	0.204 75(0.242 79)	0.213 32(0.189 74)	0.219 74(0.264 96)
		1	0.399 43(0.488 28)	0.409 49(0.485 57)	0.426 65(0.379 48)	0.439 48(0.529 93)
	2	0	0.237 73(0.269 14)	0.238 29(0.320 50)	0.242 98(0.225 86)	0.251 36(0.290 21)
		1	0.285 28(0.322 96)	0.285 94(0.384 60)	0.291 58(0.271 04)	0.301 63(0.348 25)
		2	0.427 92(0.484 45)	0.428 92(0.576 90)	0.437 37(0.406 56)	0.452 44(0.522 37)
	3	0	0.244 36(0.261 26)	0.244 16(0.326 41)	0.244 99(0.288 47)	0.249 00(0.238 21)
		1	0.266 57(0.285 01)	0.266 36(0.356 08)	0.267 26(0.314 70)	0.271 64(0.259 87)
		2	0.333 22(0.356 27)	0.332 94(0.445 10)	0.334 07(0.393 37)	0.339 55(0.324 84)
		3	0.444 29(0.475 02)	0.443 93(0.593 47)	0.445 43(0.524 49)	0.452 73(0.433 11)
$g_{nsm}^{(4)}$	0	0	0.287 40(0.380 06)	0.454 87(0.258 68)	0.500 62(1.697 56)	0.472 91(0.779 97)
	1	0	0.172 25(0.222 26)	0.312 11(0.188 53)	0.439 25(0.430 22)	0.446 97(1.412 77)
		1	0.315 19(0.463 98)	0.422 44(0.315 38)	0.525 28(0.563 89)	0.538 78(1.234 51)
	2	0	0.008 16(0.164 49)	0.073 86(−0.986 60)	0.237 85(−0.508 26)	0.301 96(−0.184 46)
		1	0.234 09(0.297 28)	0.333 31(0.340 12)	0.470 29(0.213 73)	0.568 60(1.321 73)
		2	0.330 57(0.453 67)	0.383 07(0.513 66)	0.465 10(0.192 64)	0.532 06(1.147 67)
	3	0	0.040 08(0.177 55)	0.024 64(0.109 70)	0.159 16(−1.073 90)	0.366 18(0.305 79)
		1	0.110 71(0.217 35)	0.125 24(0.211 68)	0.249 18(−0.650 71)	0.427 43(0.414 20)
		2	0.266 72(0.318 24)	0.333 72(0.447 47)	0.426 93(0.195 64)	0.537 91(0.639 27)
		3	0.340 52(0.424 66)	0.370 07(0.606 64)	0.415 51(0.195 52)	0.477 79(0.680 55)

TABLE II. Neumann conditions, $\tau = R_1/R_2 = 0.2(0.5)$, $\rho_2/\rho_1 = 0.820$, $c_2/c_1 = 0.787$.

			s			
	n	m	1	2	3	4
$(x_2^0)_{ns}$	0		0 (0)	4.593 68(5.321 42)	8.076 78(8.539 48)	11.565 52(12.299 21)
	1		2.0785 7(2.065 35)	5.935 43(6.584 94)	9.276 24(10.483 81)	12.682 46(13.641 99)
	2		3.341 69(3.329 69)	7.284 93(7.708 81)	10.613 20(12.050 94)	13.911 58(15.272 41)
	3		4.514 05(4.506 37)	8.582 49(8.834 41)	11.968 13(13.250 97)	15.248 40(17.063 92)
$g_{nsm}^{(2)}$	0	0	- (-)	0.352 66(0.409 12)	0.365 22(0.314 75)	0.364 96(0.439 78)
	1	0	0.027 51(0.023 60)	0.189 10(0.254 63)	0.203 83(0.189 61)	0.214 25(0.237 87)
		1	0.484 29(0.491 39)	0.408 48(0.541 61)	0.420 08(0.388 18)	0.435 34(0.482 28)
	2	0	0.182 70(0.181 15)	0.231 60(0.299 20)	0.237 17(0.288 05)	0.245 22(0.224 55)
		1	0.257 92(0.256 54)	0.282 16(0.363 85)	0.286 50(0.347 41)	0.295 37(0.270 29)
		2	0.483 60(0.482 68)	0.433 84(0.557 81)	0.434 47(0.525 50)	0.445 82(0.407 52)
	3	0	0.212 60(0.211 25)	0.239 93(0.289 24)	0.242 25(0.329 32)	0.245 21(0.229 13)
		1	0.242 78(0.241 35)	0.263 22(0.317 19)	0.264 97(0.360 01)	0.267 92(0.250 27)
		2	0.333 31(0.331 62)	0.333 07(0.401 04)	0.333 11(0.452 08)	0.336 04(0.313 68)
		3	0.484 21(0.482 08)	0.449 50(0.540 80)	0.446 69(0.605 54)	0.449 58(0.419 36)
$g_{nsm}^{(4)}$	0	0	- (-)	0.388 61(0.331 51)	0.509 44(0.556 64)	0.497 98(2.139 97)
	1	0	0.002 79(−0.001 42)	0.230 09(0.250 39)	0.390 53(0.186 52)	0.463 00(1.218 65)
		1	0.353 85(0.377 20)	0.368 74(0.516 54)	0.488 42(0.220 76)	0.549 29(1.239 80)
	2	0	−0.045 04(0.008 38)	0.000 61(0.068 21)	0.143 08(−1.642 10)	0.296 70(−0.031 19)
		1	0.204 57(0.205 67)	0.279 89(0.377 77)	0.401 47(0.228 16)	0.535 14(0.525 47)
		2	0.354 56(0.361 04)	0.358 28(0.573 67)	0.424 37(0.284 77)	0.507 73(0.583 57)
	3	0	0.037 14(0.059 73)	0.007 27(0.251 12)	0.066 14(−0.701 53)	0.256 99(−0.292 88)
		1	0.104 04(0.118 83)	0.101 13(0.297 71)	0.169 40(−0.350 43)	0.335 23(−0.122 20)
		2	0.257 61(0.256 91)	0.300 40(0.418 80)	0.378 67(0.377 97)	0.484 26(0.210 74)
		3	0.356 50(0.356 31)	0.358 23(0.558 40)	0.392 42(0.508 95)	0.447 05(0.168 60)

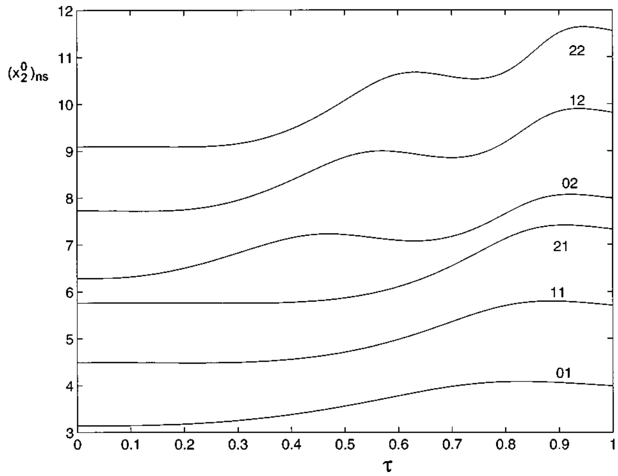

FIG. 2. Eigenfrequencies of a spherical cavity with a penetrable sphere; $\rho_2/\rho_1 = 0.820$, $c_2/c_1 = 0.787$—Dirichlet conditions.

$$\hat{u} = \left(1 - \frac{h^4}{8}\sin^2 2\theta\right)\hat{u}',$$

$$\hat{u}' = \hat{r} + \frac{h^2}{2}\sin 2\theta(\pm 1 + h^2\cos^2\theta)\hat{\theta} + O(h^6). \tag{30}$$

So

$$\hat{u}\cdot\nabla p_2 = \hat{u}'\cdot\nabla p_2 = \frac{\partial p_2}{\partial r} + \frac{h^2}{2}\sin 2\theta(\pm 1 + h^2\cos^2\theta)\frac{1}{r}\frac{\partial p_2}{\partial\theta} = 0. \tag{31}$$

We next substitute from Eq. (2) into (31), thus obtaining the equation

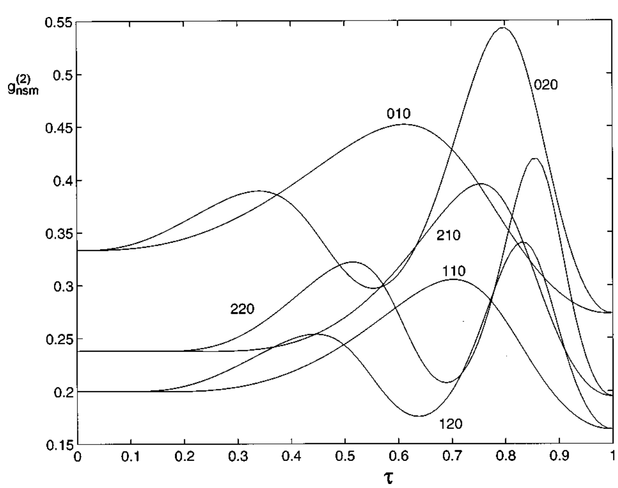

FIG. 3. First order expansion coefficients for eigenfrequencies in a spheroidal cavity with a penetrable sphere; $\rho_2/\rho_1 = 0.820$, $c_2/c_1 = 0.787$—Dirichlet conditions.

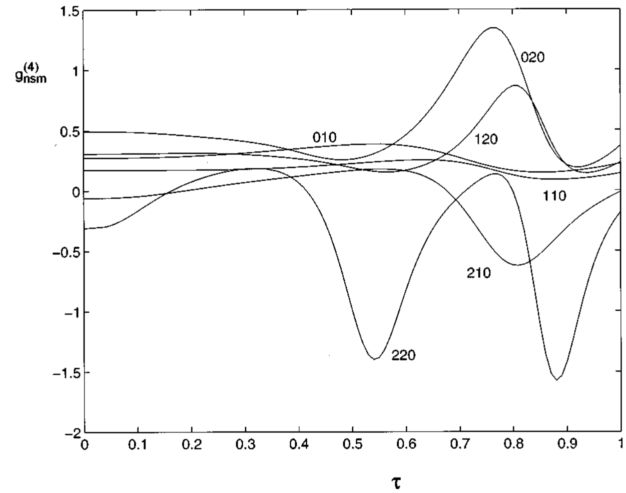

FIG. 4. Second order expansion coefficients for eigenfrequencies in a spheroidal cavity with a penetrable sphere; $\rho_2/\rho_1 = 0.820$, $c_2/c_1 = 0.787$—Dirichlet conditions.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Big\{[j_n'(k_2r) - E_n n_n'(k_2r)]P_n^m(\cos\theta) + \frac{h^2}{2}\sin 2\theta(\pm 1 + h^2\cos^2\theta)\frac{1}{k_2r}[j_n(k_2r) - E_n n_n(k_2r)] \cdot \frac{dP_n^m(\cos\theta)}{d\theta}\Big\}[A_{nm}\cos m\varphi + B_{nm}\sin m\varphi] = 0. \tag{32}$$

By using Eq. (20) we get expansions similar to Eq. (21) for $j_n'(k_2r)$ and $n_n'(k_2r)$, but with one more prime in each one of their Bessel functions. We also obtain the expansion[11]

$$\frac{j_n(k_2r)}{k_2r} = \frac{j_n(x_2)}{x_2} \mp \frac{h^2}{2}\left[-\frac{j_n(x_2)}{x_2} + j_n'(x_2)\right]\sin^2\theta + \frac{h^4}{2}\left\{\left[\frac{j_n(x_2)}{x_2} - j_n'(x_2)\right]\sin^2\theta + \frac{1}{4}\left[-\frac{j_n(x_2)}{x_2} + j_n'(x_2) + x_2 j_n''(x_2)\right]\sin^4\theta\right\} + O(h^6) \tag{33}$$

and a similar one for $n_n(k_2r)/k_2r$.

We next substitute the former expansions into Eq. (32) and we use the orthogonal properties of the associated Legendre and the trigonometric functions, thus obtaining again the set (22), certainly with different α's and so D's, which are given in Eqs. (A16)–(A19) of the Appendix.

The rest steps are identical with those in the first method, i.e., we obtain again Eqs. (10)–(14), (17) and (28), (29) with identical as their final results [$x_2^{(2)}$ is obtained from Eqs. (13), (29) and (A17), while $x_2^{(4)}$ from Eqs. (13), (14), (29), (A16)–(A19) and (A22)–(A24)], as is expected for the same problem. This is a very good check for their correctness.

The parameter υ, instead of h, can be also used in this case by keeping in mind Eqs. (19) and (23)–(26).

By using the limiting value $\rho_1\to\infty$ $(q\to\infty)$, with c_1 finite, in Eq. (4), we obtain $E_n = j_n'(x_1)/n_n'(x_1)$, corresponding

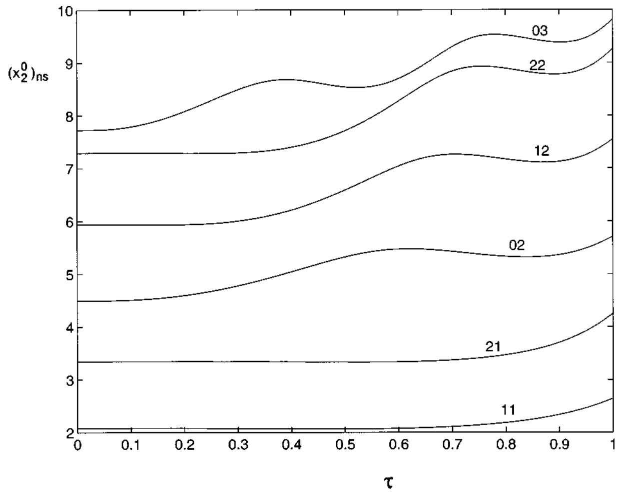

FIG. 5. Eigenfrequencies of a spherical cavity with a penetrable sphere; $\rho_2/\rho_1=0.820$, $c_2/c_1=0.787$—Neumann conditions.

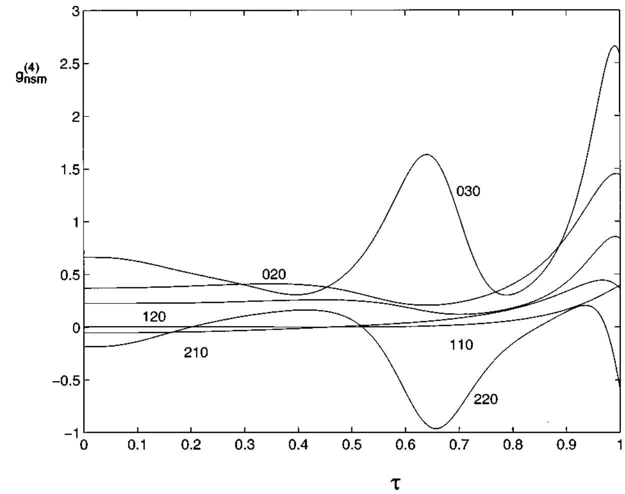

FIG. 7. Second order expansion coefficients for eigenfrequencies in a spheroidal cavity with a penetrable sphere; $\rho_2/\rho_1=0.820$, $c_2/c_1=0.787$—Neumann conditions.

to a hard inner sphere. So, the various results become identical with the corresponding ones in Refs. 10 and 11. The same is valid also for a simple spheroidal cavity, where $E_n=0$. In this last case Eq. (29) is replaced by $dD_{nn}^0(x_2^0)/dx_2=x_2^0 j_n''(x_2^0)$.

III. NUMERICAL RESULTS AND DISCUSSION

In Table I the roots $(x_2^0)_{ns}$ ($n=0-3$, $s=1-4$) of Eq. (15) as well as the corresponding values of $g_{\text{nsm}}^{(2)}$ and $g_{\text{nsm}}^{(4)}$ are given in the Dirichlet case, for $\tau=R_1/R_2=0.2,0.5,\rho_2/\rho_1=0.820,c_2/c_1=0.787$. In Table II the roots $(x_2^0)_{ns}$ of Eq. (28) are given, as well as g's in the Neumann case, for the same τ's and the values of the parameters as before. The value $(x_2^0)_{01}=0$ corresponds to the smallest eigenvalue $k_2^0=k_1^0=0$ (with constant eigenfunction) of the Helmholtz equation under Neumann conditions. As $(x_2^0)_{01}=0$, also $f_{01}(0)=0$ and $f_{010}(h)=0$, so the values of $g_{010}^{(2)}$ and $g_{010}^{(4)}$ do not matter.

Both tables are referred to the prolate cavity. For the oblate one $g^{(2)}$'s simply change their signs, while $g^{(4)}$'s remain unchanged. (The same will be valid also in Figs. 3, 4, 6 and 7, which follow.)

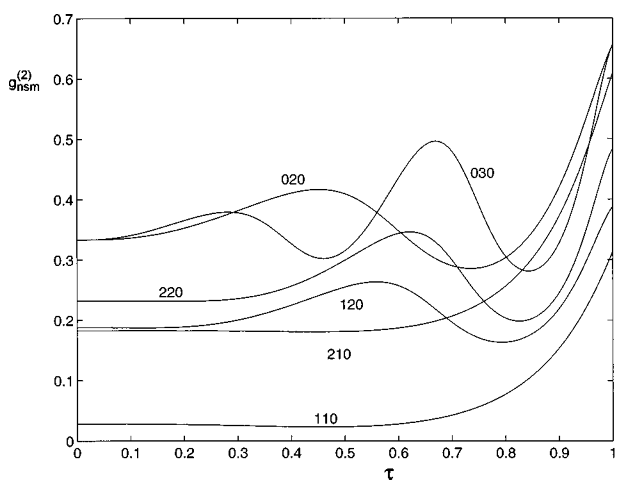

FIG. 6. First order expansion coefficients for eigenfrequencies in a spheroidal cavity with a penetrable sphere; $\rho_2/\rho_1=0.820$, $c_2/c_1=0.787$—Neumann conditions.

For the values of the parameters used, all $g^{(2)}$'s in both tables are positive. Keeping in mind Eq. (17), this means that the eigenfrequencies of the prolate/oblate cavity are greater/smaller than those of the corresponding spherical one, up to the order h^2.

From the former tables and many other available results, it is evident that $(x_2^0)_{ns}$ $(n\geqslant 0,s\geqslant 1)$ and so also $f_{ns}(0)$ for Neumann conditions are smaller than the corresponding ones for Dirichlet conditions. The same is valid for $f_{\text{nsm}}(h)$, as can be easily proved for the results given in these tables, in the case with $h\ll 1$.

In Fig. 2 we plot the roots $(x_2^0)_{ns}$ $(n=0-2, s=1,2)$ of Eq. (15) versus τ, for a concentric spherical cavity with radii R_1 and R_2 and Dirichlet conditions. The various numbers designating the curves in this and the rest of the figures correspond to the subscripts of the ordinate. For $\tau\rightarrow 0(R_1\rightarrow 0)$, $E_n\rightarrow 0$ and so $(x_2^0)_{ns}$ tend to the zeros of $j_n(x_2^0)$, corresponding to a simple spherical cavity with parameters ρ_2, c_2. For $\tau\rightarrow 1(R_1\rightarrow R_2)$, $x_1^0\rightarrow x_2^0$, so Eq. (15) is reduced to $j_n(w_1^0)=j_n(x_2^0 c_2/c_1)=0$ corresponding to a simple spherical cavity with parameters ρ_1, c_1 and $(x_2^0)_{ns}$ in this case are equal with those for $\tau\rightarrow 0$, multiplied by c_1/c_2.

In Figs. 3 and 4 we plot $g_{\text{nsm}}^{(2)}$ and $g_{\text{nsm}}^{(4)}$, respectively, versus τ, for the cavity of Fig. 1 with Dirichlet conditions. For $\tau\rightarrow 0$ the various g's tend to the corresponding ones for a simple spheroidal cavity[11] with parameters ρ_2, c_2, by taking in mind Eqs. (26). So, $g^{(2)}$'s are independent of s in this case, as it was proved in Ref. 11 and is seen in Fig. 3. For $\tau\rightarrow 1$ (for the prolate cavity is necessary that $h\rightarrow 0$, as $\tau\rightarrow 1$) the same remarks as before are valid for g's, where now the simple spheroidal cavity has parameters ρ_1, c_1. Also in this case $g^{(2)}$'s are independent of s, as is seen in Fig. 3, and are equal with those for $\tau\rightarrow 0$, multiplied by ρ_2/ρ_1. This can be proved easily by using the result[11] $g_{\text{nsm}}^{(2)}=F$ [F is given in Eq. (A10)] for $\tau\rightarrow 0$, as well as Eqs. (13), (16), (A7), (15), the Wronskian following it and (A22) for $\tau\rightarrow 1$, i.e., with $x_1^0\rightarrow x_2^0$ and $j_n(w_1^o)=0$.

In Fig. 5 the roots $(x_2^0)_{ns}$ $(n=0-2, s=1-3)$ of Eq. (28) are plotted versus τ, for a concentric spherical cavity with radii R_1 and R_2 and Neumann conditions $[(x_2^0)_{01}=0$, as in Table II]. For $\tau\to 0$, $(x_2^0)_{ns}$ tend to the zeros of $j_n'(x_2^0)$, corresponding to a simple spherical cavity with parameters ρ_2, c_2. For $\tau\to 1$, Eq. (28) is reduced to $j_n'(w_1^0) = j_n'(x_2^0 c_2/c_1)=0$ (for a simple spherical cavity with ρ_1, c_1) and $(x_2^0)_{ns}$ are equal with the corresponding ones for $\tau\to 0$, multiplied by c_1/c_2.

In Figs. 6 and 7 we plot $g_{\rm nsm}^{(2)}$ and $g_{\rm nsm}^{(4)}$, respectively, versus τ, for the cavity of Fig. 1 with Neumann conditions. For $\tau\to 0$ the various g's tend to the corresponding ones for a simple spheroidal cavity[11] with parameters ρ_2, c_2 [we keep in mind Eqs. (26)]. So, $g_{oso}^{(2)}(s\geqslant 2)$ are independent of s in this case, as is seen in Fig. 6. For $\tau\to 1$ the same remarks are valid for g's in a simple spheroidal cavity with parameters ρ_1, c_1. So $g_{oso}^{(2)}$ are independent of s also in this case (Fig. 6).

APPENDIX

The expressions for the various D's appearing in Eq. (10) and used in our calculations are the following (the upper/lower sign corresponds to the prolate/oblate cavity):

1. Dirichlet boundary conditions

A. First method (use of spheroidal wave functions)

$$D_{nn}^0=u_{nn}, \tag{A1}$$

$$D_{nn}^{(2)}=\pm\frac{x_2^2}{2(2n+1)}\left[\frac{(n+m+1)(n+m+2)}{(2n+3)^2}u_{n+2,n} - \frac{(n-m-1)(n-m)}{(2n-1)^2}u_{n-2,n}\right], \tag{A2}$$

$$\begin{aligned}D_{nn}^{(4)}&=x_2^4\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)^2(2n+7)}\\&\quad\times\left[\frac{1-4m^2}{(2n-1)(2n+3)^2}u_{n+2,n}+\frac{(n+m+3)(n+m+4)}{8(2n+5)^2}u_{n+4,n}\right]\\&\quad-x_2^4\frac{(n-m-1)(n-m)}{(2n-5)(2n-1)^2(2n+1)}\\&\quad\cdot\left[\frac{1-4m^2}{(2n-1)^2(2n+3)}u_{n-2,n}-\frac{(n-m-3)(n-m-2)}{8(2n-3)^2}u_{n-4,n}\right],\end{aligned} \tag{A3}$$

$$\begin{aligned}D_{n+2,n}^{(2)}&=\pm x_2^2\frac{(n+m+1)(n+m+2)}{2(2n+3)^2(2n+5)}u_{n+2,n},\\D_{n,n+2}^{(2)}&=\mp x_2^2\frac{(n-m+1)(n-m+2)}{2(2n+1)(2n+3)^2}u_{n,n+2},\end{aligned} \tag{A4}$$

where

$$u_{vs}=j_v(x_2)-E_s n_v(x_2). \tag{A5}$$

B. Second method (shape perturbation)

$$D_{nn}^0=u_{nn}, \tag{A6}$$

$$D_{nn}^{(2)}=\mp x_2 F u_{nn}', \tag{A7}$$

$$D_{nn}^{(4)}=x_2 G(3u_{nn}'+x_2u_{nn}'')-x_2Fu_{nn}', \tag{A8}$$

$$\begin{aligned}D_{n+2,n}^{(2)}&=\pm x_2\frac{(n+m+1)(n+m+2)}{2(2n+3)(2n+5)}u_{n+2,n+2}',\\D_{n,n+2}^{(2)}&=\pm x_2\frac{(n-m+1)(n-m+2)}{2(2n+1)(2n+3)}u_{nn}',\end{aligned} \tag{A9}$$

where

$$F=\frac{n^2+m^2+n-1}{(2n-1)(2n+3)}, \tag{A10}$$

$$\begin{aligned}G&=\frac{(n+m+1)(n+m+2)(n+m+3)(n+m+4)}{8(2n+1)(2n+3)^2(2n+5)}\\&\quad+\frac{(n-m-1)(n-m)(n+m+1)(n+m+2)}{2(2n-1)^2(2n+3)^2}\\&\quad+\frac{(n-m-3)(n-m-2)(n-m-1)(n-m)}{8(2n-3)(2n-1)^2(2n+1)},\end{aligned} \tag{A11}$$

while the number of primes over u_{vs}, in any case, denotes the number of primes over $j_v(x_2)$ and $n_v(x_2)$ (i.e., the order of their derivatives with respect to their argument x_2) in Eq. (A5).

2. Neumann boundary conditions

A. First method (use of spheroidal wave functions)

$$D_{nn}^0=x_2u_{nn}', \tag{A12}$$

$$D_{nn}^{(2)}=\mp\left[x_2u_{nn}'-mu_{nn}-x_2^3\frac{(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^2}u_{n+2,n}'+x_2^3\frac{(n-m-1)(n-m)}{2(2n-1)^2(2n+1)}u_{n-2,n}'\right], \tag{A13}$$

$$\begin{aligned}D_{nn}^{(4)}&=x_2^2\frac{(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^2}\left\{-x_2u_{n+2,n}'+mu_{n+2,n}\right.\\&\quad+\frac{2x_2^3}{2n+7}\cdot\left[\frac{1-4m^2}{(2n-1)(2n+3)^2}u_{n+2,n}'\right.\\&\quad\left.\left.+\frac{(n+m+3)(n+m+4)}{8(2n+5)^2}u_{n+4,n}'\right]\right\}\end{aligned}$$

$$-x_2^2\frac{(n-m-1)(n-m)}{2(2n-1)^2(2n+1)}\Bigg\{-x_2u'_{n-2,n}+mu_{n-2,n}+\frac{2x_2^3}{2n-5}\cdot\left[\frac{1-4m^2}{(2n-1)^2(2n+3)}u'_{n-2,n}-\frac{(n-m-3)(n-m-2)}{8(2n-3)^2}u'_{n-4,n}\right]\Bigg\}, \quad (A14)$$

$$D^{(2)}_{n+2,n}=\pm x_2^3\frac{(n+m+1)(n+m+2)}{2(2n+3)^2(2n+5)}u'_{n+2,n},$$
$$D^{(2)}_{n,n+2}=\mp x_2^3\frac{(n-m+1)(n-m+2)}{2(2n+1)(2n+3)^2}u'_{n,n+2}. \quad (A15)$$

B. Second method (shape perturbation)

$$D^0_{nn}=x_2u'_{nn}, \quad (A16)$$

$$D^{(2)}_{nn}=\mp x_2^2Fu''_{nn}\mp Mu_{nn}, \quad (A17)$$

$$D^{(4)}_{nn}=x_2^2G[3u''_{nn}+x_2u'''_{nn}]+\frac{L}{2(2n+1)}[u_{nn}+x_2u'_{nn}]-x_2^2Fu''_{nn}-Mu_{nn}, \quad (A18)$$

$$D^{(2)}_{n+2,n}=\pm\frac{(n+m+1)(n+m+2)}{2(2n+3)(2n+5)}[x_2^2u''_{n+2,n+2}-2(n+3)u_{n+2,n+2}], \quad (A19)$$

$$D^{(2)}_{n,n+2}=\pm\frac{(n-m+1)(n-m+2)}{2(2n+1)(2n+3)}[x_2^2u''_{nn}+2nu_{nn}],$$

where

$$M=\frac{1}{2n+1}\left[\frac{(n+1)(n^2-m^2)}{2n-1}-\frac{n((n+1)^2-m^2)}{2n+3}\right], \quad (A20)$$

$$L=\frac{(n-m)(n+m+1)}{2n+1}\left[\frac{(n+1)(n+m)}{2n-1}+\frac{n(n-m+1)}{2n+3}\right]\times\left(\frac{n+m}{2n-1}-\frac{n-m+1}{2n+3}\right)-\frac{n((n+1)^2-m^2)(n+m+2)(n+m+3)}{(2n+3)^2(2n+5)}+\frac{(n+1)(n-m-2)(n-m-1)(n^2-m^2)}{(2n-3)(2n-1)^2}. \quad (A21)$$

3. Two useful derivatives

The following two derivatives of E_n are very useful in Eqs. (13), (14), for the evaluation of $x_2^{(2)}$ and $x_2^{(4)}$ in any case, i.e., for Dirichel and Neumann conditions and for both methods. Various recurrence relations and Wronskians for spherical Bessel functions[13] have been used for their evaluation:

$$\frac{dE_n}{dx_2}=-\left\{\tau x_1^2\left(1-\frac{\rho_1}{\rho_2}\right)[j'_n(w_1)]^2+q^2\tau j_n^2(w_1)\times\left[x_1^2-\frac{\rho_2}{\rho_1}w_1^2-n(n+1)\left(1-\frac{\rho_2}{\rho_1}\right)\right]\right\}\Big/(x_1^4Q^2), \quad (A22)$$

$$\frac{d^2E_n}{dx_2^2}=-\frac{2\tau^2}{x_1^4Q^2}\Bigg\{x_1\left(1-\frac{\rho_1}{\rho_2}\right)j'_n(w_1)[j'_n(w_1)+w_1j''_n(w_1)]+q^2\frac{c_2}{c_1}j_n(w_1)j'_n(w_1)\cdot\left[x_1^2-\frac{\rho_2}{\rho_1}w_1^2-n(n+1)\times\left(1-\frac{\rho_2}{\rho_1}\right)\right]+q^2j_n^2(w_1)\left(x_1-\frac{w_1}{q}\right)\Bigg\}-2\tau\frac{dE_n}{dx_2}\left(\frac{2}{x_1}+\frac{Q_1+Q_2c_2/c_1}{Q}\right), \quad (A23)$$

where

$$Q=n_n(x_1)j'_n(w_1)-qj_n(w_1)n'_n(x_1),$$
$$Q_1=n'_n(x_1)j'_n(w_1)-qj_n(w_1)n''_n(x_1), \quad (A24)$$
$$Q_2=n_n(x_1)j''_n(w_1)-q\,j'_n(w_1)n'_n(x_1),$$

while $x_1=\tau x_2$, $w_1=\tau x_2c_2/c_1$ and $\tau=R_1/R_2$.

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