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# General slow viscous flows in a two-fluid system

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**Summary.** A general solution of the creeping flow equations suitable for a flow that is bounded by a non-deforming planar interface is presented. New compact representations for the velocity and pressure fields are given in terms of two scalar functions which describe arbitrary Stokes flow. A general reflection theorem is derived for a fluid-fluid interface problem containing Lorentz reflection formula as a particular case. The theorem allows a better interpretation of the image system for various singularities in the presence of a planar interface. The general solution is further used to describe the first-order approximation of the deformed interface by performing normal stress balance. It is found that the normal stress imbalance and the interface displacement are independent of the viscosity ratio of two fluids (!) and only depend on the location of initial singularity.

## 1 Introduction

The motion of a particle near the interface between two immiscible liquids is of fundamental importance in many engineering applications. The interface plays a significant role in a wide range of interesting problems including the locomotion of microorganisms, Brownian motion of colloidal particles, as well as in studies of surface phenomena in suspension mechanics. Such problems can be modelled using the creeping flow approximation. More importantly, useful information can be gained from the solutions of the problems formulated using Stokes flow approximation to understand the theory.

There are powerful methods for solving Stokes flow problems of which the boundary integral element method and the singularity method have been found more effective. The singularity method has been known for years since the pioneering work of Lorentz [1], and Oseen [2]. A detailed discussion on this technique with applications may be found in Chwang and Wu [3] and references therein. Jones et al. [4] adapted the image methods to derive the solution to Stokeslet (point force) problem with a fluid-fluid interface. Exploiting the solution they evaluated the diffusion coefficient for the motion of two polymer models (rigid rod and spherical) along the interface between two fluids. The Fourier transform technique has also been employed to solve the low Reynolds number flow problems [5]–[7]. Further, the use of Papkovitch-Neuber approach has been found convenient for constructing solutions in the case of bounded flows [8], [9].

In this paper, we present a concise method for calculating the detailed Stokes flow in the presence of a planar interface which separates two immiscible fluids. Following [10], we present an exact solution of Stokes equations in terms of two scalar functions describing arbitrary flows. In particular, this new representation is used to construct a general solution to the problem of arbitrary Stokes flow in the presence of a fluid-fluid interface under the assumption that the interface remains flat. A general theorem is developed which gives the reflection formula in the two fluid regions, extending Lorentz's [1] (see also [11]) result for the plane wall. The image systems for various point singularities are obtained in a straightforward manner by using the theorem, and their limiting cases are discussed. Although these results may be alternatively derived using the method due to Lee et al. [12], the present technique yields much more clearly the form of the image system in both regions. Furthermore, the theorem is used to perform the normal stress balance which leads to the computation of interface displacement. The features of these quantities are shown graphically. The present results may conveniently be used: i) to derive analytic expressions for the resistance and mobility functions for a particle or drop far away from a flat fluid-fluid interface; ii) to study diffusion of a polymer adsorbed at the interface between two immiscible fluids; and iii) as the basis of numerical schemes, with the point singularities distributed on the particle and drops, and interface treated with the appropriate images. Moreover, the present technique may also be employed for other divergence-free models such as transient Stokes flows [13] and flow through porous medium using the Brinkman model [14].

## 2 The problem

The governing equations for slow steady motion of a viscous incompressible fluid are

$$\nabla p = \mu \nabla^2 \mathbf{u}\,, \tag{1}$$

$$\nabla . \mathbf{u} = 0\,, \tag{2}$$

where $p, \mathbf{u}, \mu$ are the pressure, the velocity and the viscosity, respectively, and $\nabla^2$ is the Laplacian operator in three dimensions.

The general solution of (1) and (2) is (see Appendix I)

$$\mathbf{u} = \nabla \times \nabla \times (\mathbf{i}_z A) + \nabla \times (\mathbf{i}_z B)\,, \tag{3}$$

$$p = p_0 + \mu \frac{\partial}{\partial z} (\nabla^2 A) \tag{4}$$

($\mathbf{i}_z$ is the unit vector in $z$-direction). The scalar functions $A(\varrho, \phi, z)$ and $B(\varrho, \phi, z)$ satisfy

$$\nabla^4 A = 0 \quad \text{Biharmonic equation}\,, \tag{5}$$

$$\nabla^2 B = 0 \quad \text{Harmonic equation}\,, \tag{6}$$

$\varrho, \phi, z$ being cylindrical coordinates. The functions $A_0(\varrho, \phi, z)$ and $B_0(\varrho, \phi, z)$ for some motions generating singularities in the absence of boundaries are listed in Table 1. We remark that the representation (3) could also be effectively used for other divergence-free models such as unsteady creeping flows and flow through a permeable medium (Brinkman model). It is found that in these cases the functions $\nabla^2 A$ and $B$ satisfy the Helmholtz equations. We note that if we take $A = \frac{\psi(\varrho, z)}{\varrho} \cos\phi$, $B = \frac{\chi(\varrho, z)}{\varrho} \sin\phi$, then (3) and (4) reduce to the solution form adopted more recently by Feng et al. [14] to discuss the asymmetric motion of a disk in a Brinkman medium. However, we shall herein present the application of the representation (3) to steady Stokes flow problems.

Based on the above formulation, we now consider an arbitrary Stokes flow in the presence of a planar interface which separates two immiscible fluids. We use the index $i = 1, 2$

**Table 1.** Some slow viscous singularities in an unbounded flow

| Stokes flow singularities | $A_0(\varrho, \phi, z)$ | $B_0(\varrho, \phi, z)$ |
|---|---|---|
| Stokeslet at $(0, 0, c)$ | $\frac{F_1}{8\pi\mu}\frac{z-c}{\varrho}R_1\cos\phi$ | $\frac{F_1}{4\pi\mu}R_1\sin\phi$ |
| Rotlet (perpendicular) | 0 | $\frac{F_3}{8\pi\mu}\frac{1}{R_1}$ |
| Rotlet (parallel) | $-\frac{F_2}{8\pi\mu}\frac{R_1}{\varrho}\cos\phi$ | $-\frac{F_2}{8\pi\mu}\frac{z-c}{\varrho R_1}\sin\phi$ |
| Stokes-dipole | $D\frac{\varrho}{R_1}\cos\phi$ | 0 |
| Point source | $S\times\ln\left(\frac{R_1-(z-c)}{\varrho}\right)$ | 0 |
| Stokes-quadrupole (degenerate) | $D\frac{z-c}{\varrho R_1}\cos\phi$ | 0 |

to designate the two different fluids. From (3), the components of $\mathbf{u}^{(i)}$ in the directions of $\varrho, \phi, z$ are

$$u_\varrho{}^{(i)} = \frac{\partial^2 A^{(i)}}{\partial\varrho\,\partial z} + \frac{1}{\varrho}\frac{\partial B^{(i)}}{\partial\phi}\,, \tag{7}$$

$$u_\phi{}^{(i)} = \frac{\partial^2 A^{(i)}}{\partial\phi\,\partial z} - \frac{\partial B^{(i)}}{\partial\varrho}\,, \tag{8}$$

$$u_z{}^{(i)} = -\left(\frac{\partial^2 A^{(i)}}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial A^{(i)}}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2 A^{(i)}}{\partial\phi^2}\right). \tag{9}$$

If we assume $A^{(i)}(\varrho, \phi, z) = A^{(i)}(\varrho, z)$ and $B^{(i)} = 0$ and further $\frac{\partial A^{(i)}}{\partial\varrho} = \frac{\psi^{(i)}}{\varrho}$ the above equations reduce to

$$u_\varrho{}^{(i)} = \frac{1}{\varrho}\frac{\partial\psi^{(i)}}{\partial z}\,, \tag{10}$$

$$u_\phi{}^{(i)} = 0\,, \tag{11}$$

$$u^{(i)} = -\frac{1}{\varrho}\frac{\partial\psi^{(i)}}{\partial\varrho}\,, \tag{12}$$

where $\psi^{(i)}(\varrho, z)$ is the usual Stokes stream function.

### *2.1 Boundary conditions*

The boundary conditions at a stationary (planar) interface are:

(i) The normal velocity components at the interface vanish,
(ii) continuity of tangential velocity components,
(iii) continuity of tangential stresses.

If we take $z = 0$ as the planar interface, then the above conditions in terms of $A^{(i)}$ and $B^{(i)}$ read

$$A^{(1)} = 0\,, \qquad A^{(2)} = 0\,, \tag{13}$$

$$A_z{}^{(1)} = A_z{}^{(2)}\,, \tag{14}$$

$$\mu^{(1)} A_{zz}^{(1)} = \mu^{(2)} A_{zz}^{(2)}, \tag{15}$$

$$B^{(1)} = B^{(2)}, \qquad \mu^{(1)} B_z{}^{(1)} = \mu^{(2)} B_z{}^{(2)}, \tag{16}$$

where the subscripts denote partial differentiation. While the tangential stresses are continuous across the planar interface, the normal stress suffers a jump across the interface. The jump in the normal stress must be balanced by surface tension, and this determines a first-order approximation to the deviation of the interface shape from flat (i.e., interface dislacement $\zeta$). Thus, for small displacement, the equation for the interface becomes [15]

$$T {\nabla_2}^2 \zeta = \Delta\tau_{zz} = \left[ p^{(1)} - p^{(2)} + 2\left( \mu^{(2)} \frac{\partial u_z{}^{(2)}}{\partial z} - \mu^{(1)} \frac{\partial u_z{}^{(1)}}{\partial z} \right) \right]_{z=0}, \tag{17}$$

where $T$ is the surface tension, ${\nabla_2}^2$ is the two-dimensional Laplacian in $x$ and $y$, $\Delta\tau_{zz}$ is the normal stress imbalance, and $u_z{}^{(1)}$, $u_z{}^{(2)}$ are the $z$-components of velocities in the respective regions. The first-order approximation to the interface shape may be found by solving the two-dimensional Poisson equation defined in (17). For the situations where the surface tension cannot alone balance the induced normal stress on the interface, an additional term due to hydrostatic pressure difference must be included to have a meaningful solution to Eq. (17). In the following section we present a general solution for a fluid-fluid interface problem.

## 3 Reflection theorem

Let a Stokes flow of a viscous incompressible fluid in the absence of boundaries be described by the biharmonic function $A_0(\varrho, \phi, z)$ and the harmonic function $B_0(\varrho, \phi, z)$, whose singularities lie in the region $z > 0$. When the boundary $z = 0$ is introduced, the modified functions for the regions $z > 0$ and $z < 0$ satisfying the boundary conditions (13)–(16) on $z = 0$ are:

$$\begin{aligned} A^{(1)}(\varrho, \phi, z) = {} & A_0(\varrho, \phi, z) + \Lambda \left[ - A_0(\varrho, \phi, -z) + 2z \frac{\partial}{\partial z} A_0(\varrho, \phi, -z) - z^2 \nabla^2 A_0(\varrho, \phi, -z) \right] \\ & - (1 - \Lambda)\, A_0(\varrho, \phi, -z), \end{aligned} \tag{18}$$

$$B^{(1)}(\varrho, \phi, z) = B_0(\varrho, \phi, z) - \Lambda B_0(\varrho, \phi, -z) + (1 - \Lambda)\, B_0(\varrho, \phi, -z), \tag{19}$$

for $z > 0$, and

$$A^{(2)}(\varrho, \phi, z) = (1 - \Lambda) \left[ 2z \frac{\partial}{\partial z} A_0(\varrho, \phi, z) - z^2 \nabla^2 A_0(\varrho, \phi, z) \right], \tag{20}$$

$$B^{(2)}(\varrho, \phi, z) = 2(1 - \Lambda)\, B_0(\varrho, \phi, z) \tag{21}$$

for $z < 0$, where $\Lambda = \dfrac{\mu^{(2)}}{\mu^{(1)} + \mu^{(2)}}$.

The proof of the theorem is given in Appendix II. It is worth mentioning here that the reflection at the fluid-fluid interface is just a linear combination (in $\Lambda$) of the results for a rigid plane wall and a shear free surface. It may be noted that when $\Lambda = 1$, we recover the Lorentz reflection formula [1]; for $\Lambda = 0$ the results reduce to the shear-free case. It should also be pointed out that, in the axisymmetric case, the stream function $\psi^{(i)}$ has the same form as $A^{(i)}$ ($B^{(i)}$ being identically zero) in the two regions.

The velocity components in the respective flow regions can be computed directly using (7) – (9), and the pressure may be obtained from (4). The normal stresses in the two regions are, respectively,

$$\tau_{zz}^{(1)} = \mu^{(1)}\left[2\,\frac{\partial^3 A_0(\varrho,\phi,z)}{\partial z^3} - 3\,\frac{\partial}{\partial z}\,\nabla^2 A_0(\varrho,\phi,z) - 2\,\frac{\partial^3 A_0(\varrho,\phi,-z)}{\partial z^3} + 3\,\frac{\partial}{\partial z}\,\nabla^2 A_0(\varrho,\phi,-z)\right]$$

$$+ \Lambda\mu^{(1)} z\left[4\,\frac{\partial^4 A_0(\varrho,\phi,z)}{\partial z^4} - 6\,\frac{\partial}{\partial z}\,\nabla^2 A_0(\varrho,\phi,z) - 2z\,\frac{\partial^3}{\partial z^3}\,\nabla^2 A_0(\varrho,\phi,-z)\right], \tag{22}$$

$$\tau_{zz}^{(2)} = (1-\Lambda)\,\mu^{(2)} z\left[4\,\frac{\partial^4 A_0(\varrho,\phi,z)}{\partial z^4} - 4\,\frac{\partial}{\partial z}\,\nabla^2 A_0(\varrho,\phi,z) - 2\,\frac{\partial^2}{\partial z^2}\,\nabla^2 A_0(\varrho,\phi,z)\right.$$

$$\left. - 2z\,\frac{\partial^3}{\partial z^3}\,\nabla^2 A_0(\varrho,\phi,z)\right]. \tag{23}$$

Therefore, the normal stress imbalance on the interface $z = 0$ is

$$\Delta\tau_{zz} = \mu^{(1)}\left[2\,\frac{\partial^3 A_0(\varrho,\phi,z)}{\partial z^3} - 3\,\frac{\partial}{\partial z}\,\nabla^2 A_0(\varrho,\phi,z) - 2\,\frac{\partial^3 A_0(\varrho,\phi,-z)}{\partial z^3}\right.$$

$$\left. + 3\,\frac{\partial}{\partial z}\,\nabla^2 A_0(\varrho,\phi,-z)\right]_{z=0}. \tag{24}$$

From (24), we see that the normal stress imbalance, in general, does not depend on the viscosity ratio. This in turn implies that the interface displacement is independent of the viscosity ratio of the two fluids!

## 4 Images of singularities

We now use the theorem derived in the previous section to construct the image systems for various singularities in a two-fluid space with a planar interface.

### *4.1 Stokeslet at* $(0, 0, c)$

A solution to the Stokeslet problem in a two fluid system using image methods was first obtained by Jones et al. [4] in connection with their study concerning diffusion of polymers. Their work seems to be unnoticed, and the same solution was derived little later by Aderogba and Blake [7] using Fourier transform technique and by Lee et al. [12] using the reciprocal theorem. Here, we rederive their results using our method and add further flow details.

Consider a Stokeslet of strength $F_1$ located at $(0, 0, c)$ whose axis is along the positive $x$-direction, i.e., tangential to the plane boundary $z = 0$. The velocity components due to this Stokeslet in an unbounded fluid, in cylindrical coordinates, are:

$$u_{0\varrho} = \frac{F_1}{8\pi\mu^{(1)}}\left[\frac{1}{R_1} + \frac{\varrho^2}{{R_1}^3}\right]\cos\phi\,, \tag{25}$$

$$u_{0\phi} = \frac{F_1}{8\pi\mu^{(1)}}\left[-\frac{1}{R_1}\right]\sin\phi\,, \tag{26}$$

$$u_{0z} = \frac{F_1}{8\pi\mu^{(1)}}\left[\frac{\varrho(z-c)}{{R_1}^3}\right]\cos\phi\,, \tag{27}$$

where ${R_1}^2 = \varrho^2 + (z-c)^2$ and $\varrho^2 = x^2 + y^2$. The expressions for $A_0$ and $B_0$ are obtained by integrating Eq. (3) after substituting $\mathbf{u}_0$ into its right hand side. This yields

$$A_0(\varrho,\phi,z) = \frac{F_1}{8\pi\mu^{(1)}}\left[\frac{(z-c)}{\varrho}\,R_1\cos\phi\right], \tag{28}$$

$$B_0(\varrho,\phi,z) = \frac{F_1}{4\pi\mu^{(1)}}\left[\frac{R_1}{\varrho}\sin\phi\right]. \tag{29}$$

Applying the theorem we obtain for $z > 0$

$$\begin{aligned} A^{(1)}(\varrho,\phi,z) = {} & \frac{F_1}{8\pi\mu^{(1)}}\left[\frac{(z-c)}{\varrho}\,R_1\cos\phi\right] \\ & + \frac{F_1}{8\pi\mu^{(1)}}\left[(1-2\Lambda)\,\frac{(z+c)}{\varrho}\,R_2 + \Lambda\left(2c\,\frac{\varrho}{R_2} + 2c^2\,\frac{(z+c)}{\varrho R_2}\right)\right]\cos\phi, \end{aligned} \tag{30}$$

$$B^{(1)}(\varrho,\phi,z) = \frac{F_1}{4\pi\mu^{(1)}}\left[\frac{R_1}{\varrho}\sin\phi\right] + \frac{F_1}{4\pi\mu^{(1)}}\,(1-2\Lambda)\,\frac{R_2}{\varrho}\sin\phi, \tag{31}$$

where ${R_2}^2 = \varrho^2 + (z+c)^2$. For $z < 0$

$$A^{(2)}(\varrho,\phi,z) = \frac{F_1}{8\pi\mu^{(1)}}\,(1-\Lambda)\left[2\,\frac{(z-c)}{\varrho}\,R_1\cos\phi + 2c\,\frac{\varrho}{R_1}\cos\phi - 2c^2\,\frac{(z-c)}{\varrho R_1}\cos\phi\right], \tag{32}$$

$$B^{(2)}(\varrho,\phi,z) = \frac{F_1}{4\pi\mu^{(1)}}(1-\Lambda)\left[2\,\frac{R_1}{\varrho}\sin\phi\right]. \tag{33}$$

The image terms in (30), (31) can be interpreted in the following way:

(i) The second terms of (30) and (31) represent a Stokeslet of strength $(1-2\Lambda)$ times the initial Stokeslet, but located at the image point $(0,0,-c)$.
(ii) The third term on the r.h.s of (30) is a Stokes-doublet of strength $2c\Lambda F_1$ located at $(0,0,-c)$ and
(iii) the fourth term on the r.h.s of (30) is a potential-doublet (degenerate Stokes-quadrupole) of strength $-2c^2\Lambda F_1$ at $(0,0,-c)$.

The image system just obtained agrees with that obtained by Aderogba and Blake [7]. Clearly, the strength of the image singularities depends on the viscosity ratio and the location of the initial Stokeslet. It is surprising to note that the sign of the image Stokeslet is positive or negative according to whether $\Lambda >$ or $< 1/2$. When $\Lambda = 1/2$, the image Stokeslet vanishes! This in turn implies that when two identical fluids are separated by an interface, the net force (due to image singularities) acting on the interface is zero, and the image system in this case is considerably reduced. It is also found that when the Stokeslet is normal to the interface (axisymmetric case) the image Stokeslet is independent of the viscosity ratio. For the extremal values of $\Lambda$ Eqs. (30), (31) reduce to rigid and shear-free cases, respectively.

The image system in the region $z < 0$ consists of a Stokeslet, a Stokes-doublet and a potential-doublet of strengths $2(1-\Lambda)\,F_1$, $2c(1-\Lambda)\,F_1$, $-2c^2(1-\Lambda)\,F_1$, respectively. The far-field effect in both the regions is that of a Stokeslet of strength $2(1-\Lambda)\,F_1$. Now the normal stress imbalance at $z = 0$, due to the Stokeslet singularity is

$$-\Delta\tau_{zz} = \frac{F_1}{8\pi}\,\frac{12xc^2}{{R_0}^5}, \tag{34}$$

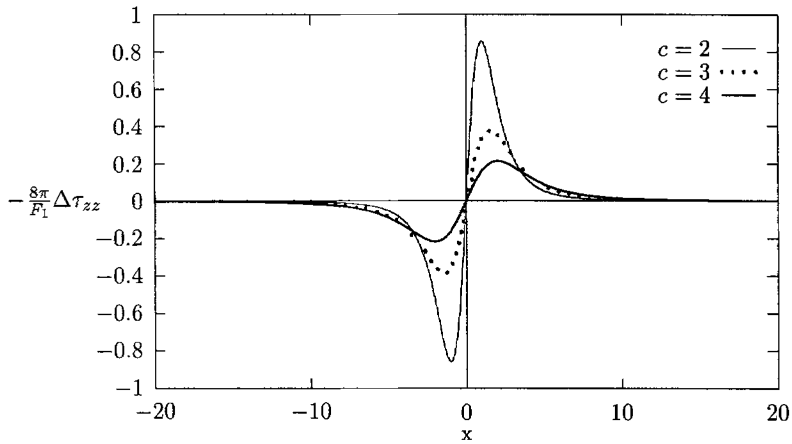


**Fig. 1.** Normal stress imbalance on the line $y = z = 0$ due to a Stokeslet

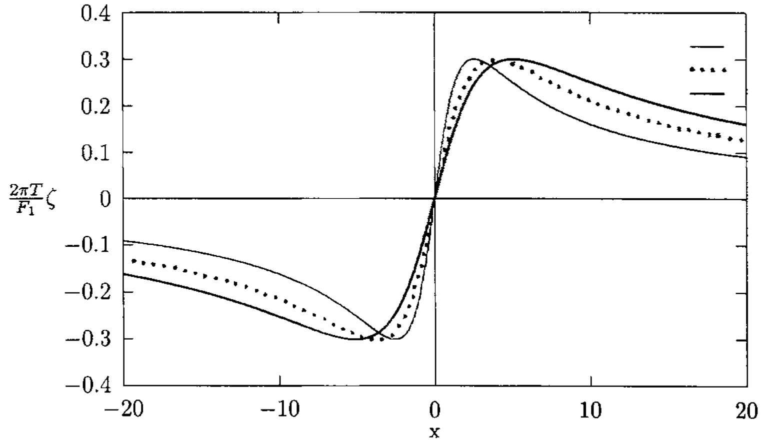


**Fig. 2.** Variation of interface displacement in $x$-direction

where ${R_0}^2 = \varrho^2 + c^2$. It follows from (17) that

$$\zeta = \frac{F_1}{2\pi T}\frac{cx}{\varrho^2}\left[1 - \frac{c}{\sqrt{\varrho^2 + c^2}}\right]. \tag{35}$$

Equations (34) and (35) are the same as those derived by Jones et al. [4], and Lee et al. [12] for the normal stress imbalance and Aderogba and Blake [7] for the interface displacement. For the sake of completeness, the normal stress imbalance and the interface displacement are plotted in Figs.1 and 2 for the values of $c = 2, 3, 4$, respectively. It may be seen that the normal stres imbalance becomes larger as the Stokeslet approaches closely to the interface. For the positive force, the fluid interface is increased in the positive direction and depressed on the negative side.

### *4.2 Rotlet at* $(0, 0, c)$

Now, consider a rotlet of strength $F_3$ located at $(0, 0, c)$ whose axis is along the $z$-direction, i.e., perpendicular to the planar interface. The corresponding expressions for $A_0$ and $B_0$ are given in Table 1. The flow fields in the two regions are described by the functions:

For $z > 0$:

$$A^{(1)}(\varrho, \phi, z) = 0\,, \tag{36}$$

$$B^{(1)}(\varrho, \phi, z) = \frac{F_3}{8\pi\mu^{(1)}}\left[\frac{1}{R_1} + \frac{F_3}{8\pi\mu^{(1)}}\,(1 - 2\Lambda)\left[\frac{1}{R_2}\right]\right], \tag{37}$$

For $z < 0$:

$$A^{(2)}(\varrho, \phi, z) = 0\,, \tag{38}$$

$$B^{(2)}(\varrho, \phi, z) = \frac{F_3}{8\pi\mu^{(1)}}\,2(1 - \Lambda)\left[\frac{1}{R_1}\right]. \tag{39}$$

The flow field in the region $z > 0$ consists of a rotlet of strength $(1 - 2\Lambda)\,F_3$ located at the image point. Here again, the sign of the image rotlet depends on the viscosity ratio (and at the critical value it vanishes). This tells that when a sphere rotates in front of a planar interface that separates two fluids, the torque on the sphere increases or decreases depending on the value of the ratio $\Lambda$. When the viscosities of the fluids are identical, the torque on the sphere is the same as that when a sphere rotates in an unbounded fluid. When $\Lambda = 1$, we recover the results of Blake and Chwang [6], and for $\Lambda = 0$ we obtain the image system for a shear-free interface. In both cases the image is a rotlet a result analogous to that in electrostatics. The region $z < 0$ has a rotlet (as an image) of strength $2(1 - \Lambda)F_3$ at the point $(0, 0, c)$. The flow fields for a rotlet whose axis is tangential to the planar interface arise from

$$\begin{aligned} A^{(1)}(\varrho, \phi, z) = \frac{F_2}{8\pi\mu^{(1)}}\Bigg[&-\frac{R_1}{\varrho}\cos\phi + (1 - 2\Lambda)\left(\frac{R_2}{\varrho}\cos\phi\right) \\ &+ \Lambda\left(\frac{2\varrho}{R_2}\cos\phi + \frac{2c(z+c)}{\varrho R_2}\cos\phi\right)\Bigg], \end{aligned} \tag{40}$$

$$B^{(1)}(\varrho, \phi, z) = \frac{F_2}{8\pi\mu^{(1)}}\left[-\frac{(z-c)}{\varrho R_1}\sin\phi + (1 - 2\Lambda)\left(\frac{(z+c)}{\varrho R_2}\sin\phi\right)\right], \tag{41}$$

$$A^{(2)}(\varrho, \phi, z) = \frac{F_2}{8\pi\mu^{(1)}}(1 - \Lambda)\left[-2\,\frac{R_1}{\varrho}\cos\phi + 2\,\frac{\varrho}{R_1}\cos\phi - \frac{2c(z-c)}{\varrho R_1}\cos\phi\right], \tag{42}$$

$$B^{(2)}(\varrho, \phi, z) = \frac{F_2}{8\pi\mu^{(1)}}(1 - \Lambda)\left[-2\,\frac{(z-c)}{\varrho R_1}\sin\phi\right]. \tag{43}$$

In this case, the image system in the two regions consists of a rotlet, a Stokes-doublet and a degenerate Stokes-quadrupole. Furthermore, the image rotlet vanishes in the region $z > 0$ at the critical value and changes sign accordingly as explained in the above cases. It may be noted that the rotlet singularity dominates in the far-field. We note that in the case of an initial rotlet perpendicular to the interface there is no jump in the normal stress, and hence there is no deformation of the interface due to this singularity. But, when the rotlet axis is parallel to the planar interface, we have

$$-\Delta\tau_{zz} = \frac{F_2}{8\pi}\frac{12xc}{{R_0}^5}\,, \tag{44}$$

$$\zeta = \frac{F_2}{2\pi T}\frac{x}{\varrho^2}\left[1 - \frac{c}{\sqrt{\varrho^2 + c^2}}\right]. \tag{45}$$

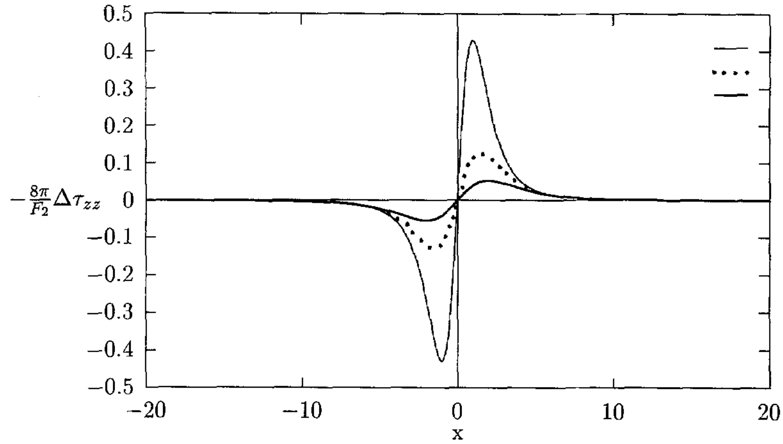


**Fig. 3.** Normal stress imbalance on the line $y = z = 0$ due to a rotlet

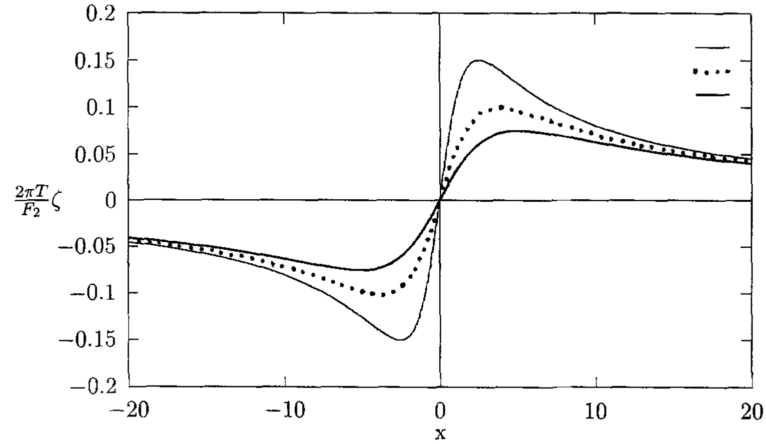


**Fig. 4.** Variation of interface displacement in $x$-direction

We observe that the normal stress imbalance and interface displacement differ from the Stokeslet case only by a factor $c$. Hence, the behavior of these quantities is qualitatively similar as can be seen from Fig. 3 and 4. However, the deformation of the interface is less in the present case.

### *4.3 Degenerate Stokes-quadrupole at* $(0, 0, c)$

The functions $A_0$ and $B_0$ for a degenerate Stokes-quadrupole (potential-doublet) located at $(0, 0, c)$ are provided in Table 1. The image system can be constructed with the aid of the theorem and is given by:

For $z > 0$:

$$A^{(1)}(\varrho, \phi, z) = \frac{S_1}{6\pi\mu^{(1)}}\left[\frac{(z-c)}{\varrho R_1}\cos\phi\right] + \frac{S_1}{6\pi\mu^{(1)}}\left[\frac{(z+c)}{\varrho R_2}\cos\phi - \Lambda\left(\frac{2\varrho(z+c)}{{R_2}^3}\cos\phi - 2c\,\frac{\varrho}{{R_2}^3}\cos\phi\right)\right], \tag{46}$$

$$B^{(1)}(\varrho, \phi, z) = 0\,. \tag{47}$$

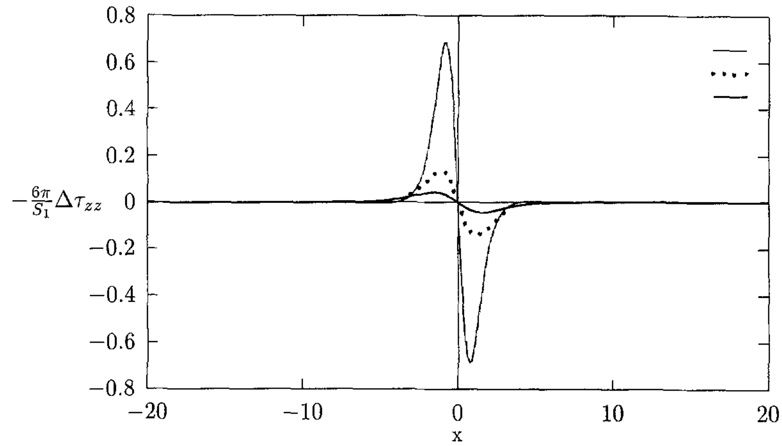


**Fig. 5.** Normal stress imbalance on the line $y = z = 0$ due to a potential-dipole

And for $z < 0$:

$$A^{(2)}(\varrho, \phi, z) = \frac{S_1}{6\pi\mu^{(1)}}(1 - \Lambda)\left[\frac{2\varrho(z - c)}{R_1{}^3}\cos\phi + 2c\frac{\varrho}{R_1{}^3}\cos\phi\right], \tag{48}$$

$$B^{(2)}(\varrho, \phi, z) = 0\,. \tag{49}$$

In the present case, the image system for the region $z > 0$ consists of a potential-doublet, a Stokes-quadrupole and a potential-quadrupole of strengths $S_1, 2\Lambda S_1, 2c\Lambda S_1$ at the image point. The image singularities in the region $z < 0$ are a Stokes-quadrupole and a potential-quadrupole at the point $(0, 0, c)$. It is important to note that, in the present case, the form of the image system remains the same for all nonzero values of $\Lambda$. It can be easily seen from (46)–(49) that the far-field effect is dominated by the potential-doublet for the region $z > 0$, while the motion in the far-field for the region $z < 0$ is influenced by the higher order singularities. The normal stress imbalance in the present case is

$$-\Delta\tau_{zz} = 2\frac{S_1}{\pi}\frac{x}{R_0{}^3}\left[1 - \frac{5c^2}{R_0{}^2}\right]. \tag{50}$$

From Fig. 5 we see that the normal stress imbalance increases as the initial singularity approaches closely to the interface and is positive or negative according to whether $x <$ or $> 0$. We note here that the latter feature is different from the other two cases discussed earlier. The interface deformation for this case may be found from (17).

## 5 Conclusions

A simple reflection theorem generalizing Lorentz's formula is offered for computing Stokes flow images systematically in a two-fluid space with a planar interface, employing a general representation of velocity and pressure fields. Exploiting this new result, the image systems for: i) a Stokeslet, ii) a rotlet, and iii) a degenerate Stokes-quadrupole, positioned in front of a flat surface separating two immiscible fluids are constructed in a straightforward manner. The flow details together with the limiting cases are also discussed in each case. It is observed that the viscosity ratio of two fluids has influence on the image systems, velocity and pressure fields. But, surprisingly, the normal stress imbalance and hence the interface displacement are independent of the viscosity ratio. However, the latter quantities are found to vary according to the location of the initial singularity in each case. Other motions generating singularities may be treated in a similar manner. The method of solution described here can also be suitably used for transient flows as well as flows through a porous medium. Finally, the present solution scheme can be applied, in principle, to motion of particles, polymers and drops in the presence of a flat interface.

## Appendix I

The representation for the velocity field **u** given by (3) satisfies the continuity equation (2), and substitution of (3) into (1) yields

$$\operatorname{grad}\left(p-\mu\frac{\partial}{\partial z}\nabla^2 A\right)=\mu[\mathbf{i_z}\nabla^4 A-(\mathbf{i_z}\times\nabla)\nabla^2 B]\,, \tag{51}$$

where the Laplacian operator in cylindrical coordinates $\varrho, \phi, z$ is defined as

$$\nabla^2=\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}\,. \tag{52}$$

Further, from (3), we have

$$LA=-\mathbf{i_z}.\mathbf{u}\,,\qquad LB=-(\mathbf{i_z}.\nabla\times\mathbf{u})\,, \tag{53}$$

where $L=\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2}{\partial\phi^2}$. Now, $\nabla^2 LA=-\nabla^2(\mathbf{i_z}.\mathbf{u})=-\mathbf{i_z}.\nabla^2\mathbf{u}$. Therefore,

$$\nabla^4 LA=-\mathbf{i_z}.\nabla^4\mathbf{u}\,. \tag{54}$$

Since $L$ and $\nabla^4$ commute and **u** is biharmonic,

$$L\nabla^4 A=0\,. \tag{55}$$

Similarly, $L\nabla^2 B=\nabla^2 LB=-\nabla^2(\mathbf{i_z}.\nabla\times\mathbf{u})=-\mathbf{i_z}.\nabla^2(\nabla\times\mathbf{u})$, and since $\nabla\times\mathbf{u}$ is harmonic,

$$L\nabla^2 B=0\,. \tag{56}$$

Therefore, $\nabla^4 A=F(z)$, where $F$ is an arbitrary function. Let $A=G(z)+A'$. This implies $\nabla^4 A=\nabla^4 G+\nabla^4 A'$. Choose $G(z)$ so that $\nabla^4 G=F(z)$. This choice is possible since $\nabla^4 G$ is an ordinary differential equation. This implies $\nabla^4 A'=0$. We can neglect $G(z)$ since $\nabla\times\mathbf{i_z}G=0$ for any $G(z)$. Therefore, without loss of generality, we assume that

$$\nabla^4 A=0\,. \tag{57}$$

By a similar argument we can assume that

$$\nabla^2 B=0\,. \tag{58}$$

Using (57) and (58) in (51), we obtain

$$p=p_0+\mu\frac{\partial}{\partial z}(\nabla^2 A)\,. \tag{59}$$

Thus, the problem of quasi-steady motion of a viscous incompressible fluid reduces to solving for the functions $A$ and $B$ that are biharmonic and harmonic, respectively. Note that the representation (3) contains three independent harmonic functions ($A$, being biharmonic, has therefore two harmonic functions) and hence it is equivalent to Lamb's general solution [16] suitable for planar interfaces. Since $z$ does not occur explicitly in the operator given by Eq. (52), it is form-invariant under the translation of the origin along the $z$-axis. We observe that if $\Omega(\varrho, \phi, z)$ is a solution of (58) then so are

$$\Omega(\varrho, \phi, 2h - z)\,, \qquad \frac{\partial}{\partial z}\Omega(\varrho, \phi, z)\,, \qquad \frac{\partial}{\partial z}\Omega(\varrho, \phi, 2h - z)\,, \tag{60}$$

where $h$ is a constant. This switches the direction of the $z$-axis and puts the origin at $z = h$. Further,

$$\nabla^2(z\Omega) = 2\,\frac{\partial \Omega}{\partial z} + z\nabla^2\Omega = 2\frac{\partial \Omega}{\partial z}\,, \tag{61}$$

so that

$$\nabla^4(z\Omega) = 0\,. \tag{62}$$

## Appendix II

Let the given flow be characterised by the biharmonic function $A_0$ and the harmonic function $B_0$. We describe the method of deriving the functions $B^{(1)}$ and $B^{(2)}$, and in a similar fashion the functions $A^{(1)}$ and $A^{(2)}$ may be obtained. We assume

$$B^{(1)}(\varrho, \phi, z) = B_0(\varrho, \phi, z) + B_1(\varrho, \phi, 2h - z)\,, \tag{63}$$

$$B^{(2)}(\varrho, \phi, z) = B_2(\varrho, \phi, z)\,, \tag{64}$$

where $B_1$, $B_2$ are harmonic functions and $h$ is a constant. We take $z = h$ as the interface and apply the boundary conditions (16) to obtain

$$B_0(\varrho, \phi, h) + B_1(\varrho, \phi, h) = B_2(\varrho, \phi, h)\,, \tag{65}$$

$$\mu^{(1)}\frac{\partial}{\partial h}\,[B_0(\varrho, \phi, h) - B_1(\varrho, \phi, h)] = \mu^{(2)}\frac{\partial}{\partial h}\,B_2(\varrho, \phi, h)\,. \tag{66}$$

From the above two equations, we obtain

$$B^{(1)}(\varrho, \phi, z) = B_0(\varrho, \phi, z) - \Lambda B_0(\varrho, \phi, 2h - z) + (1 - \Lambda)\,B_0(\varrho, \phi, 2h - z)\,, \tag{67}$$

$$B^{(2)}(\varrho, \phi, z) = 2(1 - \Lambda)\,B_0(\varrho, \phi, z)\,. \tag{68}$$

These equations reduce to (19) and (21) by putting $h = 0$. Now, since the function $A_0$ is biharmonic, it can be written as

$$A_0(\varrho, \phi, z) = P(\varrho, \phi, z) + (z - h)\,Q(\varrho, \phi, z)\,. \tag{69}$$

Here $P$ and $Q$ are harmonic functions. Choosing $A^{(1)}$ and $A^{(2)}$ in an appropriate way and using the boundary conditions (13)–(15) one could obtain (18) and (20) in a similar way as above.

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## References

[1] Lorentz, H. A.: A general theorem concerning the motion of a viscous fluid and few consequences derived from it. Versl. Konigl. Akad. Wetensch. Amst. **5,** 168–175 (1896). See also J. Engng. Math. **30,** 19–24 (1996).
[2] Oseen, C. W.: Hydrodynamik, pp. 97–107. Leipzig: Akad. Verlagsgesellschaft 1927.
[3] Chwang, A. T., Wu, T. Y.: Hydromechanics of low Reynolds-number flow. J. Fluid Mech. **67,** 787–815 (1975).
[4] Jones, R. B., Felderhof, B. U., Deutch, J. M.: Diffusion of polymers along a fluid-fluid interface. Macromolecules **8,** 680–684 (1975).
[5] Blake, J. R.: A note on the image system for a Stokeslet in a no-slip boundary. Proc. Camb. Phil. Soc. **70,** 303–310 (1971).
[6] Blake, J. R., Chwang, A. T.: Fundamental singularities of viscous flow. Part I: The image systems in the vicinity of a stationary no-slip boundary. J. Engng. Math. **8,** 23–29 (1974).
[7] Aderogba, K., Blake, J. R.: Action of force near the planar surface between two semi-infinite immiscible liquids at very low Reynolds numbers. Bull. Austral. Math. Soc. **18,** 345–356 (1978).
[8] Aderogba, K.: On Stokeslets in a two-fluid space. J. Engng. Math. **10,** 143–151 (1976).
[9] Hasimoto, H., Sano, O.: Stokeslets and eddies in creeping flow. Ann. Rev. Fluid Mech. **12,** 335–363 (1980).
[10] Palaniappan, D., Nigam, S. D., Amaranath, T., Usha, R.: Lamb's solution of Stokes's equations: a sphere theorem. Q. J. Mech. Appl. Math. **45,** 47–56 (1992).
[11] Happel, J., Brenner, H.: Low Reynolds number hydrodynamics, p. 87. The Hague: Matinus Nijhoff 1983.
[12] Lee, S. H., Chadwick, R. S., Leal, L. G.: Motion of a sphere in the presence of a plane interface. Part 1. An approximate solution by the generalization of the method of Lorentz. J. Fluid Mech. **93,** 705–726 (1979).
[13] Pozrikidis, C.: Boundary integral and singularity methods for linearised viscous flow, p. 212. Cambridge: Cambridge University Press 1992.
[14] Feng, J. Ganatos, P., Weinbaum, S.: The general motion of a disk in a Brinkman Medium. Phys. Fluids **10,** 2137–2146 (1998).
[15] Landau, L. D., Lifshitz, E. M.: Fluid mechanics. Oxford: Pergamon Press 1959.
[16] Lamb, H.: Hydrodynamics, 6th ed., p. 594. New York: Dover 1945.

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