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The fields of potential generated by a point source in the ring-shaped region

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Abstract—The problem is studied through the concept of influence (Green's) function for Laplace equation. When the classical eigenfunction expansion method (EEM) is used in elliptic PDEs for the construction of Green's functions, the latter appear in the form of non-uniformly convergent series, the ones that are not suitable for immediate computational applications. To refrain from this defect, a modified version of the EEM has earlier been proposed, the key point of which is splitting off the singular and regular components of Green's function. This radically enhances the computational effectiveness of the approach. The modified version of the EEM is applied in this study to a variety of (mixed) boundary value problems stated for the two-dimensional Laplace equation over the circular ring-shaped region. Results are obtained in the form suitable for engineering applications.

1. INTRODUCTION

The field of potential, generated by a point source in a region, is usually interpreted in physics through the influence function of a point source. In mathematics, in turn, it is associated with the Green's function of the corresponding boundary value problem for Laplace equation. Clearly, the compactness of the Green's function is a defining issue in engineering applications. Unfortunately, the representations of Green's functions for PDEs that are available in literature, are rarely appropriate for the immediate computational use. As one of the rare examples of an immediately computable form, recall the classical (Tikhonov and Samarskii, 1963) expression

$$G_1(r,\varphi;\rho,\psi)=\frac{1}{2\pi}\ln\frac{|z\bar{\zeta}-a^2|}{a|z-\zeta|} \tag{1}$$

for the Green's function of Dirichlet problem stated for Laplace equation on the disk $\{(r,\varphi)\,:\,0<r<a,\ 0\leqslant\varphi<2\pi\}$. Here $z=r(\cos\varphi+i\sin\varphi)$ and

$\zeta = \rho(\cos\psi + i\sin\psi)$ represent the field and the source point, respectively, while $\bar{\zeta}$ stands for the complex conjugate.

In the standard Green's function-related texts (see, for example, Morse and Feshbach 1953; Tikhonov and Samarskii, 1963 and Smirnov, 1964), one finds readily computable expressions for Green's functions of the Dirichlet problem for such regions as a half-plane, a circular sector, an infinite strip, and a semi-strip. For instance, the expression

$$G_2(x, y; \xi, \eta) = \frac{1}{2\pi}\ln\frac{[\cosh(x+\xi)-\cos(y-\eta)][\cosh(x-\xi)-\cos(y+\eta)]}{[\cosh(x-\xi)-\cos(y-\eta)][\cosh(x+\xi)-\cos(y+\eta)]} \tag{2}$$

represents the Green's function of the Dirichlet problem on the semi-strip $\{(x, y) : 0 < x < \infty, 0 < y < \pi\}$. Here (x, y) and (ξ, η) represent the field and the source point, respectively.

To be accurately computed, many of the existing representations of Green's functions usually require certain adjustments prior to the actual involvement in a numerical computation. Those adjustments are sometimes quite nontrivial. Recall, for example, the well-known double-series form

$$G_3(x, y; \xi, \eta) = \frac{4ab}{\pi^2}\sum_{m,l=1}^{\infty}\frac{\sin(\mu x)\sin(\mu\xi)\sin(\lambda y)\sin(\lambda\eta)}{(la)^2+(mb)^2} \tag{3}$$

of the Green's function for Dirichlet problem stated on the rectangle $\{(x, y) : 0 < x < a,\ 0 < y < b\}$, where $\mu = m\pi/a$ and $\lambda = l\pi/b$. The above expression is not appropriate for accurate computations, because the series in Eqn (3) does not uniformly converge. This computational deficiency can, however, be fixed for series representations of the type in Eqn (3). From Melnikov (1998), for example, one can find out how such a series reduces to an accurately computable form by explicitly separating its logarithmic singularity. Such an operation eventually yields a uniformly convergent series for the regular component of Green's function.

Recall another representation that is not directly suitable for computing. That is (Smirnov, 1964)

$$G_4(z, \zeta) = \frac{1}{2\pi}\left\{\frac{1}{\beta} + \ln\frac{|z\bar{\zeta}-1|}{|z-\zeta|} + 2\Re\left[\omega^{-\beta}\int_0^{\omega}\frac{t^{\beta-1}}{1-t}\,\mathrm{d}t\right]\right\}, \tag{4}$$

the Green's function of the mixed problem

$$\frac{\partial u(a,\varphi)}{\partial r} + \beta u(a,\varphi) = 0$$

stated for Laplace equation on the unit disk ($a = 1$). Here $\Re$ symbolizes the real part of a complex-valued function and $\omega = r\rho\exp[i(\varphi-\psi)]$, with (r, φ) and (ρ, ψ) representing the field and the source point, respectively.

As it follows from Eqn (4), the regular component of $G_4(z, \zeta)$ is expressed in terms of the real part of a complex-valued function written in an integral form.

Hence, the accurate computation of the entire expression in Eqn (4) represents quite a cumbersome problem itself.

A number of computable forms of Green's functions are obtained in (Melnikov, 1998) for elliptic boundary value problems of applied mechanics. Among those, an alternative form of $G_4(z,\zeta)$ was found there as

$$G_4(z,\zeta) = \frac{1}{2\pi}\left\{\frac{1}{a\beta} + \ln\frac{a^3}{|z-\zeta||z\bar{\zeta}-a^2|} - \sum_{m=1}^{\infty}\frac{2a\beta}{m(m+a\beta)}\left(\frac{r\rho}{a^2}\right)^m \cos[m(\varphi-\psi)]\right\}. \quad (5)$$

Since the regular component of this form of $G_4(z,\zeta)$ is written as a uniformly convergent series (its convergence rate is of the order of $1/m^2$), the above expression is more computationally effective compared to the form shown in Eqn (4). Indeed, to attain a required level of accuracy when computing the expression in Eqn (5), one appropriately truncates the series.

It is well known that when the classical (Morse and Feshbach, 1953) EEM is applied to the construction of Green's functions, non-uniformly convergent series appear. To fix this shortcoming, the modified version of the EEM was developed in (Melnikov and Krasnikova, 1981). Within the modified version, a special transformation was proposed allowing the singular component to be expressed analytically, while the regular component obtained in the form of a uniformly convergent series. In (Melnikov, 1998), it was shown that this method has proven to be effective for many settings in applied mechanics.

Note that it is especially difficult to construct accurately computable representations of Green's functions for multi-connected regions. In fact, none of such forms can be found in the existing literature for the ring-shaped region. The present study aims to fill this gap.

From mathematical physics (see, for example, Smirnov, 1964) it follows that if the solution of the well-posed boundary value problem

$$B_1[u(a,\varphi)] = 0, \qquad B_2[u(b,\varphi)] = 0, \quad (6)$$

stated for Poisson equation

$$\Delta u(r,\varphi) = -f(r,\varphi), \quad (r,\varphi)\in\Omega, \quad (7)$$

over the circular ring $\Omega = \{(r,\phi) : a < r < b, 0 \leqslant \phi < 2\pi\}$, is found as

$$u(r,\varphi) = \iint_{\Omega} G(r,\varphi;\rho,\psi) f(\rho,\psi)\,\mathrm{d}\Omega(\rho,\psi), \quad (8)$$

then $G(r,\varphi;\rho,\psi)$ represents the Green's function of the problem posed with Eqn (6) for Laplace equation over Ω.

Note that within this study, B_1 and B_2 are understood as continuous operators with constant coefficients. In other words, a single type of boundary conditions (either

Dirichlet, or Neumann, or mixed with constant coefficients) is imposed on each of the edges of the circular ring Ω. This creates eight different types of boundary value problems for Ω to consider.

The relation in Eqn (8) suggests, in particular, a clear way of constructing $G(r, \varphi; \rho, \psi)$. That is, of all of the methods for solving the problem in Eqns (6) and (7), we select the ones which express $u(r, \varphi)$ in a form of the integral in Eqn (8), because the latter yields the explicit expression of $G(r, \varphi; \rho, \psi)$. The EEM represents, in fact, such a method.

2. DIRICHLET PROBLEM

If B_1 and B_2 in Eqn (6) represent unit operators, then the formulation in Eqns (6) and (7) is said to be the Dirichlet problem.

Assume that the solution $u(r, \varphi)$ to the Dirichlet problem posed with Eqns (6)–(7) is expandable in Fourier series,

$$u(r, \varphi) = \frac{1}{2}u_0(r) + \sum_{n=1}^{\infty}\left[u_n^c(r)\cos(n\varphi) + u_n^s(r)\sin(n\varphi)\right]. \tag{9}$$

Upon expressing the right-hand term of Eqn (7) with Fourier series

$$f(r, \varphi) = \frac{1}{2}f_0(r) + \sum_{n=1}^{\infty}\left[f_n^c(r)\cos(n\varphi) + f_n^s(r)\sin(n\varphi)\right] \tag{10}$$

and substituting the expansions from Eqns (9) and (10) into (6) and (7), one obtains the following set of ordinary boundary value problems

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_n(r)}{\mathrm{d}r}\right) - \frac{n^2}{r}u_n(r) = -rf_n(r), \quad n = 0, 1, 2, \ldots, \tag{11}$$

$$u_n(a) = 0, \qquad u_n(b) = 0, \tag{12}$$

which holds for both the cosine and sine coefficients in Eqns (9) and (10).

It is important to note that the fundamental solution set of the homogeneous equation corresponding to (11) for the case of $n = 0$ differs of that for the general case of $n \geqslant 1$. That is why, when solving the problem posed by Eqns (11) and (12), those two cases must be considered separately.

The standard procedure based on Lagrange method of variation of parameters yields, for the problem in Eqns (11) and (12),

$$u_n(r) = \int_a^b k_n(r, \rho)\rho f_n(\rho)\,\mathrm{d}\rho, \tag{13}$$

where for the case of $n = 0$, the kernel of the integral in (13) is obtained as

$$k_0(r, \rho) = \frac{1}{\ln(a/b)}\begin{cases}\ln(r/a)\ln(b/p), & r \leqslant \rho,\\ \ln(\rho/a)\ln(b/r), & r \geqslant \rho,\end{cases}$$

whereas for $n \geqslant 1$, we have

$$k_n(r,\rho) = \frac{1}{2n(r\rho)^n(b^{2n}-a^{2n})}\begin{cases}(b^{2n}-\rho^{2n})(r^{2n}-a^{2n}), & r \leqslant \rho,\\ (b^{2n}-r^{2n})(\rho^{2n}-a^{2n}), & r \geqslant \rho.\end{cases}$$

Clearly, the symmetry of $k_0(r,\rho)$ and $k_n(r,\rho)$ reflects the self-adjointness of the formulation in Eqns (11) and (12).

According to the fundamental rule for coefficients of the Fourier series, one writes the following expressions:

$$f_n^c(\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho,\psi)\cos(n\psi)\,\mathrm{d}\psi, \quad n = 0, 1, 2, \ldots, \tag{14}$$

$$f_n^s(\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho,\psi)\sin(n\psi)\,\mathrm{d}\psi, \quad n = 1, 2, 3, \ldots, \tag{15}$$

for the coefficients $f_n^c(\rho)$ and $f_n^s(\rho)$ of the series that appears in Eqn (10).

Equation (13) can now be written, for both the cosine and sine coefficients of the series in Eqn (9), as

$$u_n^c(r) = \int_a^b k_n(r,\rho)\rho f_n^c(\rho)\,\mathrm{d}\rho, \tag{16}$$

$$u_n^s(r) = \int_a^b k_n(r,\rho)\rho f_n^s(\rho)\,\mathrm{d}\rho. \tag{17}$$

Upon substituting (14) and (15) into (16) and (17), and then the coefficients $u_0(r)$, $u_n^c(r)$, and $u_n^s(r)$ into Eqn (9), one obtains the solution $u(r,\varphi)$ of the Dirichlet problem posed by Eqns (6) and (7) in the form

$$u(r,\varphi) = \frac{1}{\pi}\int_0^{2\pi}\int_a^b\left[\frac{k_0(r,\rho)}{2} + \sum_{n=1}^{\infty} k_n(r,\rho)\cos(n\varphi)\cos(n\psi) + \sum_{n=1}^{\infty} k_n(r,\rho)\sin(n\varphi)\sin(n\psi)\right] f(\rho,\psi)\rho\,\mathrm{d}\rho\,\mathrm{d}\psi,$$

which can be rewritten as

$$u(r,\varphi) = \frac{1}{\pi}\int_0^{2\pi}\int_a^b\left[\frac{k_0(r,\rho)}{2} + \sum_{n=1}^{\infty} k_n(r,\rho)\cos(n(\varphi-\psi))\right] f(\rho,\psi)\rho\,\mathrm{d}\rho\,\mathrm{d}\psi. \tag{18}$$

Thus, the solution $u(r,\varphi)$ of the Dirichlet problem for Poisson equation stated on the ring Ω is found as an integral over Ω. Consequently, referring to Eqn (8), one concludes that the kernel

$$G(r,\varphi;\rho,\psi) = \frac{1}{2\pi}\left[k_0(r,\rho) + 2\sum_{n=1}^{\infty} k_n(r,\rho)\cos[n(\varphi-\psi)]\right] \tag{19}$$

of (18) represents the Green's function of the Dirichlet problem for Laplace equation on the ring. In other words, Eqn (19) delivers us the influence function of a point source for the field of potential in the circular ring, given that the potential function vanishes on the ring's edges.

The form of the coefficient $k_n(r, \rho)$ presented earlier implies that the trigonometric series in Eqn (19) converges non-uniformly at the rate of $1/n$. The analysis reveals, causing this, three different types of singularity: (i) the principal logarithmic singularity which shows up whenever the field point (r, φ) approaches the source point (ρ, ψ) (it is associated with the term $(rb^2/\rho)^n$ in $k_n(r, \rho)$); (ii) the near-boundary singularity showing up whenever both the field and the source points approach the outer contour, $r = b$, of the ring (this is due to the term $(r\rho)^n$ in $k_n(r, \rho)$); and (iii) another near-boundary singularity (associated with the term $[(ab)^2/(r\rho)]^n$ in $k_n(r, \rho)$), which occurs whenever both the field and the source point approach the inner contour, $r = a$.

The non-uniform convergence makes the expansion in Eqn (19) inappropriate for direct computational implementations. To illustrate this issue, the profile of $G(r, \varphi; \rho, \psi)$ is depicted in Fig. 1a for $a = 1.0$ and $b = 3.0$, with the source point fixed at (2.0, 0.0). It has been computed by truncating the series to the partial sum with ten terms counted. It is clearly seen that the contour lines are not smooth enough oscillating around their expected shapes. The smoothness of the contour lines improves somewhat but not much for higher order of partial sums (see Fig. 1b for the profile of $G(r, \varphi; \rho, \psi)$ computed with one hundred terms counted in the truncated series).

Thus, as it follows from our analysis, the costly way of involving of higher partial sums in computing the non-uniformly convergent series in Eqn (19) is not productive. There is, however, an effective way of improving the convergence of the series. This can be done by a certain decomposition of its coefficient $k_n(r, \rho)$. Note that either branch of $k_n(r, \rho)$ is appropriate for this purpose. In what follows, the decomposition is performed with the branch that is valid for $r \leqslant \rho$. This yields

$$k_n(r, \rho) = \frac{1}{2n(r\rho)^n}\left[\left(\frac{1}{b^{2n} - a^{2n}} - \frac{1}{b^{2n}}\right) + \frac{1}{b^{2n}}\right](b^{2n} - \rho^{2n})(r^{2n} - a^{2n})$$
$$= \frac{1}{2n(r\rho)^n}\left[\frac{a^{2n}}{b^{2n}(b^{2n} - a^{2n})} + \frac{1}{b^{2n}}\right](b^{2n} - \rho^{2n})(r^{2n} - a^{2n}), \quad \text{for } r \leqslant \rho.$$

With this, the series in Eqn (19) brakes down into two, the first of which, the one associated with the term

$$\frac{a^{2n}}{b^{2n}(b^{2n} - a^{2n})}$$

in $k_n(r, \rho)$, is uniformly convergent, whereas the other series, associated with the term $1/b^{2n}$, is non-uniformly convergent. It allows, however, a complete summation

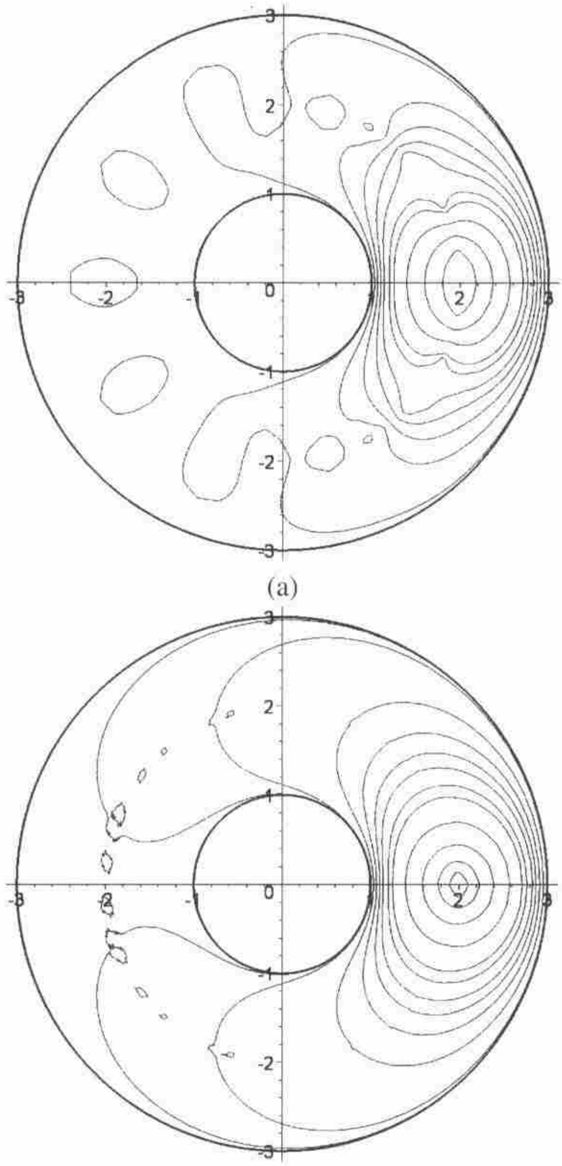

Figure 1. Profile of the Green's function in Eqn (19) with: (a) 10 terms counted; (b) 100 terms counted.

with the aid of the standard (Gradstein and Ryzhik, 1980) formula

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos(n\alpha) = -\ln \sqrt{1 - 2p\cos\alpha + p^2},$$

which is valid for $p^2 < 1$, and $0 \leqslant \alpha < 2\pi$. The summation yields

$$G(r,\varphi;\rho,\psi) = \frac{1}{2\pi}\left\{k_0(r,\rho) + \sum_{n=1}^{\infty} k_n^*(r,\rho)\cos[n(\varphi-\psi)] \right. \tag{20}$$
$$\left. + \frac{1}{2}\ln\frac{[b^4 - 2b^2 r\rho\cos(\varphi-\psi) + r^2\rho^2][a^4 - 2a^2 r\rho\cos(\varphi-\psi) + r^2\rho^2]}{[r^2 - 2r\rho\cos(\varphi-\psi) + \rho^2][b^4 r^2 - 2a^2 b^2 r\rho\cos(\varphi-\psi) + a^4\rho^2]}\right\},$$

where $k_n^*(r,\rho)$ is (similarly to $k_n(r,\rho)$) defined in two pieces and its branch that is valid for $r \leqslant \rho$, is found as

$$k_n^*(r,\rho) = \frac{a^{2n}(b^{2n} - \rho^{2n})(r^{2n} - a^{2n})}{n(b^2 r\rho)^n(b^{2n} - a^{2n})},$$

while the branch of $k^*(r,\rho)$ that is valid for $\rho \leqslant r$, can be obtained from the above one by interchanging r with ρ.

Shorthanded notation can be introduced for the logarithmic terms in Eqn (20) so that the expression for the Green's function of Dirichlet problem on the ring of radii a and b is finally rewritten as

$$G(r,\varphi;\rho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|b^2 - z\bar{\zeta}||a^2 - z\bar{\zeta}|}{|z-\zeta||b^2 z - a^2\zeta|} \right.$$
$$\left. + k_0(r,\rho) + \sum_{n=1}^{\infty} k_n^*(r,\rho)\cos[n(\varphi-\psi)]\right\}, \tag{21}$$

where z and ζ are the field and the source point, respectively.

It is important to point out that the representation just derived should inevitably be much more effective compared to that of Eqn (19), because the series in Eqn (21) converges uniformly allowing pretty accurate computation at a considerably low cost. Figure 2 convincingly supports this point. It depicts the profile of $G(r,\varphi;\rho,\psi)$ computed with the aid of the representation in Eqn (21) for the same set of initial data as that used for Fig. 1a, included the order (ten terms counted) of the partial sum for the truncated series. The smoothness of the contour lines in Fig. 2 is notably higher compared even to that of Fig. 1b, where one hundred terms have been counted in the series of Eqn (19).

3. OTHER PROBLEMS

The procedure described in Section 2 is applicable, with slight modifications, to other boundary value problems of the class indicated earlier. In what follows, the subsectional titles specify the types of boundary conditions to be imposed on the inner edge, $r = a$, and the outer edge, $r = b$, of the ring.

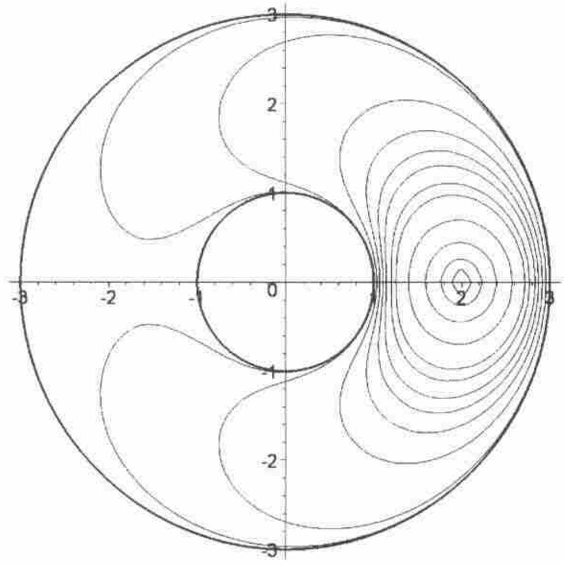

Figure 2. Profile of the Green's function in Eqn (21) with 10 terms counted.

In Section 3.1 we begin with deriving an effective representation of the influence function for the "mixed–mixed" problem. Then in Sections 3.2 through 3.7, the other problems from the class are considered.

3.1. Mixed–mixed problem

Let the following mixed boundary conditions:

$$\frac{\partial u(a,\varphi)}{\partial r} - \beta_1 u(a,\varphi) = 0, \qquad \frac{\partial u(b,\varphi)}{\partial r} + \beta_2 u(b,\varphi) = 0, \tag{22}$$

with $\beta_1 \geqslant 0$ and $\beta_2 \geqslant 0$, be imposed on the ring's edges. This formulation represents the most general (of the class specified earlier) statement of a boundary value problem for Laplace equation on the ring. The point is that all other formulations, of the class, follow from (22) as particular cases. And the only meaningless of them is that of "Neumann–Neumann" type (when both β_1 and β_2 equal zero), which simply does not allow the Green's function to exist in the classical (Smirnov, 1964) sense.

In compliance with the EEM procedure, the Green's function for the problem that is posed in Eqn (22), appears in a form of the series of Eqn (19), whose coefficients $k_0(r,\rho)$ and $k_n(r,\rho)$ are symmetric functions of r and ρ, and their $r \leqslant \rho$ branches are found in this case as

$$k_0(r,\rho) = \frac{[1 + a\beta_1 \ln(r/a)][1 - b\beta_2 \ln(\rho/b)]}{a\beta_1 + b\beta_2[1 + a\beta_1 \ln(b/a)]}, \quad r \leqslant \rho, \tag{23}$$

and

$$k_0(r,\rho) = \frac{A(a,r,n,\beta_1)B(b,\rho,n,\beta_2)}{2n(r\rho)^n D(a,b,n,\beta_1,\beta_2)}, \quad r \leqslant \rho, \tag{24}$$

where

$$A(a,r,n,\beta_1) = (n + a\beta_1)r^{2n} + (n - a\beta_1)a^{2n},$$

$$B(b,\rho,n,\beta_2) = (n + b\beta_2)b^{2n} + (n - b\beta_2)\rho^{2n},$$

and

$$D(a,b,n,\beta_1,\beta_2) = (n + a\beta_1)(n + b\beta_2)b^{2n} - (n - a\beta_1)(n - b\beta_2)a^{2n}.$$

It is clearly seen that the expressions for $k_0(r,\rho)$ and $k_n(r,\rho)$ in Eqns (23) and (24) reduce to those of Section 2 for the Dirichlet problem, if both the parameters β_1 and β_2 are taken to infinity.

Analogously to the Dirichlet problem, the expansion in Eqn (19) in the present case does not uniformly converge (its convergence rate is also of the order of $1/n$). To improve the convergence, one of the factors of the coefficient $k_n(r,\rho)$ in Eqn (24) is transformed in a manner similar to that suggested in Section 2. That is

$$\begin{aligned}\frac{1}{D(a,b,n,\beta_1,\beta_2)} &= \left[\frac{1}{D(a,b,n,\beta_1,\beta_2)} - \frac{1}{n^2b^{2n}}\right] + \frac{1}{n^2b^{2n}} \\ &= \frac{R(a,b,n,\beta_1,\beta_2)}{n^2b^{2n}D(a,b,n,\beta_1,\beta_2)} + \frac{1}{n^2b^{2n}},\end{aligned} \tag{25}$$

where

$$\begin{aligned}R(a,b,n,\beta_1,\beta_2) &= \left[n^2(n + a\beta_1)(a + b\beta_2)\right]b^{2n} + (n - a\beta_1)(n - b\beta_2)a^{2n} \\ &= (n - a\beta_1)(a - b\beta_2)a^{2n} - [n(a\beta_1 + b\beta_2) + ab\beta_1\beta_2]b^{2n}.\end{aligned}$$

Substituting (25) into (24), one obtains

$$k_n(r,\rho) = K_n(r,\rho) + \widetilde{K}_n(r,\rho),$$

where

$$K_n(r,\rho) = \frac{A(a,r,n,\beta_1)B(b,\rho,n,\beta_2)}{2n^3(b^2r\rho)^n D(a,b,n,\beta_1,\beta_2)} R(a,b,n,\beta_1,\beta_2)$$

and

$$\widetilde{K}_n(r,\rho) = \frac{A(a,r,n,\beta_1)B(b,\rho,n,\beta_2)}{2n^3(b^2r\rho)^n}.$$

By splitting off the components of the order of $1/n$, the above expression for $\widetilde{K}_n(r,\rho)$ regroups as

$$\widetilde{K}_n(r,\rho) = K_n^*(r,\rho) + \frac{1}{2n}\left[\left(\frac{r}{\rho}\right)^n + \left(\frac{a^2}{r\rho}\right)^n + \left(\frac{r\rho}{b^2}\right)^n + \left(\frac{a^2\rho}{b^2r}\right)^n\right],$$

where

$$K_n^*(r,\rho) = \frac{1}{2n^3(b^2r\rho)^n}\{n[(a\beta_1+b\beta_2)((rb)^{2n}-(a\rho)^{2n}) \\ +(a\beta_1-b\beta_2)((r\rho)^{2n}-(ab)^{2n})]+ab\beta_1\beta_2(b^{2n}-\rho^{2n})(r^{2n}-a^{2n})\}$$

is the component of the order of $1/n^2$. This transforms the coefficient $k_n(r,\rho)$ of the series in Eqn (19) to the form

$$k_n(r,\rho) = K_n(r,\rho)+K_n^*(r,\rho) \\ +\frac{1}{2n}\left[\left(\frac{r}{\rho}\right)^n+\left(\frac{a^2}{r\rho}\right)^n+\left(\frac{r\rho}{b^2}\right)^n+\left(\frac{a^2\rho}{b^2r}\right)^n\right], \quad r\leqslant\rho.$$

With this, the expansion in Eqn (19) brakes down onto two series, of which the one associated with the component $K_n(r,\rho)+K_n^*(r,\rho)$ converges uniformly allowing accurate computational implementations. The non-uniformly convergent second series in Eqn (19) (the one associated with the third component of $k_n(r,\rho)$) sums up analytically providing finally us with the form

$$G(r,\varphi;\rho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|z-\zeta||a^2-z\bar{\zeta}||b^2-z\bar{\zeta}||b^2z-a^2\zeta|}{(b^2r\rho)^2} \right. \\ \left. +k_0(r,\rho)+2\sum_{n=1}^{\infty}\left[K_n(r,\rho)+K_n^*(r,\rho)\right]\cos[n(\phi-\psi)]\right\} \quad (26)$$

of the Green's function to the "mixed–mixed" problem, which is accurately computable.

Illustrating the computational efficiency of the above representation, Fig. 3 depicts the profile of $G(r,\varphi;\rho,\psi)$, where the parameters are fixed as: $a=1.0$, $b=3.0$, $\rho=2.0$, $\psi=0$, $\beta_1=1.0$, and $\beta_2=5.0$. The series in Eqn (26) has been truncated to ten terms.

3.2. Dirichlet–Neumann problem

Clearly, the case of the boundary conditions imposed on the ring's edges as

$$u(a,\varphi)=0, \qquad \frac{\partial u(b,\varphi)}{\partial r}=0$$

follows from that stated with Eqn (22), if $\beta_1\to\infty$ and $\beta_2\to 0$. In physical terms the above conditions imply that the potential vanishes on the ring's inner edge while its outer edge is insulated. The Green's function is, in this case, also written as the series in Eqn (19), whose coefficients $k_0(r,\rho)$ and $k_n(r,\rho)$ are symmetric about r and ρ, and are found, for $r\leqslant\rho$, as

$$k_0(r,\rho)=\ln\frac{r}{a} \quad \text{and} \quad k_n(r,\rho)=\frac{(b^{2n}+\rho^{2n})(r^{2n}-a^{2n})}{2n(r\rho)^n(b^{2n}+a^{2n})}.$$

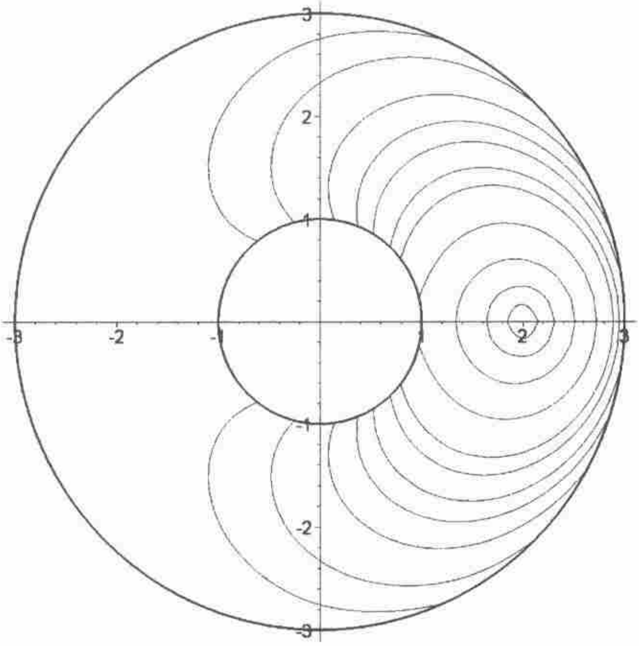

Figure 3. Profile of the Green's function in Eqn (26) with 10 terms counted.

These are obtained by taking the corresponding limits of the expressions derived earlier in Eqns (23) and (24).

Splitting off the singular components and regrouping, one finally arrives at the accurately computable representation

$$G(r,\varphi;\rho,\psi)=\frac{1}{2\pi}\left\{\ln\frac{|b^2z-a^2\zeta||a^2-z\bar{\zeta}|}{a|z-\zeta||b^2-z\bar{\zeta}|}\right.$$
$$\left.+\sum_{n=1}^{\infty}k_n^*(r,\rho)\cos[n(\varphi-\psi)]\right\} \tag{27}$$

of the Green's function for the "Dirichlet–Neumann" problem. The $r\leqslant\rho$ branch of $k_n^*(r,\rho)$ is found in this case as

$$k_n^*(r,\rho)=\frac{a^{2n}(b^{2n}+\rho^{2n})(a^{2n}-r^{2n})}{n(b^2r\rho)^n(b^{2n}+a^{2n})}.$$

3.3. Neumann–Dirichlet problem

The case of the boundary conditions imposed as

$$\frac{\partial u(a,\varphi)}{\partial r}=0,\qquad u(b,\varphi)=0$$

follows from that stated in Eqn (22), if $\beta_1\to 0$ and $\beta_2\to\infty$. The Green's function is in this case also written as the series in Eqn (19), whose coefficients $k_0(r,\rho)$

and $k_n(r, \rho)$ are found by taking the corresponding limits of the expressions derived earlier in Eqns (23) and (24). This yields, for $r \leqslant \rho$,

$$k_0(r, \rho) = \ln \frac{b}{\rho} \quad \text{and} \quad k_n(r, \rho) = \frac{(\rho^{2n} - n^{2n})(r^{2n} + a^{2n})}{2n(r\rho)^n(b^{2n} + a^{2n})}.$$

Analogously to the derivation shown in Section 3.2, we split off the singular components and regroup. This finally yields the readily computable representation

$$G(r, \varphi; \rho, \psi) = \frac{1}{2\pi}\left\{\ln \frac{|b^2 z - a^2\zeta||b^2 - z\bar{\zeta}|}{b|z - \zeta||a^2 - z\bar{\zeta}|} + \sum_{n=1}^{\infty} k_n^*(r, \rho)\cos[n(\varphi - \psi)]\right\} \tag{28}$$

of the Green's function for the "Neumann–Dirichlet" problem, where the $r \leqslant \rho$ branch of $k_n^*(r, \rho)$ is found as

$$k_n^*(r, \rho) = \frac{a^{2n}(\rho^{2n} - b^{2n})(r^{2n} + a^{2n})}{n(b^2 r\rho)^n(b^{2n} + a^{2n})}.$$

3.4. Dirichlet–mixed problem

The $r \leqslant \rho$ branches of the coefficients $k_0(r, \rho)$ and $k_n(r, \rho)$ in Eqn (19) of the Green's function for the boundary value problem stated as

$$u(a, \varphi) = 0, \qquad \frac{\partial u(b, \varphi)}{\partial r} + \beta_2 u(b, \varphi) = 0,$$

with $\beta_2 \geqslant 0$, can be obtained from Eqns (23) and (24), if β_1 is taken to infinity. That is

$$k_0(r, \rho) = \ln(r/a)\frac{1 + \beta_2 b \ln(b/\rho)}{1 + \beta_2 b \ln(b/a)}, \quad r \leqslant \rho,$$

and

$$k_n(r, \rho) = \frac{(r^{2n} - a^{2n})[n(b^{2n} + \rho^{2n}) + \beta_2 b(b^{2n} - \rho^{2n})]}{2n(r\rho)^n[n(b^{2n} + a^{2n}) + \beta_2 b(b^{2n} - a^{2n})]}, \quad r \leqslant \rho.$$

The readily computable representation of the Green's function is, in this case, finally obtained in the form

$$G(r, \varphi; \rho, \psi) = \frac{1}{2\pi}\left\{\ln \frac{|b^2 z - a^2\zeta||a^2 - z\bar{\zeta}|}{|z - \zeta||b^2 - z\bar{\zeta}|} + k_0(r, \rho) - \ln r + \sum_{n=1}^{\infty} k_n^*(r, \rho)\cos[n(\varphi - \psi)]\right\}, \quad r \leqslant \rho, \tag{29}$$

where the $r \leqslant \rho$ branch of $k_n^*(r,\rho)$ is found as

$$k_n^*(r,\rho) = \left(r^{2n} - a^{2n}\right) \frac{na^{2n}(b^{2n} + \rho^{2n}) + \beta_2 b[\rho^{2n}(b^{2n} - a^{2n}) + b^{2n}(\rho^{2n} - a^{2n})]}{n(b^2 r\rho)^n [n(b^{2n} + a^{2n}) + \beta_2 b(b^{2n} - a^{2n})]}.$$

Note that if $\beta_2 = 0$, then the expression in Eqn (29) reduces to that of the Green's function for the "Dirichlet–Neumann" problem presented with Eqn (27) in Section 3.2. On the other hand, the expression of Eqn (21) for the Green's function of the Dirichlet problem, derived in Section 2, does not directly follow from that of Eqn (29) when β_2 is taken to infinity. An additional algebraic transformation is required for that.

3.5. Mixed–Dirichlet problem

The $r \leqslant \rho$ branches of the coefficients $k_0(r,\rho)$ and $k_n(r,\rho)$ in Eqn (19) of the Green's function for the boundary value problem stated as

$$u(b,\varphi) = 0, \qquad \frac{\partial u(a,\varphi)}{\partial r} + \beta_1 u(a,\varphi) = 0,$$

with $\beta_1 \geqslant 0$, can be obtained from Eqns (23) and (24), if β_2 is taken to infinity. That is

$$k_0(r,\rho) = \ln(b/\rho)\frac{1 + \beta_1 a \ln(r/a)}{1 + \beta_1 a \ln(b/a)}, \quad r \leqslant \rho,$$

and

$$k_n(r,\rho) = \frac{(b^{2n} - p^{2n})[n(r^{2n} + a^{2n}) + \beta_1 a(r^{2n} - a^{2n})]}{2n(r\rho)^n[n(b^{2n} + a^{2n}) + \beta_1 a(b^{2n} - a^{2n})]}, \quad r \leqslant \rho.$$

Splitting off the singular components, one obtains the effective representation

$$G(r,\varphi;\rho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|b^2 z - a^2\zeta||b^2 - z\bar{\zeta}|}{|z - \zeta||a^2 - z\bar{\zeta}|} + k_0(r,\rho) + \sum_{n=1}^{\infty} k_n^*(r,\rho)\cos[n(\varphi - \psi)]\right\}, \quad r \leqslant \rho, \quad (30)$$

of the Green's function under consideration. The $r \leqslant \rho$ branches of $k_0^*(r,\rho)$ and $k_n^*(r,\rho)$ are found in this case as

$$k_0^*(r,\rho) = \frac{\beta_1 a \ln(b/r)\ln(\rho/b)}{1 + \beta_1 a \ln(b/a)}$$

and

$$k_n^*(r,\rho) = a^{2n}\left(\rho^{2n} - b^{2n}\right)\frac{n(r^{2n} + a^{2n}) + \beta_1 a[(b^{2n} - a^{2n}) + (b^{2n} - r^{2n})]}{n(b^2 r\rho)^n[n(b^{2n} + a^{2n}) + \beta_1 a(b^{2n} - a^{2n})]}.$$

Note that if $\beta_1 = 0$, then the expression in Eqn (30) reduces to that of the Green's function for the "Neumann–Dirichlet" problem (see Eqn (28)). However, analogously to the situation in the previous section, some additional algebraic transformation is required to reduce it to that of Eqn (21).

3.6. Neumann–mixed problem

The expression in Eqn (26) can be considered as a efficient representation of the Green's function for the boundary value problem

$$\frac{\partial u(a,\varphi)}{\partial r} = 0, \qquad \frac{\partial u(b,\varphi)}{\partial r} + \beta_2 u(b,\varphi) = 0,$$

with $\beta_2 \geqslant 0$, if the components $k_0(r,\rho)$, $K_n(r,\rho)$, and $K_n^*(r,\rho)$ are obtained from those shown in Section 3.1 by letting β_1 be zero. This yields

$$k_0(r,\rho) = \frac{1 + \beta_2 b \ln(b/\rho)}{\beta_2 b}, \quad r \leqslant \rho,$$

$$\begin{aligned} &K_n(r,\rho) \\ &\quad = \left[(n - \beta_2 b)a^{2n} - \beta_2 b^{2n+1}\right] \frac{(r^{2n} + a^{2n})[(n + \beta_2 b)b^{2n} + (n - \beta_2 b)\rho^{2n}]}{2n^2(b^2 r\rho)^n[(n + \beta_2 b)b^{2n} - (n - \beta_2 b)a^{2n}]}, \\ &\quad r \leqslant \rho, \end{aligned}$$

and

$$K_n^*(r,\rho) = \frac{\beta_2 b}{2n^3(b^2 r\rho)^n}\left\{n\left[(rb)^{2n} - (ap)^{2n}\right] - \left[(r\rho)^{2n} - (ab)^{2n}\right]\right\}, \quad r \leqslant \rho.$$

3.7. Mixed–Neumann problem

Analogously to Section 3.6, the efficient representation of the Green's function for the boundary value problem

$$\frac{\partial u(a,\varphi)}{\partial r} - \beta_1 u(a,\varphi) = 0, \qquad \frac{\partial u(b,\varphi)}{\partial r} = 0,$$

with $\beta_2 \geqslant 0$, can also be given by Eqn (26). The components $k_0(r,\rho)$, $K_n(r,\rho)$, and $K_n^*(r,\rho)$ are obtained, in this case, from those shown in Section 3.1 by letting β_2 be zero. This yields

$$k_0(r,\rho) = \frac{1 + \beta_1 a \ln(r/a)}{\beta_1 a}, \quad r \leqslant \rho,$$

$$\begin{aligned} &K_n(r,\rho) \\ &\quad = \left[(n - \beta_1 a)a^{2n} - \beta_1 a b^{2n}\right] \frac{(b^{2n} + \rho^{2n})[(n + \beta_1 a)r^{2n} + (n - \beta_1 a)a^{2n}]}{2n^2(b^2 r\rho)^n[(n + \beta_1 a)b^{2n} - (n - \beta_1 a)a^{2n}]}, \\ &\quad r \leqslant \rho, \end{aligned}$$

and

$$K_n^*(r,\rho) = \frac{\beta_1 a}{2n^3(b^2 r\rho)^n}\left\{n\left[(rb)^{2n} - (a\rho)^{2n}\right] - \left[(r\rho)^{2n} - (ab)^{2n}\right]\right\}, \quad r \leqslant \rho.$$

4. CONCLUSION

The emphasis in this study is on the providing of engineers and other practitioners, who work in industry, with computationally effective representations of influence functions for potential problems formulated in elements of constructions that are annular in shape. In addition, the expressions of influence functions derived herein, can also be of help in solving problems formulated on regions that are not quite annular but close to that, like a circular ring with cutoffs or extra apertures. The point is that Green's functions, being used as kernels of harmonic potentials (see Melnikov, 1977), may in such cases, significantly enhance the productivity of the boundary element method.

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