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Multiple Scattering of Elastic Waves by Parallel Cylinders

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The formal solution resulting from the scattering of a plane, time-harmonic, compressional elastic wave impinged on a group of parallel circular cylindrical inclusions in a finite domain is obtained. The inclusions are rigid as well as immovable and the geometry of their configuration is arbitrary. The technique of "multiple scattering" which was developed in acoustic and electromagnetic wave propagation is applied. The stress field around two identical circular cylindrical inclusions at a finite separation is studied in detail.

Introduction

In the recent past, the scattering of elastic waves by a single simple discontinuity has been extensively studied [1–4][1] and the stress concentration as well as the displacements in the near field are carefully examined. However, the scattering of elastic waves by more than one object has not been fully explored. This paper is an attempt to fulfill this need.

Due to the presence of lattices and gratings, the work on the diffraction of acoustic or electromagnetic waves by a configuration of elementary scatterers is numerous. Rayleigh [5] first put forward the "single scattering" hypothesis in which the scatterers are so far apart such that their excitation can be taken as the incident wave. More recently, Miles [6], Heaviside [7], and Row [8] have examined this type of problem but their approach cannot be easily extended to the case of elastic waves. However, Twersky [9, 10] used the method of multiple scattering which, in essence, is an iteration process based on systematically improving the results of single scattering approximation. This technique can be applied directly to the problems of elastic wave propagation.

In the present paper, the formal solution resulting from the scattering of a plane, time-harmonic elastic compressional wave impinged on a group of parallel circular cylindrical inclusions in a finite domain is obtained. These inclusions are rigid and immovable in space. Their geometry is arbitrary. An example of two identical circular cylinders at finite separation is analyzed in detail.

General Theory

In an isotropic homogeneous elastic medium, the displacement **u** for the case of plane strain may be expressed in terms of two scalar potentials Φ and Ψ which satisfy the wave equations

$$\nabla^2\Phi = \frac{1}{C_\alpha^2}\ddot{\Phi}, \tag{1}$$

$$\nabla^2\Psi = \frac{1}{C_\beta^2}\ddot{\Psi} \tag{2}$$

and

$$\mathbf{u} = \nabla\Phi + \nabla \times (\Psi\mathbf{k}) \tag{3}$$

where

$$C_\alpha^2 = (\lambda + 2\mu)/\rho, \qquad C_\beta^2 = \mu/\rho$$

k is the unit vector in the z-direction, λ, μ are the Lamé constants, and ρ is the density of the medium.

The stress tensor $\bar{\tau}$ is related to the displacement vector by

$$\bar{\tau} = \lambda(\nabla\cdot\mathbf{u})\bar{\bar{I}} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla) \tag{4}$$

where $\bar{\bar{I}}$ is the identity tensor.

Multiple Scattering

The analyses of scattering of elastic and acoustic waves by a group of parallel cylinders are quite similar. However, for the sake of clarity and completeness, this scheme is detailed next with appropriate changes made to accommodate the partial mode conversion of the elastic waves at the discontinuities.

A plane compressional wave with potential ϕ_i is incident on $N + 1$ cylinders with radii a_s, $s = 0, 1, 2, 3, \ldots\ldots N$. Each of these cylinders is held stationary in space by a finite force in the x-y plane [2]. The sth cylinder is excited first by the incident wave ϕ_i, and the compressional wave ${}^s\phi^1$ as well as shear wave ${}^s\psi^1$ is scattered. The potentials

$$\Phi^1 = {}^s\phi^1 + \phi_i,$$

$$\Psi^1 = {}^s\psi^1$$

are known as "the first order of scattering." Both Φ^1 and Ψ^1 satisfy the governing wave equations and the boundary conditions. Further, the sth cylinder is excited by the waves $\sum_{s'}{}' \, {}^{s'}\phi^1$ and $\sum_{s'}{}' \, {}^{s'}\psi^1$, the first order of scattering from the remaining cylinders. $\sum_{s'}{}'$ denotes that $s = s'$ is not included in the summation. In response to these waves it reflects ${}^s\phi^2$ and ${}^s\psi^2$ and thus,

$$\Phi^2 = \sum_{s'}{}' \, {}^{s'}\phi^1 + {}^s\phi^2,$$

$$\Psi^2 = \sum_{s'}{}' \, {}^{s'}\psi^1 + {}^s\psi^2$$

are the waves of second order of scattering which in turn also satisfy boundary conditions. To proceed in this manner, the total waves due to sth cylinder are

$${}^s\phi = \sum_{m=1}^{\infty} {}^s\phi^m, \tag{5}$$

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[1] Numbers in brackets designate References at end of paper.

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$$ {}^s\psi = \sum_{m=1}^{\infty} {}^s\psi^m. \quad (6) $$

Next, the sum of scattered fields of the entire geometry of $N + 1$ cylinders are

$$ \phi = \sum_{s=0}^{N} \sum_{m=1}^{\infty} {}^s\phi^m, \quad (7) $$

$$ \psi = \sum_{s=0}^{N} \sum_{m=1}^{\infty} {}^s\psi^m. \quad (8) $$

The total fields, incident and scattered, are

$$ \Phi = \phi_i + \phi = \sum_{m=1}^{\infty} \Phi^m, \quad (9) $$

$$ \Psi = \psi = \sum_{m=1}^{\infty} \Psi^m. \quad (10) $$

Φ and Ψ satisfy the wave equations and the required boundary conditions since each of the Φ^m and Ψ^m does so. The displacement and stress components from (3) and (4) may now be written as

$$ u_r = \sum_{m=1}^{\infty} u_r^m = \sum_{m=1}^{\infty} \frac{\partial \Phi^m}{\partial r} + \frac{1}{r}\frac{\partial \Phi^m}{\partial \theta}, \quad (11) $$

$$ u_\theta = \sum_{m=1}^{\infty} u_\theta^m = \sum_{m=1}^{\infty} \frac{1}{r}\frac{\partial \Phi^m}{\partial \theta} - \frac{\partial \Phi^m}{\partial r}, \quad (12) $$

$$ \tau_{rr} = \sum_{m=1}^{\infty} \tau_{rr}^m = \sum_{m=1}^{\infty} \lambda \nabla^2 \Phi^m + 2\mu \left[\frac{\partial^2 \Phi^m}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \Psi^m}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \Psi^m}{\partial \theta} \right], \quad (13) $$

$$ \tau_{\theta\theta} = \sum_{m=1}^{\infty} \tau_{\theta\theta}^m = \sum_{m=1}^{\infty} \lambda \nabla^2 \Phi^m + 2\mu \left[\frac{1}{r^2}\frac{\partial^2 \Phi^m}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi^m}{\partial r} - \frac{1}{r}\frac{\partial^2 \Psi^m}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 \Psi^m}{\partial \theta^2} \right], \quad (14) $$

$$ \tau_{r\theta} = \sum_{m=1}^{\infty} \tau^{r\theta m} = \sum_{m=1}^{\infty} 2\mu \left[\frac{1}{r}\frac{\partial^2 \Phi^m}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \Phi^m}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \Psi^m}{\partial \theta^2} - \frac{1}{2}\nabla^2 \Psi^m \right]. \quad (15) $$

τ_{rr}, $\tau_{\theta\theta}$ are the two-dimensional normal stresses and $\tau_{r\theta}$, the shear stress. τ_{rr} and $\tau_{\theta\theta}$ are related by $\tau_{\theta\theta} = \nu \tau_{rr}$ where ν is the Poisson ratio.

Incident and Scattered Waves

Let cylinder $s = 0$ be chosen as the reference cylinder and its polar coordinates to any point P are r, θ, Fig. 1. Similarly the coordinates from any cylinder s and s' are r_s, $\theta_{s'}$, and $r_{s'}$, θ_s, respectively. Moreover,

$$ \mathbf{r} = r(\cos\theta \mathbf{i} + \sin\theta \mathbf{j}), $$

$$ \bar{\mathbf{r}} = r_s(\cos\theta_s \mathbf{i} + \sin\theta_s \mathbf{j}), $$

$$ \mathbf{r}_{0s} = \mathbf{r}_{0s'} - \mathbf{r}_{ss'} = \mathbf{r} - \mathbf{r}_s, \text{ etc.} $$

If the incident compressional plane wave, ϕ_i, with propagation vector, $\boldsymbol{\alpha}$, amplitude, ϕ_0, and circular frequency, ω, is expressed in the coordinates of the 0th cylinder

$$ \phi_i = \phi_0 e^{i\boldsymbol{\alpha}\cdot\mathbf{r} - i\omega t} $$

and where

$$ \boldsymbol{\alpha} = \alpha(\cos\gamma \mathbf{i} - \sin\gamma \mathbf{j}), \qquad \alpha = \frac{\omega}{c_\alpha} $$

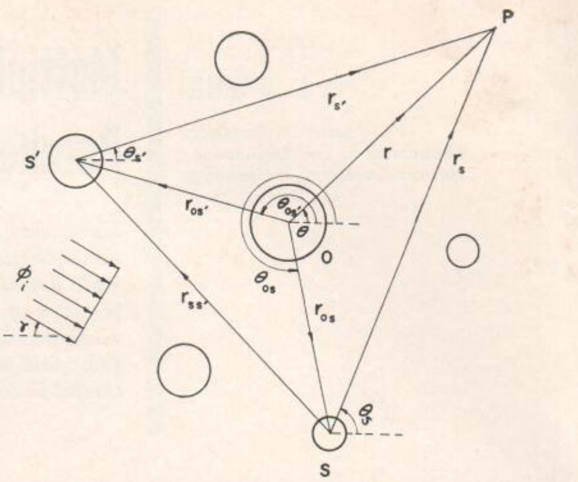

Fig. 1 Plane compressional elastic wave incident on a group of parallel cylinders

Furthermore, ϕ_i in the coordinates of the sth cylinder will be

$$ \phi_i = \phi_0 e^{i\boldsymbol{\alpha}\cdot(\mathbf{r}_{0s} + \mathbf{r}_s)} $$

$$ = \phi_0 e^{i\alpha r_{0s}\cos(\theta_{0s} + \gamma)} \sum_{n=-\infty}^{\infty} J_n(\alpha r_s) e^{in(\theta_s + \gamma + \pi/2)} $$

The factor $e^{i\omega t}$ will be omitted in all field quantities hereon.

The scattered waves of the mth order may be written in terms of the well-known cylindrical wave functions with respect to the sth cylinder

$$ {}^s\phi^m = \sum_{n=-\infty}^{\infty} {}^sA_n^m H_n(\alpha r_s) e^{in(\theta_s + \gamma + \pi/2)} $$

$$ {}^s\psi^m = \sum_{n=-\infty}^{\infty} {}^sB_n^m H_n(\beta r_s) e^{in(\theta_s + \gamma + \epsilon)} $$

$$ \epsilon = \pi/2n $$

for even n if $n > 0$ with

$$ \epsilon = 0 $$

for odd n and even n if $n < 0$

$$ {}^sB_0^m = 0 $$

for $n = 0$ where ${}^sA_n^m$ and ${}^sB_n^m$ are scattering coefficients of the mth order, H_n, the Hankel function of the first kind and $\beta = \omega/C_\beta$.[2] From (7) and (8) the total scattered wave may be written as

$$ \phi = \sum_{s=0}^{N} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} {}^sA_n^m H_n(\alpha r_s) e^{in(\theta_s + \gamma + \pi/2)}, \quad (16) $$

$$ \psi = \sum_{s=0}^{N} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} {}^sB_n^m H_n(\beta r_s) e^{in(\theta_s + \gamma + \epsilon)}. \quad (17) $$

${}^sA_n^m$ and ${}^sB_n^m$ are to be determined from boundary conditions which in this case are

$$ u_r^1\big|_{r_s = a_s} = u_r^2\big|_{r_s = a_s} = \ldots\ldots \sum_{m=1}^{\infty} u_r^m\big|_{r_s = a_s} = 0, \quad (18) $$

$$ u_\theta^1\big|_{r_s = a_s} = u_\theta^2\big|_{r_s = a_s} = \ldots\ldots \sum_{m=1}^{\infty} u_\theta^m\big|_{r_s = a_s} = 0 \quad (19) $$

[2] ϵ is so chosen that ${}^s\phi^m$ and ${}^s\psi^m$ are $\pi/2$ radians apart for all n.

and $s = 0, 1, 2 \ldots\ldots N$. Therefore the coefficients of the mth order scattering will make use of

$$u_r{}^m\big|_{r_s=a_s} = 0; \qquad u_\theta{}^m\big|_{r_s=a_s} = 0. \tag{20}$$

The mth order coefficients ${}^sA_n{}^m$ and ${}^sB_n{}^m$ may be shown in terms of those of the $(m-1)$th order

$$\begin{aligned} {}^sA_n{}^m &= {}^sA_n \sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'}{}^{m-1} H_{n-n'}(\alpha r_{ss'}) e^{-i(n-n')\theta_{ss'}} \\ &\quad - {}^sB_n e^{in\epsilon} \sum_{s'}{}' \sum_{n'} [{}^{s'}B_{n'}{}^{m-1} H_{n-n'}(\beta r_{ss'}) e^{-i(n-n')\theta_{ss'}}] e^{in'\epsilon}, \end{aligned} \tag{21}$$

$$\begin{aligned} {}^sB_n{}^m &= {}^sB_n \sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'}{}^{m-1} H_{n-n'}(\alpha r_{ss'}) e^{-i(n-n')\theta_{ss'}} \\ &\quad + {}^sC_n \sum_{s'}{}' \sum_{n'} [{}^{s'}B_{n'}{}^{m-1} H_{n-n'}(\beta r_{ss'}) e^{-i(n-n')\theta_{ss'}}] e^{in'\epsilon} \end{aligned} \tag{22}$$

where

$${}^sA_n = [\alpha a_s J_n'(\alpha a_s)\beta a_s H_n'(\beta a_s) - n^2 J_n(\alpha a_s) H_n(\beta a_s)] e^{in\epsilon}/G,$$

$${}^sB_n = -2in/\pi G$$

$${}^sC_n = [\beta a_s J_n'(\beta a_s)\alpha a_s H_n'(\alpha a_s) - n^2 J_n(\beta a_s) H_n(\alpha a_s)]/G$$

and

$$G = [-\alpha a_s H_n'(\alpha a_s)\beta a_s H_n'(\beta a_s) + n^2 H_n(\alpha a_s) H_n(\beta a_s)] e^{in\epsilon}.$$

Physically, the second-order scattering may be interpreted as contributions from the plane wave first scattered by all the cylinders and then scattered for the second time by the sth cylinder. The mth order scattering, of course, implies the waves have been diffracted m times. It should also be noted that the mode conversion of elastic waves is obvious here; i.e., the mth order compressional wave (${}^sA_n{}^m$) is derived from two $m-1$th order waves, one compressional (${}^sA_n{}^{m-1}$) and one shear (${}^sB_n{}^{m-1}$) as does the mth order shear wave (${}^sB_n{}^m$). After the scattering coefficients are obtained to the desired order, they may be substituted in the field components in (11)–(15). Thus a formal solution to the problem of scattering by an arbitrary configuration of parallel cylinders satisfying the boundary conditions simultaneously at the surface of each cylinder is obtained. From the physical viewpoint these infinite series will converge since the influence of the scattered waves will diminish with the higher order of scattering. However, the rate of convergence of these coefficients is dependent upon the geometry of the configuration; i.e., $\mathbf{r}_{ss'}$ and αa_s. It is of interest to note that if all $\alpha r_{ss'} \to \infty$ and all $H_{n-n'}(\alpha r_{ss'}) \to 0$ then ${}^sA_n{}^m = {}^sB_n{}^m \to 0$ for all $m \geq 2$. The solution is reduced to that of a single scatterer [2].

Numerical Results

The problem of two identical circular cylinders of radius a at a separation $2Z$ is shown in Fig. 2. In this configuration $r_s = \sqrt{x^2 + (y - y_s)^2}$, $s = 1, 2$ and y_s is the distance from the origin to sth cylinder; i.e., $y_1 = Z$ and $y_2 = -Z$. The separations $r_{ss'}$ of the scatterers are $r_{12} = r_{21} = 2Z$ and $\theta_{12} = -\pi/2$, $\theta_{21} = \pi/2$. For the incident wave $r_{01} = Z$, $r_{02} = -Z$, $\theta_{01} = \pi/2$, and $\theta_{02} = -\pi/2$. Also $\boldsymbol{\alpha}\cdot\mathbf{r}_{01} = \alpha r_{01}\cos(\theta_{01} + \gamma) = -\alpha Z \sin\gamma$ and $\boldsymbol{\alpha}\cdot\mathbf{r}_{02} = \alpha Z \sin\gamma$. The computation is performed on a CDC 6600 computer. The influences of wave number αa, ratio Z/a, and angle γ on the stress concentration are explored. Most of the calculations are done for $m = 3$ whose solutions are checked against those of $m = 2$. At $m = 3$ the convergence of the infinite series is slow for $Z/a < 1.0$. The accuracy is such that the end term of the series before truncation is less than five percent of the sum of the series. The Poisson ratio ν is chosen to be 0.25 throughout this study.

For computational convenience, the stress field is nondimen-

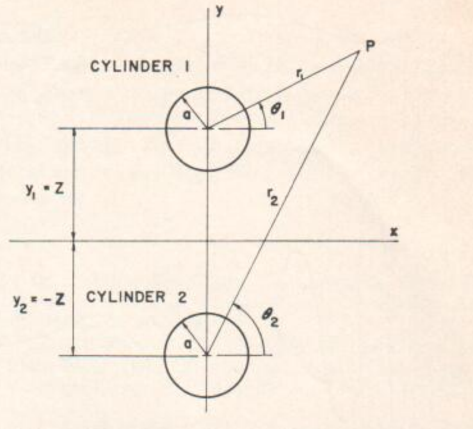

Fig. 2 Coordinate system for two identical cylinders

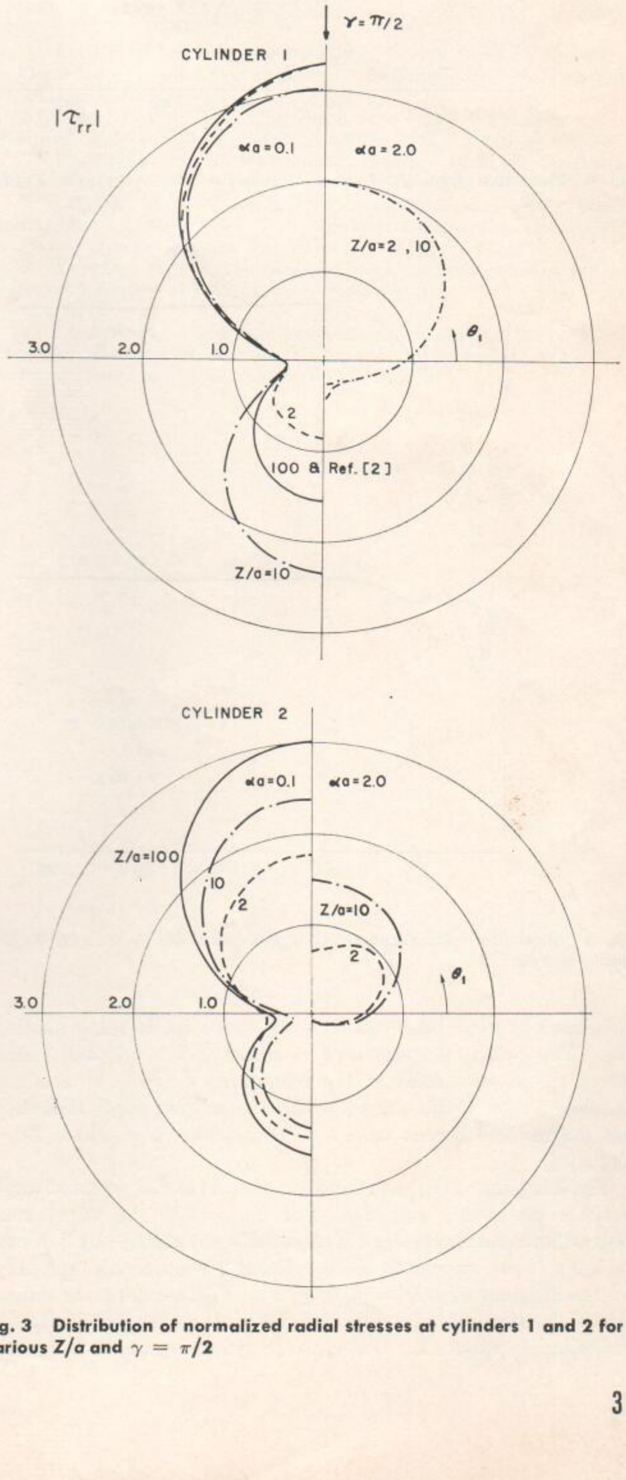

Fig. 3 Distribution of normalized radial stresses at cylinders 1 and 2 for various Z/a and $\gamma = \pi/2$

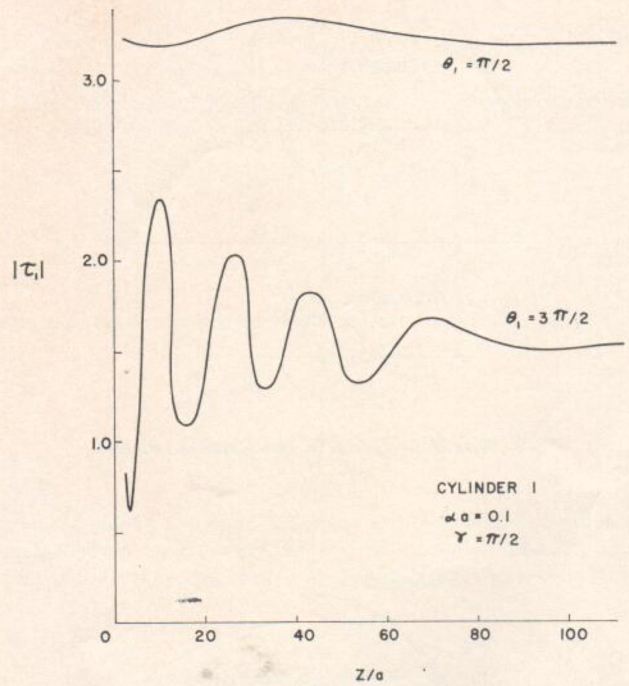

Fig. 4 Maximum normalized stress at cylinder 1 for $\theta_1 = \pi/2$, $3\pi/2$, and $\gamma = \pi/2$

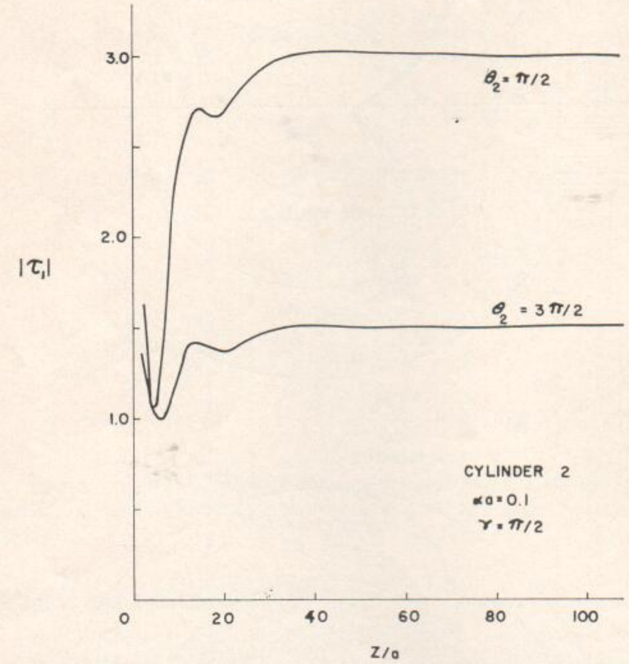

Fig. 5 Maximum normalized stress at cylinder 2 for $\theta_2 = \pi/2$, $3\pi/2$, and $\gamma = \pi/2$

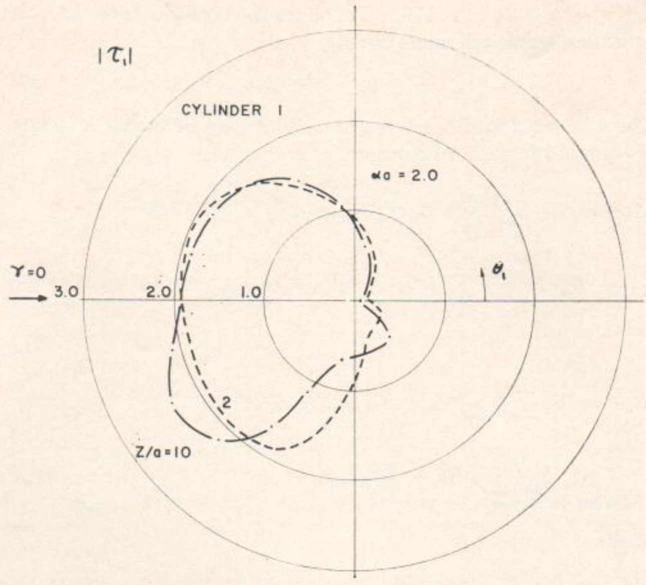

Fig. 6 Maximum normalized stress at cylinder 1 for $\alpha a = 0.5$ and $\gamma = 0$

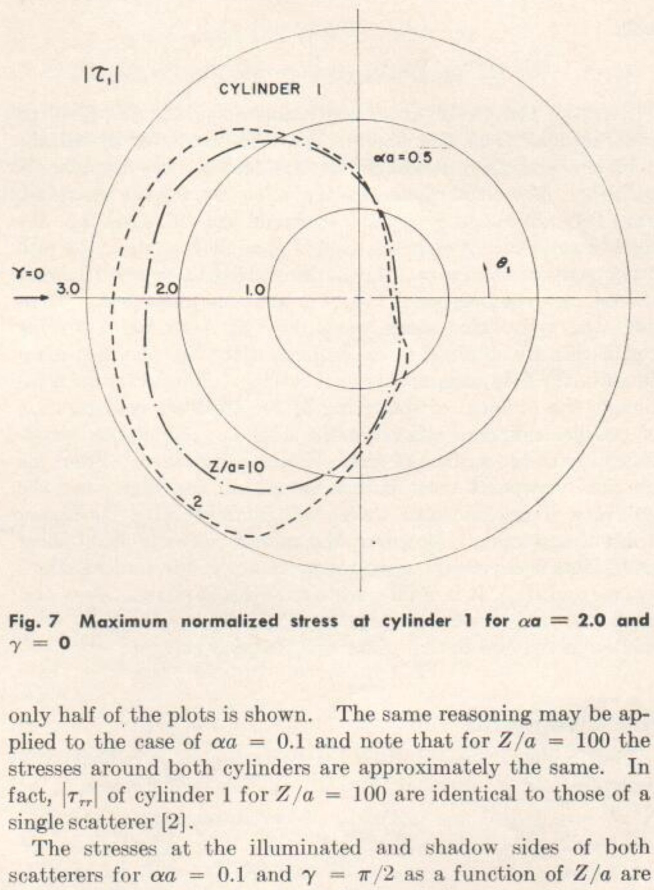

Fig. 7 Maximum normalized stress at cylinder 1 for $\alpha a = 2.0$ and $\gamma = 0$

sionalized by $\tau_0 = \mu\phi_0\beta^2$ which is the stress due to static loading [2]. The principal stresses τ_1, τ_2 where $|\tau_1| > |\tau_2|$ and radial stress τ_{rr} are evaluated at the boundaries $r_1 = r_2 = a$. The absolute values of the stresses which are complex imply that they are maximum between time $t = 0$ and $t = T/4$ where $T = 2\pi/\omega$ [2].

Fig. 3 is plots of $|\tau_{rr}|$ as a function of Z/a for $r = \pi/2$ and wave number $\alpha a = 0.1$ and 2.0. For $\alpha a = 2.0$, the stress concentration around cylinder 1 is essentially not affected by Z/a and its influence on the values about cylinder 2 is noticeable especially on the illuminated side ($0 \leq \theta_2 \leq \pi$). Furthermore, the reduction of the stress concentration is larger for smaller Z/a due to the shielding of cylinder 1. Owing to the symmetry about the y-axis only half of the plots is shown. The same reasoning may be applied to the case of $\alpha a = 0.1$ and note that for $Z/a = 100$ the stresses around both cylinders are approximately the same. In fact, $|\tau_{rr}|$ of cylinder 1 for $Z/a = 100$ are identical to those of a single scatterer [2].

The stresses at the illuminated and shadow sides of both scatterers for $\alpha a = 0.1$ and $\gamma = \pi/2$ as a function of Z/a are shown in Figs. 4 and 5. Let it be noted that the value at the illuminated side of cylinder 1 is a constant of 3.2 unaffected by the ratio Z/a and, also, the solutions of cylinder 1 coincide with those of cylinder 2 for large separation, i.e., $Z/a > 50$. For $Z/a < 50$ the stress for both scatterers fluctuates somewhat. At $\theta_1 = 3\pi/2$ a maximum is reached at $Z/a = 15.0$ and minimum at $Z/a = 3.0$. However, for parameters within the range of investigation, the stress concentrations of cylinder 1 on the shadow side do not exceed that of the illuminated side; i.e., 3.2 in this case.

The variations of stress around either cylinders 1 and 2 for $\gamma = 0$ and $\alpha a = 0.5$, 2.0 are given in Figs. 6 and 7. Due to symmetry about x-axis only cylinder 1 is shown. The effects of one inclusion on another may be seen. For the parameters chosen, the stresses are slightly higher for smaller values of Z/a especially on the "sunny" side of the scatterers; i.e., at $\pi < \theta_1 < 3\pi/2$ and $\pi/2 < \theta_2 < \pi$.

Conclusion

Within the parametric range of this investigation and by comparison with single scatterer's solution it is indicated that for $\gamma = \pi/2$ the stress concentration on cylinder 1 due to the presence of cylinder 2 increases somewhat on the shadow side. However, the magnitude does not exceed that of the illuminated side; i.e., maximum stress for a single scatterer. For $\gamma = 0$, the stresses on the illuminated sides increase beyond those of a single scatterer. This increase depends on the values of the parameters chosen.

In this paper, the case of immovable rigid cylinders is chosen for the simplicity of its boundary conditions. However, there is no conceptual difficulty in extending this problem to cases of movable rigid inclusions or cylindrical cavities. For the problem of elastic cylinders composed of material different than that of the infinite medium the solution also may be obtained with the introduction of two refracted wave potentials ${}^s\phi_R{}^m$, ${}^s\psi_R{}^m$ where

$$ {}^s\phi_R{}^m = \sum_{n=-\infty}^{\infty} {}^sD_n{}^m H_n(\alpha_1 r_s) e^{in(\theta_s + \gamma + \pi/2)} $$

and

$$ {}^s\psi_R{}^m = \sum_{n=-\infty}^{\infty} {}^sE_n{}^m H_n(\beta_1 r_s) e^{in(\theta_s + \gamma + \epsilon)} $$

where α_1, β_1 denote these parameters in the transmitted medium. ${}^sA_n{}^m$, ${}^sB_n{}^m$, ${}^sC_n{}^m$, and ${}^sD_n{}^m$ may be obtained from conditions of continuity of displacement and stress fields at all the interfaces simultaneously.

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Results

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