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Mohr's wisdom

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=[Q]\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{bmatrix}I_{\hat{x}\hat{x}} & I_{\hat{x}\hat{y}}\\ I_{\hat{y}\hat{x}} & I_{\hat{y}\hat{y}}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}I_{xx} & I_{xy}\\ I_{yx} & I_{yy}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=[Q]\begin{bmatrix}I_{xx} & I_{xy}\\ I_{yx} & I_{yy}\end{bmatrix}[Q]^T$$

$$=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}I_{xx}\cos\theta+I_{xy}\sin\theta & -I_{xx}\sin\theta+I_{xy}\cos\theta\\ I_{yx}\cos\theta+I_{yy}\sin\theta & -I_{yx}\sin\theta+I_{yy}\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}I_{xx}\cos^2\theta+2I_{xy}\sin\theta\cos\theta+I_{yy}\sin^2\theta & I_{xy}(\cos^2\theta-\sin^2\theta)+(I_{yy}-I_{xx})\sin\theta\cos\theta\\ I_{xy}(\cos^2\theta-\sin^2\theta)+(I_{yy}-I_{xx})\sin\theta\cos\theta & I_{xx}\sin^2\theta-2I_{xy}\sin\theta\cos\theta+I_{yy}\cos^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}\frac{(I_{xx}+I_{yy})}{2}+\frac{(I_{xx}-I_{yy})}{2}\cos 2\theta+I_{xy}\sin 2\theta & \frac{(-I_{xx}+I_{yy})}{2}\sin 2\theta+I_{xy}\cos 2\theta\\ \frac{(-I_{xx}+I_{yy})}{2}\sin 2\theta+I_{xy}\cos 2\theta & \frac{(I_{xx}+I_{yy})}{2}-\frac{(I_{xx}-I_{yy})}{2}\cos 2\theta-I_{xy}\sin 2\theta\end{bmatrix}$$

$$I_{\hat{x}\hat{x}}=\frac{(I_{xx}+I_{yy})}{2}+\frac{(I_{xx}-I_{yy})}{2}\cos 2\theta+I_{xy}\sin 2\theta$$

$$I_{\hat{x}\hat{y}}=\frac{(-I_{xx}+I_{yy})}{2}\sin 2\theta+I_{xy}\cos 2\theta$$

$$I_{\hat{y}\hat{y}}=\frac{(I_{xx}+I_{yy})}{2}-\frac{(I_{xx}-I_{yy})}{2}\cos 2\theta-I_{xy}\sin 2\theta$$

兩個不變量

1. $I_{\hat{x}\hat{x}}+I_{\hat{y}\hat{y}}=I_{xx}+I_{yy}$

2. $\begin{vmatrix}I_{xx} & I_{xy}\\ I_{yx} & I_{yy}\end{vmatrix}=\begin{vmatrix}I_{\hat{x}\hat{x}} & I_{\hat{x}\hat{y}}\\ I_{\hat{y}\hat{x}} & I_{\hat{y}\hat{y}}\end{vmatrix}$

$$R^2 = [I_{xx} - \frac{1}{2}(I_{xx} + I_{yy})]^2 + I_{xy}{}^2$$

$$= [\frac{1}{2}(I_{xx} - I_{yy})]^2 + I_{xy}{}^2$$

$$= [\frac{1}{2}(I_{\hat{x}\hat{x}} - I_{\hat{y}\hat{y}})]^2 + I_{\hat{x}\hat{y}}{}^2$$

$$I_{\hat{x}\hat{x}} - I_{\hat{y}\hat{y}}$$

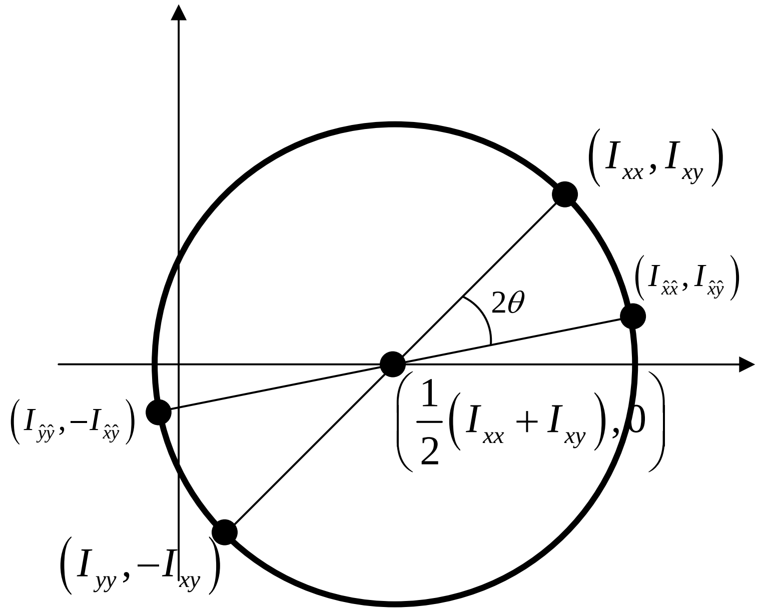
$$= I_{xx}(\cos^2\boldsymbol{\theta} - \sin^2\boldsymbol{\theta}) + I_{yy}(\sin^2\boldsymbol{\theta} - \cos^2\boldsymbol{\theta}) + 4I_{xy}\sin 2\boldsymbol{\theta}$$

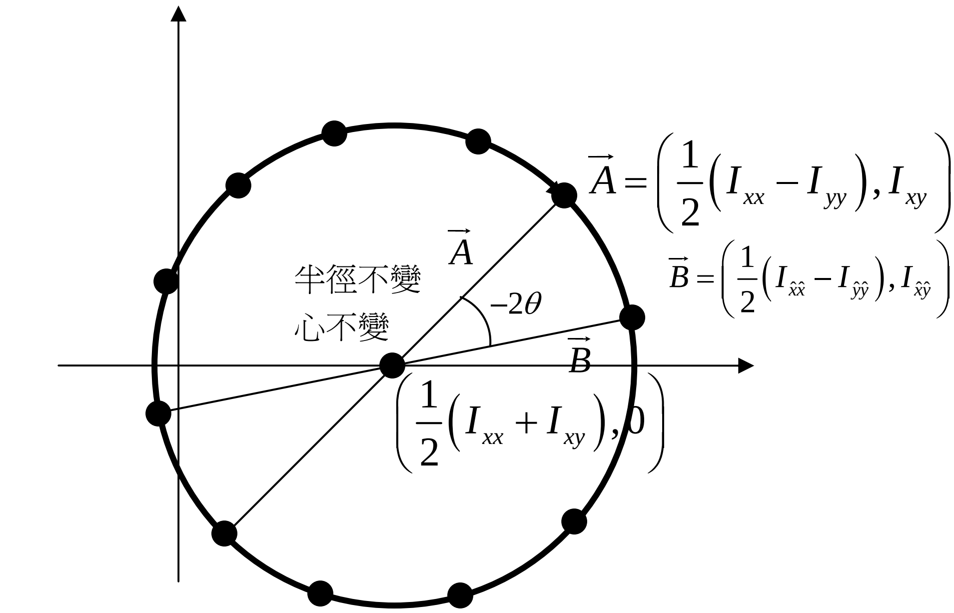
$$= I_{xx}\cos 2\boldsymbol{\theta} - I_{yy}\cos 2\boldsymbol{\theta} + 2I_{xy}\sin 2\boldsymbol{\theta}$$

$$\therefore \frac{1}{2}(I_{\hat{x}\hat{x}} - I_{\hat{y}\hat{y}}) = \frac{1}{2}(I_{xx} - I_{yy})\cos 2\boldsymbol{\theta} + I_{xy}\sin 2\boldsymbol{\theta}$$

$$\therefore I_{\hat{x}\hat{y}} = \frac{1}{2}\sin 2\boldsymbol{\theta}(I_{yy} - I_{xx}) + I_{xy}\cos 2\boldsymbol{\theta}$$

$$\begin{pmatrix} \frac{1}{2}\left(I_{\hat{x}\hat{x}} - I_{\hat{y}\hat{y}}\right) \\ I_{\hat{x}\hat{y}} \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2}\left(I_{xx} - I_{yy}\right) \\ I_{xy} \end{pmatrix}$$





$$\begin{pmatrix} \frac{1}{2}\left(I_{\hat{x}\hat{x}}-I_{\hat{y}\hat{y}}\right) \\ I_{\hat{x}\hat{y}} \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2}\left(I_{xx}-I_{yy}\right) \\ I_{xy} \end{pmatrix}$$

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