# 矩　　　陣

## 線性代數:

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

## 利用高斯消去法可求解

例. $$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 18 & 9 \\ 0 & 18 & 40 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \\ 82 \\ 171 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 18 & 9 \\ 0 & 0 & 4 & 16 \\ 0 & 0 & 16 & 80 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \\ 28 \\ 144 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 9 & 18 & 9 \\ 0 & 0 & 4 & 16 \\ 0 & 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \\ 28 \\ 32 \end{bmatrix}$$

$p = -4$ ，　$q = 3$ ，　$r = -1$ ，　$s = 2$

## 矩陣操作

降階:　$[\ ]_{4\times4} \times [\ ]_{4\times1} = [\ ]_{4\times1}$

左右通乘 $A_{3\times4}$ ，且 $A_{4\times3} \times [\ ]_{3\times1} = [\ ]_{4\times1}$

即　$A_{3\times4} \times [\ ]_{4\times4} \times A_{4\times3} \times [\ ]_{3\times1} = A_{3\times4} \times [\ ]_{4\times1}$

$\Rightarrow$　$[\ ]_{3\times3} \times [\ ]_{3\times1} = [\ ]_{3\times1}$

## $A_{4\times3}$ 如何獲得

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 4 & 25 & 26 & 9 \\ 2 & 26 & 44 & 34 \\ 0 & 9 & 34 & 89 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 51 \\ 94 \\ 171 \end{bmatrix}$$

$\because p + 4q + 2r + 0s = 6$

$\therefore p = (-4)q + (-2)r + 0s + 6$

$$\Rightarrow \quad 令\, a_{4\times3}=\begin{bmatrix}-4 & -2 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}p\\q\\r\\s\end{bmatrix}=\begin{bmatrix}-4 & -2 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}q\\r\\s\end{bmatrix}+\begin{bmatrix}6\\0\\0\\0\end{bmatrix}$$

經過降階原式:

$$\Rightarrow \begin{bmatrix}-4 & 1 & 0 & 0\\ -2 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 4 & 2 & 0\\ 4 & 25 & 26 & 9\\ 2 & 26 & 44 & 34\\ 0 & 9 & 34 & 89\end{bmatrix}\begin{bmatrix}-4 & -2 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}p\\q\\r\\s\end{bmatrix}=\begin{bmatrix}-4 & 1 & 0 & 0\\ -2 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}6\\51\\94\\171\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}9 & 18 & 9\\ 18 & 40 & 34\\ 9 & 34 & 89\end{bmatrix}\begin{bmatrix}q\\r\\s\end{bmatrix}=\begin{bmatrix}27\\82\\171\end{bmatrix}$$

HOMEWOREK:

使上三角形為零，並寫出 $b_{4\times3}$ ， $b_{3\times2}$ ， $b_{2\times1}$？

$$\Rightarrow \begin{bmatrix}1 & 4 & 2 & 0\\ 4 & 25 & 26 & 9\\ 2 & 26 & 44 & 34\\ 0 & 9 & 34 & 89\end{bmatrix}\begin{bmatrix}p\\q\\r\\s\end{bmatrix}=\begin{bmatrix}6\\51\\94\\171\end{bmatrix} \qquad 令\, b_{4\times3}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & \frac{-9}{89} & \frac{-34}{89}\end{bmatrix}$$

$$b_{3\times4}\times[\ ]_{4\times4}\times b_{4\times3}\times[\ ]_{3\times1}=b_{3\times4}\times[\ ]_{4\times1}$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & \frac{-9}{89}\\ 0 & 0 & 1 & \frac{-34}{89}\end{bmatrix}\begin{bmatrix}1 & 4 & 2 & 0\\ 4 & 25 & 26 & 9\\ 2 & 26 & 44 & 34\\ 0 & 9 & 34 & 89\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & \frac{-9}{89} & \frac{-34}{89}\end{bmatrix}\begin{bmatrix}p\\q\\r\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & \frac{-9}{89}\\ 0 & 0 & 1 & \frac{-34}{89}\end{bmatrix}\begin{bmatrix}6\\51\\94\\171\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 4 & 2\\ 4 & \frac{2114}{89} & \frac{2008}{89}\\ 2 & \frac{2008}{89} & \frac{2760}{89}\end{bmatrix}\begin{bmatrix}p\\q\\r\end{bmatrix}=\begin{bmatrix}6\\ \frac{3000}{89}\\ \frac{2552}{89}\end{bmatrix} \qquad 令\, b_{3\times2}=\begin{bmatrix}1 & 0\\ 0 & 1\\ \frac{-178}{2760} & \frac{-2008}{2760}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{601}{690} & \frac{878}{345} \\ \frac{878}{345} & \frac{235672}{30705} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \frac{127448}{30705} \\ \frac{394448}{30705} \end{bmatrix}$$

令 $b_{2\times 1} = \begin{bmatrix} -878 \\ 2648 \\ \\ 1 \end{bmatrix}$

$$\Rightarrow \left[ \frac{276345}{10163355} \right] [p] = \left[ \frac{-1105380}{10163355} \right]$$

$\Rightarrow$ $p=-4$ , $q=3$ , $r=-1$ , $s=2$

## 矩陣在土力，材力，結構上之應用
## 勁度矩陣 × 位移 = 力

數學式: 物理意義:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ …單位為 $1$ 之力作用在 $u_2$ 點上。

$$\Rightarrow \begin{bmatrix} \frac{14}{5} & \frac{-16}{5} & 1 \\ \frac{-16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{15}{7} & \frac{-20}{7} \\ \frac{-20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ \frac{-5}{14} \end{bmatrix}$$ …預使 $u_3$ ，$u_4$ 產生同樣之位移，

可在 $u_3$ ，$u_4$ 各施 $8/7$ 及 $-5/14$ 之力即可。

$$\Rightarrow \left[ \frac{5}{6} \right] [u_4] = \left[ \frac{7}{6} \right]$$ …預使 $u_4$ 產生同樣之位移，可在 $u_4$ 上施 $7/6$ 的力，但其它各點的位移，不一定和原先的情況相同，僅有 $u_4$ 之位移可確定。

## Direct Shear Box (直剪盒)

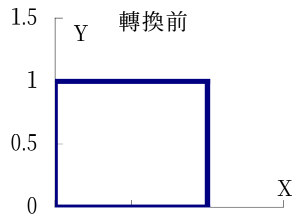
物體受力而產生形變

$\Rightarrow$ $\vec{r}' = [\ ] \times \vec{r}$ ， 式中之[ ]為轉換矩陣

HOMEWOREK:

令轉換矩陣 $F=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ，求下列圖形之變形。

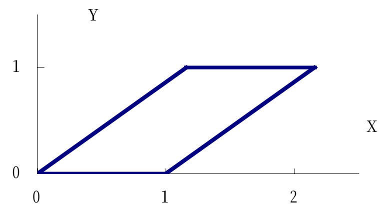
解: 原圖形




$\because \quad X'=FX$

$$\Rightarrow X_1'=FX_1=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad , \quad X_2'=FX_2=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$X_3'=FX_3=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt{3}}\\1\\0\end{bmatrix} \quad , \quad X_4'=FX_4=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1+\frac{2}{\sqrt{3}}\\1\\0\end{bmatrix}$$




轉換後之圖形

$$A=\begin{bmatrix}\frac{1}{\sqrt{3}}\\1\\0\end{bmatrix} \quad , \quad A'=\begin{bmatrix}1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}}\\1\\0\end{bmatrix}=\begin{bmatrix}\sqrt{3}\\1\\0\end{bmatrix}$$

※角度變成 $30^\circ$ ，長度變成 $\sqrt{3}$ 倍

**任一矩陣必可分解為下列情形 ， $F=RU=VR$**

**其中 $R$ 為旋轉矩陣，$UV$ 為拉伸矩陣**

※ $R$ 旋轉矩陣(即僅有方向改變 $\alpha$ 度，長度不變):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\cos^2\alpha - (-\sin^2\alpha) = 1$ 長度不變……$R$ 純粹為使原向量旋轉

若 $y = Rx$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \quad x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

$$x^T x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = {x_1}^2 + {x_2}^2 + {x_3}^2 = x \cdot x = |x|^2$$ ……長度的平方

同理轉換後

$y^T y = |y|^2$ 又 $y = Rx$

$\Rightarrow \quad y^T = x^T R^T \quad \Rightarrow \quad |y|^2 = y^T y = (x^T R^T)(Rx) = x^T R^T Rx$

$|x|^2 = x^T x$

長度不變 $|y|^2 = |x|^2 \quad \Rightarrow \quad x^T R^T Rx = x^T x$

**$R^T R = I$ ……即旋轉矩陣之條件**

※拉伸矩陣(即僅有長度改變 $\lambda$ 倍，方向不變):

$x' = Ax = \lambda x \quad \Rightarrow \quad (A - \lambda I)x = 0$

$\Rightarrow \quad \det|A - \lambda I| = 0$

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det\left|A-\lambda I\right| = \begin{bmatrix} \frac{\sqrt{3}}{2}-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad \lambda_1 = \sqrt{3} \qquad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cos\,60° \\ sin\,60° \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{\sqrt{3}} \qquad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} cos\,150° \\ sin\,150° \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1 \qquad \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

向量分解:　任一向量 $x$ 必可分解作 $\alpha\begin{bmatrix} cos\,60° \\ sin\,60° \\ 0 \end{bmatrix} + \beta\begin{bmatrix} cos\,150° \\ sin\,150° \\ 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

又 $Ax = \lambda x$

$$x' = Ax = \sqrt{3}\alpha\begin{bmatrix} cos\,60° \\ sin\,60° \\ 0 \end{bmatrix} + \frac{1}{\sqrt{3}}\beta\begin{bmatrix} cos\,150° \\ sin\,150° \\ 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

※ $F = RU = VR$ ...I　且　$R^T R = I$

$\Rightarrow$ $F^T = U^T R^T$ ...II

II ×I　$\Rightarrow$　得　$F^T F = U^T R^T R U = U^T U$

又令 $U$ 為一對稱矩陣　$\Rightarrow$　$U^T = U$

$\Rightarrow$ $U^T U = UU = U^2 = F^T F$

$U = \sqrt{F^T F}$

$R^T R = I$

同乘 $R^{-1} \Rightarrow R^T R R^{-1} = I R^{-1}$

又　$RR^{-1} = I$　$\Rightarrow$　$R^T = R^{-1}$

$\Rightarrow$　$\because RR^{-1} = I, R^{-1} = R^T$

$\Rightarrow \quad \therefore RR^T = I$

$F = RU = VR$ ...I

$F^T = R^T V^T$ ...II

II ×I $\Rightarrow$ 得 $FF^T = VRR^T V^T = VV^T$

又令 $V$ 為一對稱矩陣 $V^T = V$

$\Rightarrow \quad VV^T = VV = V^2 = FF^T$

$\Rightarrow \quad V = \sqrt{FF^T}$

| |
|---|
| ※$U = \sqrt{F^T F}$ ， $V = \sqrt{FF^T}$ |

若給定一 $F$ 則 $F^T$ 也為已知

$U^2 = F^T F = C \quad \Rightarrow \quad U = \sqrt{F^T F}$

求出 $U$ 後 $F = RU \quad \Rightarrow \quad FU^{-1} = R$ 即可得 $R$

$V^2 = FF^T = \quad \Rightarrow \quad \mathrm{V} = \sqrt{FF^T}$

| |
|---|
| ※$A^{-1} = \dfrac{adj[A]}{\lvert A\rvert}$ ， $adj[A] = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T$ |

例. $F = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad F^T = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\sqrt{3}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad U^2 = F^T F = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{7}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \qquad \because A = CDC^{-1}$$

$$\Rightarrow \quad \lambda = \frac{1}{3},1,3 \quad , \quad D = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad , \quad C = \begin{bmatrix} -\sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{3}} & 1 & 0 \end{bmatrix}$$

# 慣量

$$\Rightarrow \quad C^{-1} = \begin{bmatrix} \frac{-\sqrt{3}}{4} & 0 & \frac{\sqrt{3}}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad U = C\sqrt{D}C^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad U^{-1} = \begin{bmatrix} \frac{5}{2\sqrt{3}} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R = FU^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad V = FR^{-1} = \begin{bmatrix} \frac{5}{2\sqrt{3}} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Tensor 張量

| Order | 0 | 1 | 2 |
|---|---|---|---|
| | 純量 | 向量 | 矩陣 |
| | scalar | vector | tensor |

**慣量** $\begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}$

$$I_{xx} = \iint y^2 dA$$

$$I_{yy} = \iint x^2 dA$$

$$I_{xy} = -\iint xy\,dA$$

若座標軸旋轉

其不變量: 1.$I_{xx} + I_{yy} = I_{\overline{xx}} + I_{\overline{yy}}$

2.行列式值不變…即取 $det$ 相同

證明: $I_{xx} + I_{yy} = \iint y^2 dA + \iint x^2 dA$

$$= \iint (x^2 + y^2) dA = \iint r^2 dA$$

$$I_{\overline{xx}} + I_{\overline{yy}} = \iint y^2 dA + \iint x^2 dA$$

$$= \iint (x^2 + y^2) dA = \iint r^2 dA$$

$$I_{xx} + I_{yy} = I_{\overline{xx}} + I_{\overline{yy}}$$ 故得證

## ill conditioned problems (病態問題)
## 有效數字的取決與否

$$\begin{bmatrix} 1 & 1 \\ 1 & -1.014 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{aligned} x &= 1.007 \\ y &= 0.993 \end{aligned}$$

若四捨五入

$$\begin{bmatrix} 1 & 1 \\ 1 & -1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{aligned} x &= 1.005 \\ y &= 0.995 \end{aligned}$$ 則兩解相差不多

此為 Well-behaved(健康型)

但若題目為

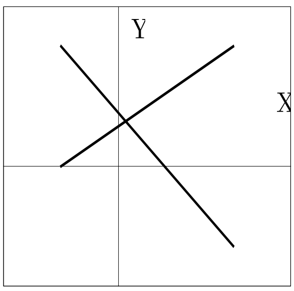
$$\begin{bmatrix} 1 & 1 \\ 1 & 1.014 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{aligned} x &= 144.9 \\ y &= -142.9 \end{aligned}$$

若四捨五入

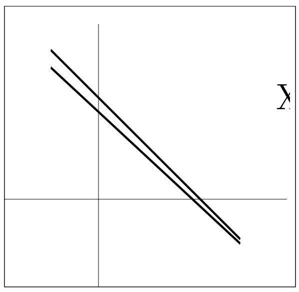
$$\begin{bmatrix} 1 & 1 \\ 1 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \begin{aligned} x &= 200.2 \\ y &= 0 - 200 \end{aligned}$$ 則兩解相差太多

此為 ill-conditioned(病態型)

(健康型)




(病態型)



判定原則

$Ax_1 = \lambda_1 x_1$　　$\lambda_1, \lambda_2$　為　eigenvalue

$Ax_2 = \lambda_2 x_2$　　$x_1, x_2$　為　eigenvector

$Ax = y$　　$A(x + \Delta x) = (y + \Delta y)$　　若 $\frac{\Delta x}{\Delta y} \rangle\rangle\rangle 1$　則為病態

依向量分解

$Ax = y = \alpha x_1 + \beta x_2$　同除 $A$

$$x = \frac{\alpha}{\lambda_1} x_1 + \frac{\beta}{\lambda_2} x_2 = (\frac{\lambda_2}{\lambda_1} \alpha x_1 + \beta x_2) \frac{1}{\lambda_2}$$

$$A(x + \Delta x) = (y + \Delta y)$$

$$x + \Delta x = \frac{\alpha + \Delta\alpha}{\lambda_1} x_1 + \frac{\beta + \Delta\beta}{\lambda_2} x_2 = (\frac{\lambda_2}{\lambda_1}(\alpha + \Delta\alpha) x_1 + (\beta + \Delta\beta) x_2) \frac{1}{\lambda_2}$$

其中 $\Delta\alpha, \Delta\beta \to 0$　，　故影響不太，其主因在於 $\frac{\lambda_2}{\lambda_1}$

即為 $\frac{|\lambda_m|_{max}}{|\lambda_n|_{min}}$ 其值越接近 $1$，則越健康，值大則影響越大

$\frac{|\lambda_m|_{max}}{|\lambda_n|_{min}}$ 可代表 eigenvalue 的分散情況

## 矩陣的應用之範圍

1. 力平衡　$F_1 + F_2 = P$　，$F_1 L = Pa$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ L & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} P \\ Pa \end{bmatrix}$$

2. Flow Equilibrium
3. Least Square
4. Grapher (View Point)
5. 連體力學變形機制 $F = RU = VR$

## ※矩　陣　運　算

1. 加法　$[A]_{n\times m} = [B]_{n\times m} + [C]_{n\times m}$
2. 乘法　$[A]_{n\times p} = [B]_{n\times m} \cdot [C]_{m\times p}$

3. 分解 $F = RU = VR$

4. 轉置 Transpose $A^T$

$$A_{n\times m} \to A_{m\times n} \quad , \quad A_{ij} \to A_{ji}$$

5. 特徵值 eigenvalues

$$Ax = \lambda x \quad \Rightarrow (A-\lambda)x = 0$$

所以給一個$A$我們可以找到$C$和$D$(對稱)

使 $A = CDC^{-1}$ 成立

例.

$$\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ Ax_2 &= \lambda_2 x_2 \\ Ax_3 &= \lambda_3 x_3 \end{aligned} \quad \Rightarrow D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad , \quad C = [x_1 \quad x_2 \quad x_3]_{3\times 3}$$

$\Rightarrow \quad AC = CD$

$\Rightarrow \quad A = CDC^{-1}$ 即 $f(A) = C \cdot f(D) \cdot C^{-1}$

例. $A^n = (C \cdot D \cdot C^{-1} \cdot C \cdot D \cdot C^{-1} \cdot C \cdot D \cdot C^{-1} \ldots\ldots \cdot C \cdot D \cdot C^{-1})$

又 $C^{-1}C = I \qquad A^n = C \cdot D^n \cdot C^{-1}$

同理 $\sqrt{A} = C\sqrt{D}C^{-1}$ 等等

Degenerate Problem(重根問題；退化問題)

Jordan Form(解重根) $AC = CJ$

$$J = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & c \end{bmatrix}$$

例. $A = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$

$\because (A-\lambda I)x = 0$

$$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 & 3 \\ -1 & 0-\lambda & -3 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 32 = 0 \qquad \Rightarrow \lambda = -2,4,4$$

$$\Rightarrow \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$AC = CD$ 存在，但 $A = CDC^{-1}$ 不存在($\because \det C = 0$ ，$C^{-1}$ 不存在)

所以令 $J = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ 即有重根時多加一個1來彌補。

$$A[x_1 \quad x_2 \quad x_3] = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

即 $Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$Ax_3 = \lambda_2 x_3 + x_2$

HOMEWOREK:

求 $x_3$ 及 $C^{-1}$

$\because (A-\lambda I)x = 0$

$$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 & 3 \\ -1 & 0-\lambda & -3 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 32 = 0 \qquad \Rightarrow \lambda = -2,4,4$$

$$\Rightarrow \quad \lambda = -2 \qquad x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda = 4 \qquad x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$Ax_3 = \lambda_2 x_3 + x_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} t \\ 1-t \\ t-1 \end{bmatrix} \quad \text{令 } t=1 \quad \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad C^{-1} = \frac{1}{|C|} adj \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}^T$$

$$\Rightarrow \quad = \frac{1}{-2}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad CJC^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} = A$$

故得證

若 $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow J^2 = \begin{bmatrix} 3^2 & 0 & 0 \\ 0 & 2^2 & 4 \\ 0 & 0 & 2^2 \end{bmatrix}$

$$\Rightarrow \quad J^3 = \begin{bmatrix} 3^3 & 0 & 0 \\ 0 & 2^3 & 12 \\ 0 & 0 & 2^3 \end{bmatrix}$$

....

$$\Rightarrow \quad J^{n-1} = \begin{bmatrix} 3^{n-1} & 0 & 0 \\ 0 & 2^{n-1} & P_{n-1} \\ 0 & 0 & 2^{n-1} \end{bmatrix}$$

$$\Rightarrow \quad J^n = \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 2^n & P_n \\ 0 & 0 & 2^n \end{bmatrix}$$

$\Rightarrow \quad P_n = 2^{n-1} + 2P_{n-1}$

註:

解一. $P_n = 2^{n-1} + 2(2^{n-2} + 2P_{n-2})$

$= 2^{n-1} + 2^{n-1} + 4P_{n-2}$

$= 2^{n-1} + 2^{n-1} + 4(2^{n-3} + 2P_{n-3})$

.......

$= 2^{n-1} + 2^1 \times 2^{n-1} + 2^2 \times 2^{n-1} + 2^3 \times 2^{n-1} + \ldots\ldots + 2^{n-1}P_1$

$= (n-1)(2^{n-1}) + 2^{n-1}P_1$

又 $P_1 = 1 \quad \Rightarrow \quad P_n = n \times 2^{n-1}$

解二. $P_n = 2^{n-1} + 2P_{n-1}$

$\Rightarrow \quad P_n - 2P_{n-1} = 2^{n-1}$

$P_n - 2P_{n-1} = 0$ ......補解

$P_n - 2P_{n-1} = 2^{n-1}$ ...特解

令 $P_n = f(n)$ , $f(1) = 1$

$P_n - 2P_{n-1} = 0$ 再令 $P = \rho^n$ 代入

$\Rightarrow \rho^n - 2\rho^{n-1} = 0 \quad \Rightarrow \rho - 2 = 0 \quad \Rightarrow \rho = 2$ ......補解

令特解 $= q_n \times 2^{n-1}$ $q_n 2^{n-1} - 2q_{n-1}2^{n-2} = 2^{n-1}$

$q_n - q_{n-1} = 1 \quad \Rightarrow q_n = n + k$ (等差級數)

※ $P_n = k \times 2^n + q_n 2^{n-1}$ 由 $P_1 = 1$ 求出 $k$ 值

解三. 級數解

$P_n = C_n 2^n \quad \Rightarrow P_n - 2P_{n-1} = 2^{n-1}$

$\Rightarrow C_n 2^n - 2C_{n-1}2^{n-1} = 2^{n-1}$

$\Rightarrow C_n - C_{n-1} = \dfrac{1}{2}$ ......成等差

$$\Rightarrow \quad C_n = C_0 + (n-1) \times \frac{1}{2}$$

$$\Rightarrow \quad P_n = C_n 2^n = C_0 2^n + \frac{(n-1)}{2} \times 2^n$$

$$P_1 = 1 \quad \Rightarrow \quad C_0 \times 2^1 + 0 = 1$$

$$\Rightarrow \quad C_0 = \frac{1}{2}$$

$$\Rightarrow \quad C_n = \frac{1}{2} + (n-1) \times \frac{1}{2} = \frac{n}{2}$$

$$\Rightarrow \quad P_n = \frac{n}{2} 2^n = n \cdot 2^{n-1}$$ ……特解

故我們可以得知

$$※ J^n = \begin{bmatrix} a & o & o & o & o & o \\ o & b & n \cdot b^{n-1} & o & o & o \\ o & o & b & o & o & o \\ o & o & o & c & n \cdot c^{n-1} & o \\ o & o & o & o & c & n \cdot c^{n-1} \\ o & o & o & o & o & c \end{bmatrix}$$

## Eigen Problem

運用於彈簧 $m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$

$$m_2 y_2'' = -k_2 (y_2 - y_1)$$

$$\Rightarrow \quad y_1'' = \frac{-(k_1 + k_2)}{m_1} y_1 + \frac{k_2}{m_1} y_2$$

$$y_2'' = \frac{k_1}{m_2} y_1 + \frac{-k_2}{m_2} y_2$$

$$\Rightarrow \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} \frac{-(k_1 + k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

若 $k_1 = 3$ ， $k_2 = 2$ ， $m_1 = m_2 = 1$

$$\Rightarrow \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

令 $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{i\omega t} \quad \Rightarrow \quad \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = -\omega^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{i\omega t}$

代入原式 $\Rightarrow$ $$-\omega^2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}e^{i\omega t}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}e^{i\omega t}$$

$$\Rightarrow \quad -\omega^2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}-5+\omega^2 & 2\\ 2 & -2+\omega^2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=0$$

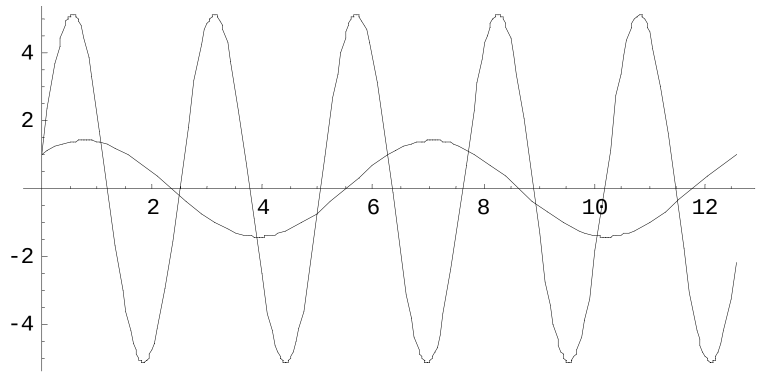
$$\Rightarrow \quad {\omega_1}^2=1 \quad , \quad \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

$$\Rightarrow \quad {\omega_2}^2=6 \quad , \quad \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}e^{it}+\begin{bmatrix}2\\ -1\end{bmatrix}e^{i\sqrt{6}t}$$

低頻　高頻

(共振):




HOMEWORK:

三條彈簧

$$m_1y_1''=-k_1y_1+k_2(y_2-y_1)$$

$$m_2y_2''=-k_2(y_2-y_1)+k_3(y_3-y_2)$$

$$m_3y_3''=-k_3(y_3-y_2)$$

$$\Rightarrow \quad y_1''=\frac{-(k_1+k_2)}{m_1}y_1+\frac{k_2}{m_1}y_2+0y_3$$

$$y_2''=\frac{k_1}{m_2}y_1+\frac{-k_2}{m_2}y_2+\frac{k_3}{m_2}y_3$$

$$y_3''=0y_1+\frac{k_2}{m_3}y_2-\frac{k_3}{m_3}y_3$$

$$\Rightarrow \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & \frac{-k_3}{m_3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

## 餘　式　定　理

HOMEWORK:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ 求 $e^A$ 及 $e^{At}$

$$det|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = 2,2,3$$

餘式定理: $f(x) = (2-x)^2(3-x)Q(x) + px^2 + qx + r$

運用在矩陣: $f(A) = (2-A)^2(3-A)Q(A) + pA^2 + qA + rI$

微分: $f'(A) = 2(2-A)(-1)(-1)Q'(A) + 2pA + q$

又 $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \Rightarrow \quad A^2 = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$e^x = (2-x)^2(3-x)Q(x) + px^2 + qx + r$$

微分 $e^x = 2(2-x)Q'(x) + 2px + q$

$x = 2,2,3$ 代入 $\Rightarrow$ $\begin{aligned} p &= -2e^2 + e^3 \\ q &= 9e^2 - 4e^3 \\ r &= 4e^3 - 9e^2 \end{aligned}$

$$e^A = (2-A)^2(3-A)Q(A) + (-2e^3 + e^3)A^2 + (9e^2 - 4e^3)A + (4e^3 - 9e^2)I$$

$$= \begin{bmatrix} e^2 & 0 & e^3 - e^2 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}$$

$$e^{At}=\begin{bmatrix} e^{2t} & 0 & e^{3t}-e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

## 對　稱、反對稱矩陣

$[A]_{n\times m}$　　對稱　$a_{ij}=a_{ji}$　即　$A^T=A$

例.　$A_{3\times 3}=\begin{bmatrix} d & a & b \\ a & e & c \\ b & c & f \end{bmatrix}$

反對稱　$a_{ij}=-a_{ji}$　即　$A^T=-A$　且　$i=j$　時　$a_{ij}=0$

例.　$A_{3\times 3}=\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

## ※任意矩陣必可分解為對稱及反對稱矩陣

$C=S+A$，$S$ 為對稱矩陣，$A$ 為反對稱矩陣

$$S=\frac{1}{2}(C+C^T)\quad,\quad A=\frac{1}{2}(C-C^T)$$

證明:

$$S^T=\frac{1}{2}(C+C^T)^T=\frac{1}{2}(C^T+C^{T^T})=\frac{1}{2}(C^T+C)=S$$

$$A^T=\frac{1}{2}(C-C^T)^T=\frac{1}{2}(C^T-C^{T^T})=\frac{1}{2}(C^T-C)=-A$$

故得證

HOMEWOREK:

$v=\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$，$\omega=\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$，$\omega\times v=Av$　求 $A$ 矩陣

$$\Rightarrow\quad \omega\times v=\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ v_1 & v_2 & v_3 \end{vmatrix}=(\omega_2 v_3-\omega_3 v_2)i+(\omega_3 v_1-\omega_1 v_3)j+(\omega_1 v_2-\omega_2 v_1)k$$

$\Rightarrow$ 令 $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$　$Av=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} av_1+bv_2+cv_3 \\ dv_1+ev_2+fv_3 \\ gv_1+hv_2+iv_3 \end{bmatrix}$

兩式相等

$\Rightarrow \quad a = e = i = 0$ ， $b = -\omega_3$ ， $c = \omega_2$ ， $d = \omega_3$

$f = -\omega_1$ ， $g = -\omega_2$ ， $h = \omega_1$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

HOMEWOREK:

$x' = \omega x$ ， $x_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ， $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ 且 $\omega_1^2 + \omega_2^3 + \omega_3^2 = 1$

試找出一 $x[t]$ ，使 $x'(t) = (\omega_1, \omega_2, \omega_3) \times (x_1, x_2, x_3)$

又 $x(t) = e^{At} x_0$ 證明 $e^{At}(e^{At})^T = I$ 。

$$x' = \omega \times x = Ax$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\varpi_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{bmatrix}$$

$\Rightarrow \quad det|A - \lambda I| = 0 \qquad \Rightarrow \quad \lambda = 0, \pm i$

利用餘式定理

$$e^{0t} = 0p + 0q + r$$
$$e^{it} = (-1)p + iq + r$$
$$e^{-it} = (-1)p + (-i)q + r$$

我們可以得到

$$p = 1 - (\frac{e^t + e^{-t}}{2})$$
$$q = \frac{e^t - e^{-t}}{2i}$$
$$r = 1$$

泰勒展開式:

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \ldots$$

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \ldots$$

$$e^{it} = 1 + \frac{1}{1!}it + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \ldots$$
$$= 1 + \frac{1}{1!}it - \frac{1}{2!}t^2 - \frac{1}{3!}it^3 + \frac{1}{4!}t^4 + \frac{1}{5!}it^5 - \frac{1}{6!}t^6 + \ldots$$

$$= \cos t + i \sin t$$

$$e^{-it} = 1 + \frac{1}{1!}(-it) + \frac{1}{2!}(-it)^2 + \frac{1}{3!}(-it)^3 + \ldots$$

$$= 1 - \frac{1}{1!}it - \frac{1}{2!}t^2 + \frac{1}{3!}it^3 + \frac{1}{4!}t^4 - \frac{1}{5!}it^5 - \frac{1}{6!}t^6 + \ldots$$

$$= \cos t - i \sin t$$

$$p = 1 - cos\, t$$

$\Rightarrow$ $$q = sin\, t$$

$$r = 1$$

$\Rightarrow$ $$e^{At} = (1 - cos\, t)A^2 + (sin\, t)A + I$$

解一.

$$e^{At} = \begin{bmatrix} (1-cos\,t)(\omega_1^2 - 1) + 1 & (1-cos\,t)\omega_1\omega_2 + \omega_3\, sin\,t & (1-cos\,t)\omega_1\omega_3 - \omega_2\, sin\,t \\ (1-cos\,t)\omega_1\omega_2 - \omega_3\, sin\,t & (1-cos\,t)(\omega_1^2 - 1) + 1 & (1-cos\,t)\omega_2\omega_3 + \omega_1\, sin\,t \\ (1-cos\,t)\omega_1\omega_3 + \omega_2\, sin\,t & (1-cos\,t)\omega_2\omega_3 - \omega_1\, sin\,t & (1-cos\,t)(\omega_3^2 - 1) + 1 \end{bmatrix}$$

$$(e^{At})^T = \begin{bmatrix} (1-cos\,t)(\omega_1^2 - 1) + 1 & (1-cos\,t)\omega_1\omega_2 - \omega_3\, sin\,t & (1-cos\,t)\omega_1\omega_3 + \omega_2\, sin\,t \\ (1-cos\,t)\omega_1\omega_2 + \omega_3\, sin\,t & (1-cos\,t)(\omega_1^2 - 1) + 1 & (1-cos\,t)\omega_2\omega_3 - \omega_1\, sin\,t \\ (1-cos\,t)\omega_1\omega_3 - \omega_2\, sin\,t & (1-cos\,t)\omega_2\omega_3 + \omega_1\, sin\,t & (1-cos\,t)(\omega_3^2 - 1) + 1 \end{bmatrix}$$

$$e^{At}(e^{At}) =$$

乘開即可得其答案 $e^{At}(e^{At})^T = I$

但過程十分煩雜，計算要小心

解二. $e^{At} = (1 - cos\, t)A^2 + (sin\, t)A + I$

$$(e^{At})^T = (1 - cos\, t)(A^2)^T + (sin\, t)(A)^T + I^T$$

$\because A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$...為一反對稱矩陣 $A^T = -A$

$\because A^2 = \begin{bmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\varpi_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{bmatrix}$...為一對稱矩陣 $(A^2)^T = A^2$

故

$$e^{At}(e^{At})^T = (1-cos\,t)^2 A^2(A^2) + (sin^2\,t)AA^T + I^2 + (1-cos\,t)(sin\,t)A^2A^T +$$

$$(1-cos\,t)A^2 + (1-cos\,t)(sin\,t)A(A^2)^T + A\,sin\,t + (1-cos\,t)(A^2)^T + A^T\, sin\,t$$

$$= (1-cos\,t)^2 A^4 - (sin^2\,t)A^2 + I^2 - (1-cos\,t)(sin\,t)A^3 - A^3(1-cos\,t)\,sin\,t + A\,sin\,t$$

$+(1-\cos t)A^2 - A\sin t$

又 $A^3 + A = 0$ $A^3 = -A$ $\Rightarrow$ $A^4 = -A^2$

$\Rightarrow$ $e^{At}(e^{At})^T = I$

## 正 交 矩 陣

正交即代表 $R^T R = I$

$R_{2\times 2} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ $\alpha$為任意值皆成立。

$R_{3\times 3}$： $R = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$RR^T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a^2 + b^2 + c^2 = 1$

$d^2 + e^2 + f^2 = 1$

$g^2 + h^2 + i^2 = 1$

$ad + be + cf = 0$

$dg + eh + fi = 0$

$ga + hb + ic = 0$

九個未知數，六條方程式 $\Rightarrow$ 無限多解

## Householder 矩 陣

$$H = I - \frac{2VV^T}{V^T \cdot V} \qquad V^T \cdot V \text{ 為內積}$$

鏡射(Mirror)原理:

$Hy = p$ 又 $H(Hy) = y$

$H^2 = I$ 即 $HH^T = I$

故任意取一向量$V$，即可找到一相應的$H$，且$H$對稱並正交

令 $V=\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{bmatrix} \Rightarrow V^T=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix} \Rightarrow V^T\cdot V=1$

$\Rightarrow VV^T=\begin{bmatrix}\frac{1}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\end{bmatrix}$

$\Rightarrow H=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-2\begin{bmatrix}\frac{1}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9}\end{bmatrix}=\begin{bmatrix}\frac{7}{9} & \frac{-4}{9} & \frac{-4}{9}\\ \frac{-4}{9} & \frac{1}{9} & \frac{-8}{9}\\ \frac{-4}{9} & \frac{-8}{9} & \frac{1}{9}\end{bmatrix}$

$HH^T=I$ ， 且 $H=H^T$ 正交且對稱

證明: $H=I-\frac{2VV^T}{V^T\cdot V}$

$H^T=I^T-(\frac{2VV^T}{V^T\cdot V})^T=I-\frac{2(VV^T)^T}{V^T\cdot V}$ $V^T\cdot V$ 內積為純量不影響

$=I-\frac{2}{V^T\cdot V}(V^T)^T V^T$

$=I-\frac{2VV^T}{V^T\cdot V}=H$ 對稱

$HH^T=(I-\frac{2VV^T}{V^T\cdot V})(I-\frac{2VV^T}{V^T\cdot V})$

$=I^2-\frac{4VV^T}{V^T\cdot V}+4\frac{VV^TVV^T}{V^T\cdot V\times V^T\cdot V}$

$\because V^TV=V\cdot V=|V|^2$ 為長度的平方

原式 $=I^2-\frac{4VV^T}{V^T\cdot V}+\frac{4VV^T}{V^T\cdot V}$

$=I^2=I$ 故得證

例: 令 $V=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

$$H=I-\frac{2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\begin{bmatrix}\alpha & \beta\end{bmatrix}}{\alpha^2+\beta^2}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-2\begin{bmatrix}\frac{\alpha^2}{\alpha^2+\beta^2} & \frac{\alpha\beta}{\alpha^2+\beta^2}\\ \frac{\alpha\beta}{\alpha^2+\beta^2} & \frac{\beta^2}{\alpha^2+\beta^2}\end{bmatrix}$$

若 $\alpha^2+\beta^2=1$ 即 $V$ 為一單位向量

$$H=\begin{bmatrix}\beta^2-\alpha^2 & -2\alpha\beta\\ -2\alpha\beta & \alpha^2-\beta^2\end{bmatrix}$$

Homework:

若給一 $V=\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$ 且 $\omega_1^2+\omega_2^3+\omega_3^2=1$

求 $H$？ $H=I-\dfrac{2VV^T}{V^T\cdot V}$

$$=I-2\frac{VV^T}{\omega^2{}_1+\omega^2{}_2+\omega^2{}_3}$$

$$=I-2\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-2\begin{bmatrix}\omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3\\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3\\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2\end{bmatrix}$$

$$=\begin{bmatrix}1-2\omega_1^2 & -2\omega_1\omega_2 & -2\omega_1\omega_3\\ -2\omega_1\omega_2 & 1-2\omega_2^2 & -2\omega_2\omega_3\\ -2\omega_1\omega_3 & -2\omega_2\omega_3 & 1-2\omega_3^2\end{bmatrix}$$

若 $A=\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$

$$e^{At}=\begin{bmatrix}(1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_1\omega_2+\omega_3\sin t & (1-\cos t)\omega_1\omega_3-\omega_2\sin t\\ (1-\cos t)\omega_1\omega_2-\omega_3\sin t & (1-\cos t)(\omega_1^2-1)+1 & (1-\cos t)\omega_2\omega_3+\omega_1\sin t\\ (1-\cos t)\omega_1\omega_3+\omega_2\sin t & (1-\cos t)\omega_2\omega_3-\omega_1\sin t & (1-\cos t)(\omega_3^2-1)+1\end{bmatrix}$$

給 $t=\pi$ 時

$$e^{A\pi}=\begin{bmatrix}2\omega_1^2-1 & 2\omega_1\omega_2 & 2\omega_1\omega_3\\ 2\omega_1\omega_2 & 2\omega_2^2-1 & 2\omega_2\omega_3\\ 2\omega_1\omega_3 & 2\omega_2\omega_3 & 2\omega_3^2-1\end{bmatrix}$$

令 $e^{A\pi}=M \quad\Rightarrow\quad -M=H$

$MM^T=I \quad\Rightarrow\quad (-M)(-M)^T=(-1)(-1)MM^T=MM^T=I$