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THE POWER OF A MATRIX*

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Abstract. In this note an elementary method of calculating any power of a given matrix is described. The method is quite similar to that presented by I. E. Leonard [*SIAM Rev.*, 38 (1996), pp. 507–512].

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In the note by I. E. Leonard [1] the author has presented an elementary method of calculating the exponential matrix e^{tA} which uses a minimum mathematical prerequisites. Namely, the method uses only the students' knowledge of homogeneous linear differential equations with constant coefficients and the Cayley–Hamilton theorem. The present note is only a supplement to Leonard's note. It deals with the analogous problem for linear difference equations with constant coefficients.

Consider the initial value problem for the system of linear homogeneous first order difference equations with constant coefficients

$$x(k+1) = Ax(k), \quad k = 0, 1, \ldots, \tag{1}$$

$$x(0) = x_0. \tag{2}$$

The unique solution of the problem (1)–(2) has the form

$$k \longrightarrow A^k x_0, \quad k = 0, 1, \ldots\ .$$

Here A is an $n \times n$ matrix. Take the characteristic polynomial of the matrix A

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0. \tag{3}$$

The Cayley–Hamilton theorem says that $p(A) = 0$. As a simple consequence of this one has the following results.

THEOREM 1. *The matrix function* $k \longrightarrow A^k$, $k = 0, 1, \ldots,$ *is the unique solution of the nth order matrix difference initial value problem*

$$\begin{aligned} L(X)(k) \equiv X(n+k) + c_{n-1}X(n+k-1) + \cdots \\ + c_1X(k+1) + c_0X(k) = 0, \quad k = 0, 1, \ldots\ , \end{aligned} \tag{4}$$

$$X(0) = I,\ X(1) = A,\ \ldots\ , X(n-1) = A^{n-1}. \tag{5}$$

Proof. One can see easily that

$$L(A^k)(k) = A^k p(A) = 0, \quad k = 0, 1 \ldots\ ,$$

which means that the theorem is true. ∎

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THEOREM 2. *If x_i, $i = 1, 2, \dots n$, are the solutions of the equation*

$$l(x)(k) \equiv x(n+k) + c_{n-1}x(n+k-1) + \cdots + c_1x(k+1) + c_0x(k) = 0, \quad k = 0, 1, \dots, \tag{6}$$

with the initial conditions

$$x_i(j-1) = \delta_{ij}, \quad i, j = 1, 2, \dots, n, \tag{7}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

then

$$A^k = x_1(k)I + x_2(k)A + \ldots + x_n(k)A^{n-1}, \quad k = 0, 1, \ldots. \tag{8}$$

Proof. Put

$$\Phi(k) = x_1(k)I + x_2(k)A + \ldots + x_n(k)A^{n-1}, \quad k = 0, 1, \ldots.$$

One has

$$\begin{aligned} L(\Phi)(k) &= L(x_1(k)I) + L(x_2(k)A) + \ldots + L(x_n(k)A^{n-1}) \\ &= l(x_1)(k)I + l(x_2)(k)A + \ldots + l(x_n)(k)A^{n-1} \\ &= 0 \cdot I + 0 \cdot A + \ldots + 0 \cdot A^{n-1} = 0, \quad k = 0, 1, \ldots. \end{aligned}$$

Moreover,

$$\begin{gathered} \Phi(0) = x_1(0)I + x_2(0)A + \ldots + x_n(0)A^{n-1} = I, \\ \Phi(1) = x_1(1)I + x_2(1)A + \ldots + x_n(1)A^{n-1} = A, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Phi(n-1) = x_1(n-1)I + x_2(n-1)A + \ldots + x_n(n-1)A^{n-1} = A^{n-1}. \end{gathered}$$

This shows that Φ is a solution of the problem (4)–(5), which, according to Theorem 1, implies that $\Phi(k) = A^k$, $k = 0, 1, \ldots$. □

Example. Consider the system of difference equations (analogous to that of the paper by I. E. Leonard)

$$\begin{aligned} x(k+1) &= 2\,x(k) + 0\,y(k) + 1\,z(k), \\ y(k+1) &= 0\,x(k) + 2\,y(k) + 0\,z(k), \\ z(k+1) &= 0\,x(k) + 0\,y(k) + 3\,z(k). \end{aligned}$$

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

has the form $p(\lambda) = (\lambda - 2)^2(\lambda - 3)$. The general solution of the corresponding scalar third-order difference equations is expressed by the formula

$$x(k) = C_1 2^k + C_2 k 2^k + C_3 3^k, \quad k = 0, 1, \ldots.$$

The solutions x_1, x_2, x_3 satisfying the conditions of Theorem 2 have the form

$$\begin{aligned} x_1(k) &= -3\,2^k - 3\,k2^k + 4\,3^k, \\ x_2(k) &= 4\,2^k + \frac{5}{2}\,k2^k - 4\,3^k, \\ x_3(k) &= -2^k - \frac{1}{2}\,k2^k + 3^k. \end{aligned}$$

According to Theorem 2 we have

$$A^k = x_1(k)I + x_2(k)A + x_3(k)A^2, \quad k = 0, 1, \ldots,$$

and after some algebra we find

$$A^k = \begin{pmatrix} 2^k & 0 & 3^k - 2^k \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix}, \quad k = 0, 1, \ldots .$$

For more information concerning other methods of calculating the matrix power of any order, an interested reader is referred to any standard textbook on difference equations.

REFERENCE

[1] I. E. Leonard, *The matrix exponential*, SIAM Rev., 38 (1996), pp. 507–512.