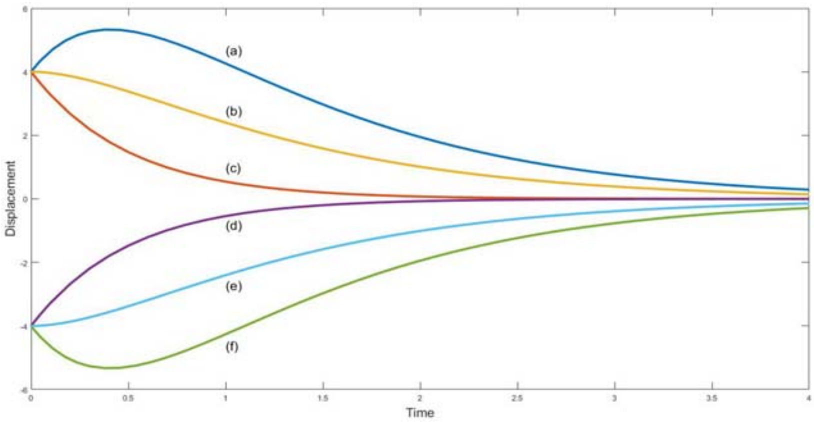
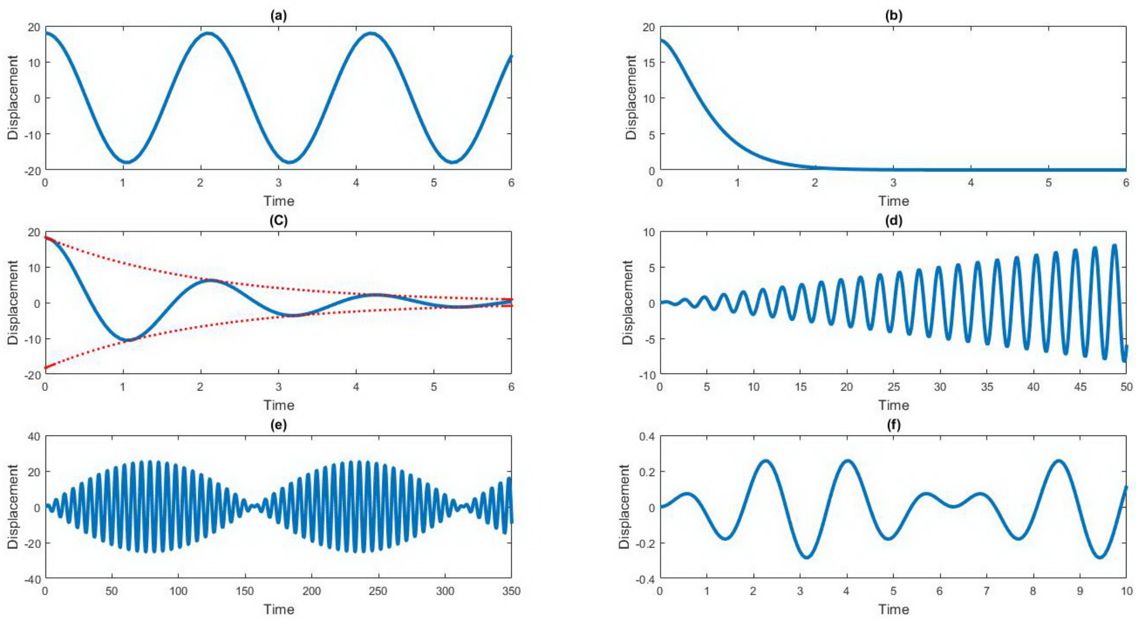
國立臺灣海洋大學河海工程學系 2023 工程數學(一) 第二次大考參考解答

系級：______________　　學號：____________　　姓名：___________

1. 給一單自由度振動系統，其控制方程式為 $m\ddot{y}+c\dot{y}+ky=0$，已知相關參數 $m=1$, $c=3$, $k=2$，試問不同初始條件所對應之位移圖為何？(9%)
   (1) 初始條件： $y(0)=4$, $\dot{y}(0)=0$
   (2) 初始條件： $y(0)=-4$, $\dot{y}(0)=8$
   (3) 初始條件： $y(0)=-4$, $\dot{y}(0)=-8$



2. 考慮下述三條微分方程式，試問其所對應之位移圖為何？(9%)
   (1) $y''(t)+y'(t)+9y(t)=0$, $y(0)=18$, $\dot{y}(0)=0$
   (2) $y''(t)+9y(t)=\cos(3t)$, $y(0)=0$, $\dot{y}(0)=0$
   (3) $y''(t)+9y(t)=0$, $y(0)=18$, $\dot{y}(0)=0$



3. 給一非齊次線性微分方程如下:

$$xy''(x)+ay'(x)+\frac{b}{x}y(x)=x^{-3}$$

已知此微分方程的兩個補解為 x^{-2} 與 $x^{-2}\ln x$

(1) 試求常數 a 、b 為何? (4%)

(2) 以變數變換，令 $t=\ln x$，則 $y(x)=Y(t)$ ，試求轉換後以 $Y(t)$ 表示的微分方程式。 (6%)

(3) 試求轉換後微分方程的補解 $Y_h(t)=?$ (4%)

(4) 試求轉換後微分方程的特解 $Y_p(t)=?$ (4%)

(5) 試將 $Y(t)$ 轉換回 $y(x)$ 。 (4%)

4. 試求下述微分方程之通解

(1) $y^{(6)}+2y^{(4)}+y''=0$ (8%)

(2) $y''-2y'+y=\frac{e^x}{x}$ (8%)

(3) $(1-x^2)y''-4xy'-2y=x$ (8%)

(4) $2(3x+1)^2y''+21(3x+1)y'+18y=0$ (8%)

5. 已知微分方程式 $xy''-(2x+1)y'+(x+1)y=(x^2+x-1)e^{2x}$

(1) 已知 $y_1=e^{ax}$ 為上述 ODE 之一補解，試問 $a=?$ (2%)

(2) 試問:另一補解與特解分別為何? (8%)

6. 已知單自由度振動系統其數學表示為 $m\ddot{y}(t)+c\dot{y}(t)+ky(t)=f(t)$，若給定質量塊 $m=2$，阻尼係數 $c=0$ 與彈簧常數 $k=8$ 並且質量塊為靜止狀態即其初始條件 $y(0)=0$ 與 $\dot{y}(0)=0$，給一外力為 $f(t)=4\cos\omega t$ ，試問:

(1) 此系統的自然振動頻率 $\omega_n=?$ (2%)

(2) 若 y_1, y_2 為其兩補解，試求: $W(y_1,y_2)=?$ (4%)

(3) 當 $\omega=2$ ，其解為何? (4%) 此時系統的運動行為稱為什麼? (2%)

(4) 當 $\omega=1.99$，其解為何? (4%) 此時系統的運動行為稱為什麼? (2%)

參考解答:

1. 給一單自由度振動系統，其控制方程式為 $m\ddot{y}+c\dot{y}+ky=0$ ，已知相關參數 $m=1$，$c=3$，$k=2$ ，試問不同初始條件所對應之位移圖為何? (9%)

(1) 初始條件: $y(0)=4$，$\dot{y}(0)=0$

(2) 初始條件: $y(0)=-4$，$\dot{y}(0)=8$

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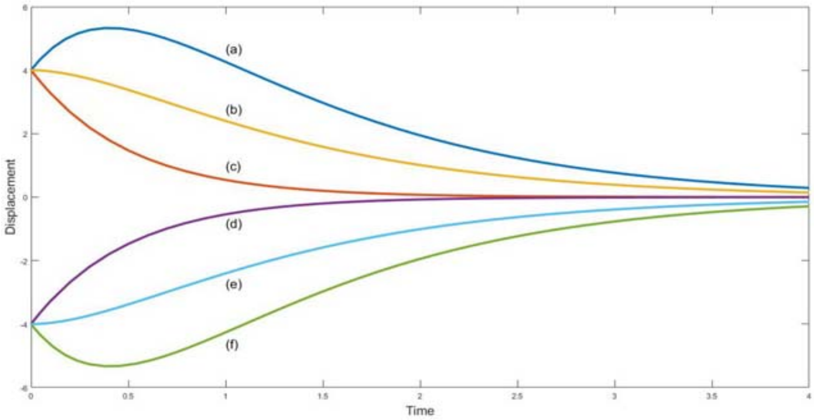


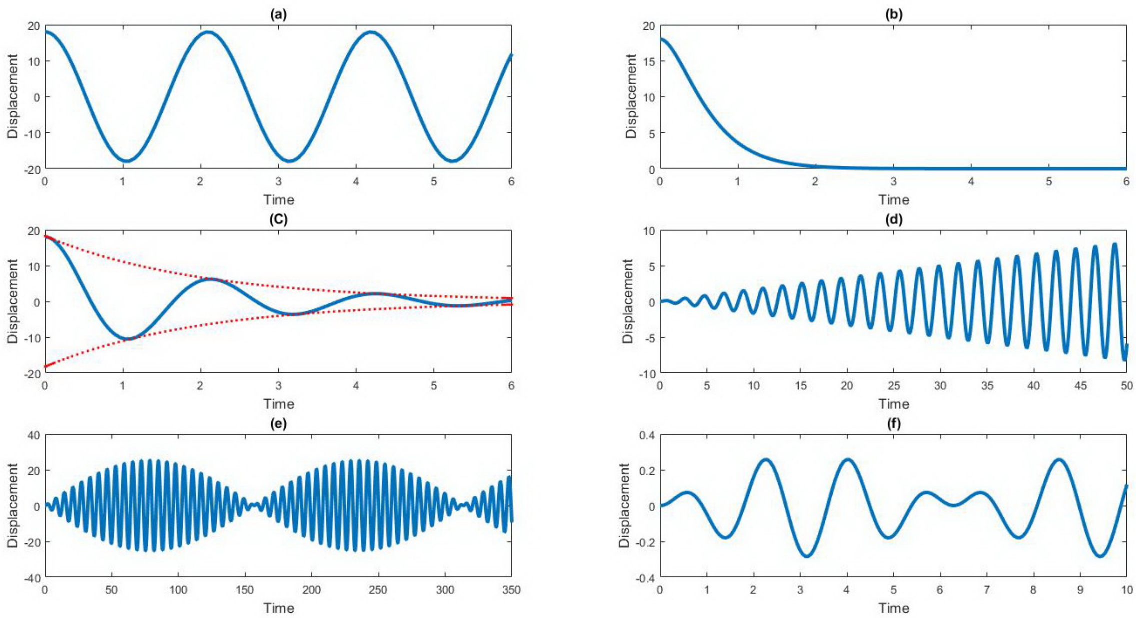
(1) b　(2) d　(3) e

2. 考慮下述三條微分方程式，試問其所對應之位移圖為何? (9%)

(1) $y''(t)+y'(t)+9y(t)=0$，$y(0)=18$，$\dot{y}(0)=0$

(2) $y''(t)+9y(t)=\cos(3t)$，$y(0)=0$，$\dot{y}(0)=0$

(3) $y''(t)+9y(t)=0$，$y(0)=18$，$\dot{y}(0)=0$



(1) c　(2) d　(3) a

3. 給一非齊次線性微分方程如下:

$$xy''(x) + ay'(x) + \frac{b}{x}y(x) = x^{-3}$$

已知此微分方程的兩個補解為 x^{-2} **與** $x^{-2}\ln x$

(1) 試求常數 a **、** b **為何? (4%)**

(2) 以變數變換，令 $t = \ln x$ **，則** $y(x) = Y(t)$ **，試求轉換後以** $Y(t)$ **表示的微分方程式。 (6%)**

(3) 試求轉換後微分方程的補解 $Y_h(t) = ?$ **(4%)**

(4) 試求轉換後微分方程的特解 $Y_p(t) = ?$ **(4%)**

(5) 試將 $Y(t)$ **轉換回** $y(x)$ **。 (4%)**

(1) $xy''(x) + ay'(x) + \frac{b}{x}y(x) = x^{-3} \quad \Rightarrow x^2y''(x) + axy'(x) + by(x) = x^{-2}$

$\because$ a 、b 為常數

$\therefore$ 可知此為 Euler-Cauchy ODE

又 x^{-2} 與 $x^{-2}\ln x$ 為此微分方程的 2 個補解 ($m = -2$ 重根)

故可知此 ODE 之特徵方程為 $(m+2)^2 = 0$

$\therefore$ $m(m-1) + am + b = (m-1)^2$

$\Rightarrow m^2 + (a-1)m + b = m^2 + 4m + 4$

比較係數後可得 $a = 5$ 與 $b = 4$

可知此 ODE 為 $x^2y''(x) + 5xy'(x) + 4y(x) = x^{-2}$

(2) 令 $t = \ln x \quad \Rightarrow x = e^t$

$\therefore y(x) = y(e^t) = Y(t)$

$$y'(x) = \frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dt}{dx} \cdot \frac{dY(t)}{dt} = \frac{1}{x}Y'(t)$$

$$y''(x) = \frac{dy'(x)}{dx} = \frac{d}{dx}(\frac{1}{x}Y'(t)) = -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'(t)}{dx} = -\frac{1}{x^2}Y'(t) + \frac{1}{x^2}Y''(t)$$

將 $y'(x)$ 與 $y''(x)$ 代回 ODE 可得

$$x^2[-\frac{1}{x^2}Y'(t) + \frac{1}{x^2}Y''(t)] + 5x \cdot \frac{1}{x}Y'(t) + 4Y(t) = e^{-2t}$$

$$\Rightarrow Y''(t) + 4Y'(t) + 4Y(t) = e^{-2t}$$

(3) $\because$ 此為常係數 ODE

$\therefore$ 令 $y = e^{\lambda x}$ 代入 ODE 可得

$(\lambda^2 + 4\lambda + 4)e^{\lambda x} = 0$

$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow \lambda = -2, -2$ (重根)

$\therefore$ $Y_h = c_1e^{-2t} + c_2e^{-2t} \cdot t$

(4) 由待定係數法，令$Y_P = At^2 e^{-2t}$代回 ODE 可得

$Y_P' = 2Ate^{-2t} - 2At^2 e^{-2t} = A(2t - 2t^2)e^{-2t}$

$Y_P'' = A(2-4t)e^{-2t} - 2A(2t-2t^2)e^{-2t} = A(2-8t+4t^2)e^{-2t}$

$$A(2-8t+4t^2)e^{-2t} + 4A(2t-2t^2)e^{-2t} + 4At^2 e^{-2t} = e^{-2t} \quad \Rightarrow A = \frac{1}{2}$$

$$Y_P = \frac{1}{2}t^2 e^{-2t}$$

(5) $Y(t) = Y_h(t) + Y_p(t) = c_1 e^{-2t} + c_2 e^{-2t} \cdot t + \frac{1}{2}t^2 e^{-2t}$

$$\Rightarrow y(x) = c_1 x^{-2} + c_2 x^{-2} \cdot \ln x + \frac{1}{2}x^{-2} \cdot (\ln x)^2$$

4. 試求下述微分方程之通解

(1) $y^{(6)} + 2y^{(4)} + y'' = 0$ **(8%)**

(2) $y'' - 2y' + y = \dfrac{e^x}{x}$ **(8%)**

(3) $(1-x^2)y'' - 4xy' - 2y = x$ **(8%)**

(4) $2(3x+1)^2 y'' + 21(3x+1)y' + 18y = 0$ **(8%)**

(1) 令 $y = e^{\lambda x}$ 代入 ODE 可得

$(\lambda^6 + 2\lambda^4 + \lambda^2)e^{\lambda x} = 0$

$\Rightarrow \lambda^2(\lambda^2+1)^2 = 0$

$\Rightarrow \lambda = 0, 0, \pm i, \pm i$

$\therefore \quad y = C_1 + C_2 x + (C_3 \cos x + C_4 \sin x) + x(C_5 \cos x + C_6 \sin x)$

(2) 求補解

$\because$ 此為常係數 ODE

$\therefore$ 令 $y = e^{\lambda x}$ 代入 ODE 可得

$(\lambda^2 - 2\lambda + 1)e^{\lambda x} = 0$

$\Rightarrow (\lambda - 1)^2 = 0$

$\Rightarrow \lambda = 1, 1$

$\therefore \quad y_h = c_1 e^x + c_2 x e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

令其特解 $y_p(x) = u_1 y_1 + u_2 y_2 = u_1 e^x + u_2 x e^x$ 代回 ODE 後可得

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x + xe^x \end{vmatrix}}{e^{2x}} = -1 \quad \Rightarrow u_1 = -x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{e^x}{x} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix}}{e^{2x}} = \frac{1}{x} \quad \Rightarrow u_2 = \ln|x|$$

$$\therefore \quad y_p(x) = u_1 e^x + u_2 \cdot xe^x = -xe^x + \ln|x| \cdot xe^x = xe^x(-1 + \ln|x|)$$

通解 $y = y_h(x) + y_p(x) = C_1 e^x + C_2 xe^x + xe^x \cdot \ln|x|$

(3) $(1-x^2)y'' - 4xy' - 2y = x$

令 $a_2 = (1-x^2)$，$a_1 = -4x$，$a_0 = -2$

由判斷式: $a_2'' - a_1' + a_0 = 0$ 可知此為正合式

$$(1-x^2)y'' - 4xy' - 2y = \frac{d}{dx}[b_1(x)y' + b_0(x)y]$$

$$\Rightarrow b_1(x)y'' + [b_1'(x) + b_0(x)] + b_0'(x)y = (1-x^2)y'' - 4xy' - 2y$$

$$\Rightarrow b_1 = 1 - x^2,\ b_0 = -2x$$

$$\therefore \quad (1-x^2)y'' - 4xy' - 2y = \frac{d}{dx}[(1-x^2)y' - 2\mathrm{x}y] = x$$

$\Rightarrow \quad (1-x^2)y' - 2\mathrm{x}y = \frac{1}{2}x^2 + C_1$ 此為一階線性 ODE

$$\Rightarrow \quad y' - \frac{2\mathrm{x}}{1-x^2}y = \frac{1}{2}\frac{x^2}{1-x^2} + \frac{C_1}{1-x^2}$$

積分因子: $\mu = e^{\int \frac{-2\mathrm{x}}{1-x^2}dx} = e^{\int \frac{1}{1-x^2}d(1-x^2)} = e^{\ln|1-x^2|} = 1 - x^2$

同乘積分因子: $(1-x^2)y' - 2\mathrm{x}y = \frac{1}{2}x^2 + C_1$

$$\Rightarrow \frac{d}{dx}[(1-x^2)y] = \frac{1}{2}x^2 + C_1$$

$$\Rightarrow (1-x^2)y = \frac{1}{6}x^3 + C_1 x + C_2$$

$$\Rightarrow y = \frac{x^3}{6(1-x^2)} + \frac{C_1 x}{1-x^2} + \frac{C_2}{1-x^2}$$

(4) $2(3x+1)^2 y'' + 21(3x+1)y' + 18y = 0$

令 $t = 3x+1 \quad \Rightarrow \dfrac{dy(x)}{dx} = \dfrac{dt}{dx}\cdot\dfrac{dY(t)}{dt} = 3Y'(t)$

$$\Rightarrow \frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right) = 3\frac{dt}{dx}\cdot\frac{dY'(t)}{dt} = 9Y''(t)$$

$2(3x+1)^2 y'' + 21(3x+1)y' + 18y = 0 \quad \Rightarrow 18t^2Y'' + 63tY' + 18Y = 0$

令 $Y = t^m \quad \Rightarrow [18m(m-1) + 63m + 18]t^m = 0$

$\Rightarrow 18m^2 + 45m + 18 = 0$

$\Rightarrow (2m+1)(9m+18) = 0$

$\Rightarrow m = -\dfrac{1}{2}$ or -2

$\therefore \quad Y(t) = C_1 t^{-\frac{1}{2}} + C_2 t^{-2}$

$\Rightarrow y(x) = C_1(3x+1)^{-\frac{1}{2}} + C_2(3x+1)^{-2}$

5. 已知微分方程式 $xy'' - (2x+1)y' + (x+1)y = (x^2 + x - 1)e^{2x}$

(1) 已知 $y_1 = e^{ax}$ **為上述 ODE 之一補解，試問** $a = ?$ **(2%)**

(2) 試求另一補解。 (8%)

(1) $xy'' - (2x+1)y' + (x+1)y = 0$

令 $y_1 = e^{ax}$ 代入 ODE 可得

$a^2xe^{ax} - a(2x+1)e^{ax} + (x+1)e^{ax} = 0$

$\Rightarrow a^2x - 2ax - a + x + 1 = 0$

$\Rightarrow a = 1$

故可得一補解為 $y_1 = e^x$

(2) 令另一補解 $y_2 = vy_1 \quad \Rightarrow y_2' = v'y_1 + vy_1'$

$\Rightarrow y_2'' = v''y_1 + 2v'y_1' + vy_1''$

代入 ODE: $xy'' - (2x+1)y' + (x+1)y = (x^2 + x - 1)e^{2x}$

可得 $x(v''y_1 + 2v'y_1' + vy_1'') - (2x+1)(v'y_1 + vy_1') + (x+1)vy_1 = (x^2 + x - 1)e^{2x}$

$\Rightarrow x(v''y_1 + 2v'y_1') - (2x+1)v'y_1 = (x^2 + x - 1)e^{2x}$

$\Rightarrow (xy_1)v'' + (2xy_1' - 2xy_1 - y_1)v' = (x^2 + x - 1)e^{2x}$

令 $z = v' \quad \Rightarrow z' + \dfrac{2xy_1' - 2xy_1 - y_1}{xy_1}z = \dfrac{(x^2 + x - 1)e^{2x}}{xy_1}$

$\Rightarrow z' - \dfrac{1}{x}z = (x + 1 - \dfrac{1}{x})e^x \quad \longrightarrow$ 一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$

$$\Rightarrow \frac{1}{x}z' - \frac{1}{x^2}z = (1 + \frac{1}{x} - \frac{1}{x^2})e^x$$

$$\Rightarrow \frac{d}{dx}(\frac{1}{x}z) = (1 + \frac{1}{x} - \frac{1}{x^2})e^x$$

$$\Rightarrow \int d(\frac{1}{x}z) = \int (1 + \frac{1}{x} - \frac{1}{x^2})e^x dx$$

$$\Rightarrow \frac{1}{x}z = (e^x + \frac{e^x}{x}) + C_1$$

$$\Rightarrow z = (xe^x + e^x) + C_1 x$$

$$\Rightarrow v' = (xe^x + e^x) + C_1 x$$

$$\Rightarrow v = xe^x + \frac{C_1}{2}x^2 + C_2$$

$$y_2 = vy_1 = (xe^x + \frac{C_1}{2}x^2 + C_2)e^x = xe^{2x} + C_3 x^2 e^x + C_2 e^x$$

$\therefore$ 另一補解為 $x^2 e^x$ ，特解為 xe^{2x}

6. 已知單自由度振動系統其數學表示為 $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t)$ **，若給定質量塊** $m = 2$ **，阻尼係數** $c = 0$ **與彈簧常數** $k = 8$ **並且質量塊為靜止狀態即其初始條件** $y(0) = 0$ **與** $\dot{y}(0) = 0$ **，給一外力為** $f(t) = 4\cos\omega t$ **，試問:**

(1) 此系統的自然振動頻率 $\omega_n = ?$ **(2%)**

(2) 若 y_1, y_2 **為其兩補解，試求:** $W(y_1, y_2) = ?$ **(4%)**

(3) 當 $\omega = 2$ **，其解為何? (4%) 此時系統的運動行為稱為什麼? (2%)**

(4) 當 $\omega = 1.99$ **，其解為何? (4%) 此時系統的運動行為稱為什麼? (2%)**

(1) $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t)$ 又 $m = 2,\ c = 0,\ k = 8$ 與 $f(t) = 4\cos\omega t$

$\therefore$ $2\ddot{y}(t) + 8y(t) = 4\cos\omega t \ \Rightarrow \ddot{y}(t) + 4y(t) = 2\cos\omega t$

$$\omega_n = \sqrt{\frac{k}{m}} = 2$$

(2) 令 $y_1 = \cos 2t,\ y_2 = \sin 2t$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$$

(3) 當 $\omega = \omega_n = 2$ 時，外力對系統產生共振行為

$\ddot{y}(t) + 4y(t) = 2\cos 2t$

令 $y_p = t(A\cos 2t + B\sin 2t) = t \cdot y_h$

$$\dot{y}_p = y_h + t\,\dot{y}_h$$

$$\ddot{y}_p = \dot{y}_h + \dot{y}_h + t\,\ddot{y}_h = 2\dot{y}_h + t\,\ddot{y}_h$$ 代回 ODE 可得

$$(2\dot{y}_h + t\cdot\ddot{y}_h) + t\cdot y_h = 2\cos 2t$$

$$\Rightarrow 2\dot{y}_h = 2\cos 2t$$

$$\Rightarrow (-2A\sin 2t + 2B\cos 2t) = \cos 2t$$

$$\therefore \quad A = 0\,,\ \ B = \frac{1}{2}$$

$$y(t) = y_h(t) + y_p(t) = C_1\cos 2t + C_2\sin 2t + \frac{1}{2}t\cdot\sin 2t$$

又 $y(0) = 0 \quad \Rightarrow C_1 = 0$

$\dot{y}(0) = 0 \quad \Rightarrow C_2 = 0$

$$\therefore \quad y(t) = \frac{1}{2}t\cdot\sin 2t$$

(4) 當 $\omega = 1.99 \approx \omega_n$ 時，外力對系統產生拍擊行為

$$\ddot{y}(t) + 4y(t) = 2\cos(1.99t)$$

$$\Rightarrow y(t) = \frac{2}{(\omega_n^2 - \omega^2)}(\cos\omega t - \cos\omega_n t)$$

$$= \frac{4}{m(\omega_n^2 - \omega^2)}\sin\frac{\omega_n + \omega}{2}t\cdot\sin\frac{\omega_n - \omega}{2}t$$

$$= \frac{2}{2^2 - 1.99^2}(\cos 1.99t - \cos 2t)$$

$$= \frac{4}{2^2 - 1.99^2}\sin\frac{3.99}{2}t\cdot\sin\frac{0.01}{2}t$$