使用變數異動法求解特解(通式)

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$$y'' + a(x)y' + b(x)y = f(x)$$
$$y_c = c_1 y_1 + c_2 y_2$$

$$\Rightarrow y_1'' + a(x)y_1' + b(x)y_1 = 0 \quad \text{---(1)}$$

$$y_2'' + a(x)y_2' + b(x)y_2 = 0 \quad \text{---(2)}$$

$$(1) * y_2 - (2) * y_1 = 0$$

$$y_1'' y_2 + a(x)y_1' y_2 + b(x)y_1 y_2 - y_2'' y_1 - a(x)y_2' y_1 - b(x)y_1 y_2 = 0$$

$$y_1 y_2'' - y_1'' y_2 + a(x)\left[y_1 y_2' - y_1' y_2 \right] = 0$$

$$w'(x) + a(x)w(x) = 0$$

$$\Rightarrow I = e^{\int a(x)dx}$$

$$\therefore w(x) = ke^{-\int a(x)dx} \qquad \Rightarrow w(x)e^{\int a(x)dx} = k$$

令 $y_p = u_1 y_1 + u_2 y_2$

$$y_p' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$$

$$c(x) = u_1' y_1 + u_2' y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + c(x)$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + c'(x)$$

$\Rightarrow$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + c'(x) + a(x)\left[u_1 y_1' + u_2 y_2' + c(x)\right] + b(x)\left[u_1 y_1 + u_2 y_2\right] = f(x)$$

$$u_1 \left[y_1'' + a(x)y_1' + b(x)y_1\right] + u_2\left[y_2'' + a(x)y_2' + b(x)y_2\right]$$

$$+u_1' y_1' + u_2' y_2' + a(x)c(x) + c'(x) = f(x)$$

$$\therefore u_1' y_1' + u_2' y_2' = f(x) - a(x)c(x) - c'(x)$$

$u_1' y_1 + u_2' y_2 = c(x)$ --(1)

$u_1' y_1' + u_2' y_2' = f(x) - a(x)c(x) - c'(x)$ --(2)

(1) 課堂上選 c(x)=0

(2) 作業選　c(x)=c

(3) 在這裡選 c(x)=c(x)　(general)

$(1) * y_2' - (2) * y_1'$

$$\Rightarrow u_1' y_1 y_2' - u_1' y_1' y_2 = c(x)y_2' - f(x)y_2 + a(x)c(x)y_2 + c'(x)y_2$$

$$u_1' w(x) = -f(x)y_2 + c(x)\left[y_2' + a(x)y_2\right] + c'(x)y_2$$

$$u_1' = \frac{-f(x)y_2}{w(x)} + c(x)\frac{\left[y_2' + a(x)y_2\right]}{w(x)} + \frac{c'(x)y_2}{w(x)}$$

$(2) * y_1 - (1) * y_1'$

$$\Rightarrow u_2' y_1 y_2' - u_2' y_1' y_2 = f(x)y_1 - a(x)c(x)y_1 - c'(x)y_1 - c(x)y_1'$$

$$u_2{}' w(x) = f(x) y_1 - c(x)\left[y_1{}' + a(x) y_1 \right] - c'(x) y_1$$

$$u_2{}' = \frac{f(x) y_1}{w(x)} - c(x)\frac{\left[y_1{}' + a(x) y_1 \right]}{w(x)} - \frac{c'(x) y_1}{w(x)}$$

$$u_1{}' = \frac{-f(x) y_2}{w(x)} + c(x)\frac{\left[y_2{}' + a(x) y_2 \right]}{w(x)} \frac{e^{\int a(x)dx}}{e^{\int a(x)dx}} + \frac{c'(x) y_2}{w(x)} \frac{e^{\int a(x)dx}}{e^{\int a(x)dx}}$$

$$u_1{}' = \frac{-f(x) y_2}{w(x)} + \frac{c(x)}{k}\left[y_2 e^{\int a(x)dx} \right]' + \frac{1}{k} c'(x) y_2 e^{\int a(x)dx}$$

$$u_2{}' = \frac{f(x) y_1}{w(x)} - c(x)\frac{\left[y_1{}' + a(x) y_1 \right]}{w(x)} \frac{e^{\int a(x)dx}}{e^{\int a(x)dx}} - \frac{c'(x) y_1}{w(x)} \frac{e^{\int a(x)dx}}{e^{\int a(x)dx}}$$

$$u_2{}' = \frac{f(x) y_1}{w(x)} - \frac{c(x)}{k}\left[y_1 e^{\int a(x)dx} \right]' - \frac{1}{k} c'(x) y_1 e^{\int a(x)dx}$$

$$u_2{}' = \frac{f(x) y_1}{w(x)} - \frac{1}{k}\left[c(x) y_1 e^{\int a(x)dx} \right]'$$

$\therefore$

$$u_1 = \int \frac{-f(x) y_2}{w(x)} dx + \frac{1}{k} \int \left(c(x)\left[y_2 e^{\int a(x)dx} \right] \right)' dx$$

$$u_1 = \int \frac{-f(x) y_2}{w(x)} dx + \frac{1}{k} c(x) y_2 e^{\int a(x)dx}$$

$$u_2 = \int \frac{f(x) y_1}{w(x)} dx - \frac{1}{k} \int \left(c(x)\left[y_1 e^{\int a(x)dx} \right] \right)' dx$$

$$u_2 = \int \frac{f(x)y_1}{w(x)}dx - \frac{1}{k}c(x)y_1 e^{\int a(x)dx}$$

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \left[\int \frac{-f(x)y_2}{w(x)}dx + \frac{1}{k}c(x)y_2 e^{\int a(x)dx}\right]y_1 + \left[\int \frac{f(x)y_1}{w(x)}dx - \frac{1}{k}c(x)y_1 e^{\int a(x)dx}\right]y_2 \\ &= y_1 \int \frac{-f(x)y_2}{w(x)}dx + y_2 \int \frac{f(x)y_1}{w(x)}dx \end{aligned}$$

$\therefore$跟$c(x)$為多少無關，因此用 0 最方便