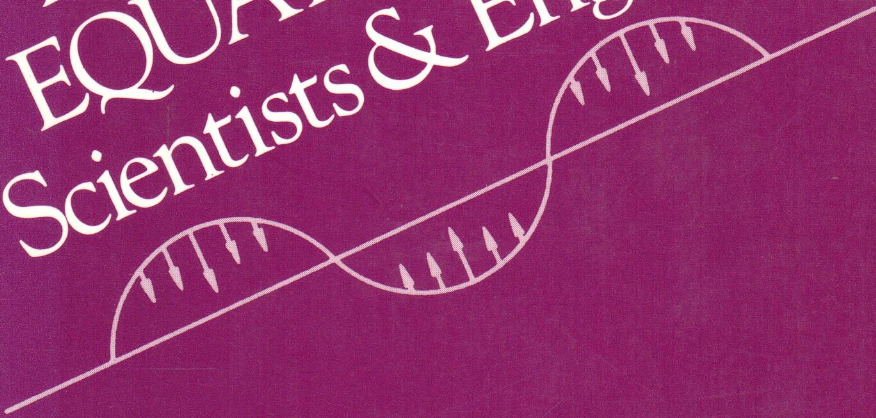
PARTIAL DIFFERENTIAL EQUATIONS for Scientists & Engineers



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LESSON 18

More on the D'Alembert Solution

PURPOSE OF LESSON: To illustrate how the D'Alembert solution can be used to find the wave motion of a *semi-infinite*-string problem

$$\begin{array}{llll} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < x < \infty & 0 < t < \infty \\ \text{BC} & u(0,t) = 0 & 0 < t < \infty & \\ \text{ICs} & \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} & 0 \le x < \infty & \end{array}$$

In addition, the D'Alembert solution is interpreted in the xt-plane.

In the previous lesson, we found that the expression

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi \tag{18.1}$$

describes the displacement u of an infinite string in terms of its initial displacement $u(x,0) = f(x)$ and velocity $u_t(x,0) = g(x)$. This lesson will show the reader some interesting interpretations of this equation in the xt-plane (space-time plane) and how the equation can be modified to find the solution of the vibrating *semi-infinite* string.

We start with our interpretation of (18.1) in the xt-plane.

The Space-Time Interpretation of D'Alembert's Solution

We proved in the last lesson the solution of the pure initial-value problem

$$\begin{array}{llll} \text{PDE} & u_{tt} = c^2 u_{xx} & -\infty < x < \infty & 0 < t < \infty \\ \text{ICs} & \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} & -\infty < x < \infty & \end{array} \tag{18.2}$$

is given by

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

We now present an interpretation of this solution in the xt-plane looking at two specific cases.

CASE 1 (Initial position given; initial velocity zero)
Suppose the string has initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

Here, the D'Alembert solution is

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

and the solution u at a point (x_0,t_0) can be interpreted via Figure 18.1 as being the average of the initial displacement $f(x)$ at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ found by backtracking along the lines

$$\begin{aligned} x - ct &= x_0 - ct_0 \\ x + ct &= x_0 + ct_0 \end{aligned} \qquad \text{(characteristic curves)}$$

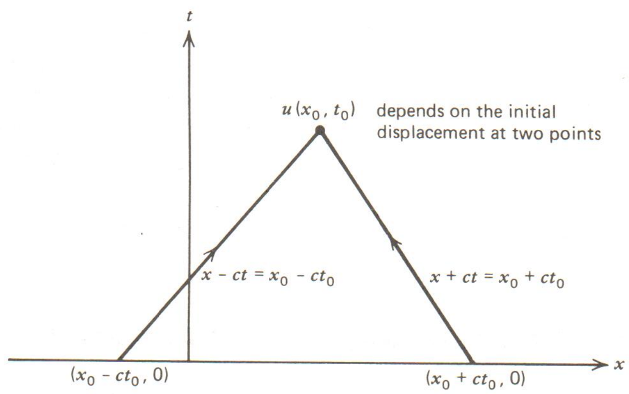

FIGURE 18.1 Interpretation of $u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$ in the xt-plane.

For example, using this interpretation, the initial-value problem

$$\text{PDE} \quad u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

(18.3)

$$\text{ICs} \quad \begin{cases} u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

would give us the solution in the xt-plane shown in Figure 18.2.

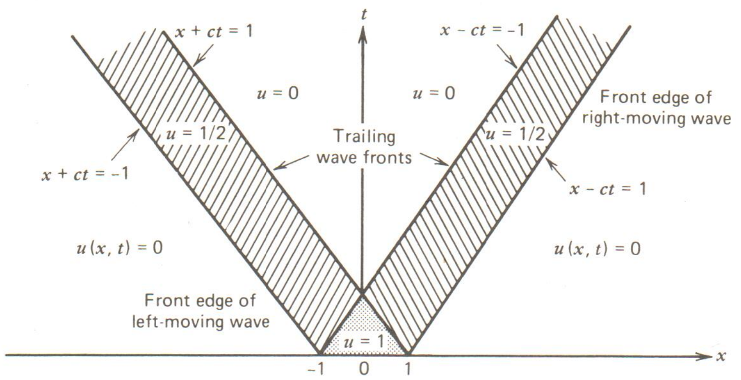

FIGURE 18.2 Solution of the initial-value problem (18.1) in the xt-plane.

Figure 18.2 is the xt-plane version of the solution graphed in Figure 17.2 in the previous lesson.

We now interpret the D'Alembert solution when the initial position is zero, but the velocity is arbitrary.

CASE 2 (Initial displacement zero; velocity arbitrary)
Consider now the IC

$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$

Here, the solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

and, hence, the solution u at (x_0,t_0) can be interpreted as integrating the initial velocity between $x_0 - ct_0$ and $x_0 + ct_0$ on the initial line $t = 0$ (Figure 18.3).

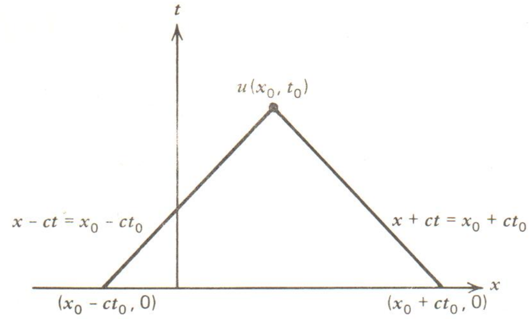

FIGURE 18.3 Interpretation of initial velocity in the *xt*-plane.

Again, using this interpretation, the solution to the initial-value problem

$$
(18.4)\qquad \begin{array}{ll} \text{PDE} & u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty \\ \text{ICs} & \begin{cases} u(x,0) = 0 \\ u_t(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases} \end{cases} \end{array}
$$

has a solution in the *xt*-plane illustrated in Figure 18.4.

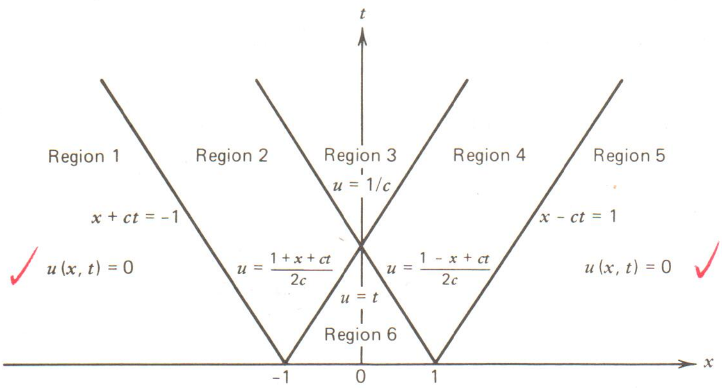

FIGURE 18.4 Solution to problem (18.4) in the *xt*-plane.

Problem 18.4 corresponds to imposing an initial *impulse* (velocity = 1) on the string for $-1 < x < 1$ and watching the resulting wave motion (as in the piano string). To find the displacement, we compute the D'Alembert solution

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} 0\, d\xi \qquad (x,t)\varepsilon \text{ Region 1}
\end{aligned}
$$

$$= \frac{1}{2c}\int_{-1}^{x+ct} d\xi \qquad (x,t)\varepsilon \text{ Region 2}$$

(18.5)
$$= \frac{1}{2c}\int_{-1}^{1} d\xi \qquad (x,t)\varepsilon \text{ Region 3}$$

$$= \frac{1}{2c}\int_{x-ct}^{1} d\xi \qquad (x,t)\varepsilon \text{ Region 4}$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} 0\, d\xi \qquad (x,t)\varepsilon \text{ Region 5}$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi \qquad (x,t)\varepsilon \text{ Region 6}$$

Diamond rule To D'Alembert's sol.

This solution is graphed at various values of time in Figure 18.5.

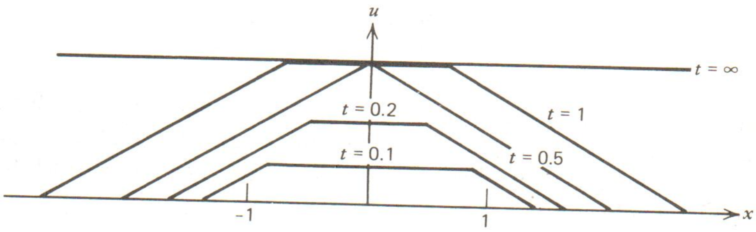

FIGURE 18.5 Solution of problem (18.4) for various values of time.

This completes our interpretation of the D'Alembert solution in the xt-plane. In the remainder of the lesson, we will solve the initial-boundary-value problem for the semi-infinite string

(18.6)

PDE	$u_{tt} = c^2 u_{xx}$	$0 < x < \infty$	$0 < t < \infty$
BC	$u(0,t) = 0$	$0 < t < \infty$	
ICs	$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$	$0 < x < \infty$	

by *modifying* the D'Alembert formula.

Solution of the Semi-infinite String via the D'Alembert Formula

The object now is to find the wave motion of the vibrating string whose left end is fixed at *zero* and has given initial conditions. To find the solution of (18.6), we proceed in a manner similar to that used with the infinite string, which is to find the general solution to the PDE