1. Solve the PDE $u_{xx} = u_{tt}$ for $-\infty < x < \infty$, $t > 0$

With initial conditions $u(x,0) = 0$, $\dot{u}(x,0) = \frac{1}{a}[H(x-a) - H(x+a)]$

Where H(t) is Heaviside function

(1) As $a = 1$, check the same problem of homework

(2) Disscuss the limiting case for $a \to 0$

(3) Plot the 3-D plot of $u(x,t)$

(4) Plot the contour of $u(x,t)$ in $x-t$ plane

Sol：

D'Alembert's solution：

$$u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(\tau)d\tau$$
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{a}[H(\tau-a) - H(\tau+a)]d\tau$$
$$= \frac{1}{2ac}[(\tau-a)H(\tau-a) - (\tau+a)H(\tau+a)]|_{x-ct}^{x+ct}$$
$$= \frac{1}{2ac}[(x+ct-a)H(x+ct-a) - (x+ct+a)H(x+ct+a) -$$
$$(x-ct-a)H(x-ct-a) + (x-ct+a)H(x-ct+a)]$$

(1) $a = 1$ ：

$$u(x,t) = \frac{1}{2ac}[(x+ct-1)H(x+ct-1) - (x+ct+1)H(x+ct+1) -$$
$$(x-ct-1)H(x-ct-1) + (x-ct+1)H(x-ct+1)]$$

(2) $a \to 0$ ：

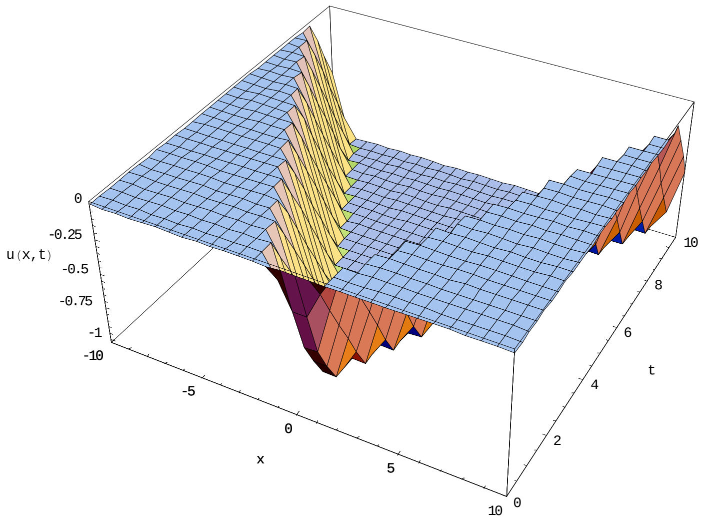
$$\dot{u}(x,0) = \lim_{a\to 0}\frac{1}{a}[H(x-a) - H(x+a)]$$
$$= \lim_{a\to 0}[-\delta(x-a) - \delta(x+a)]$$
$$= -2\delta(x)$$

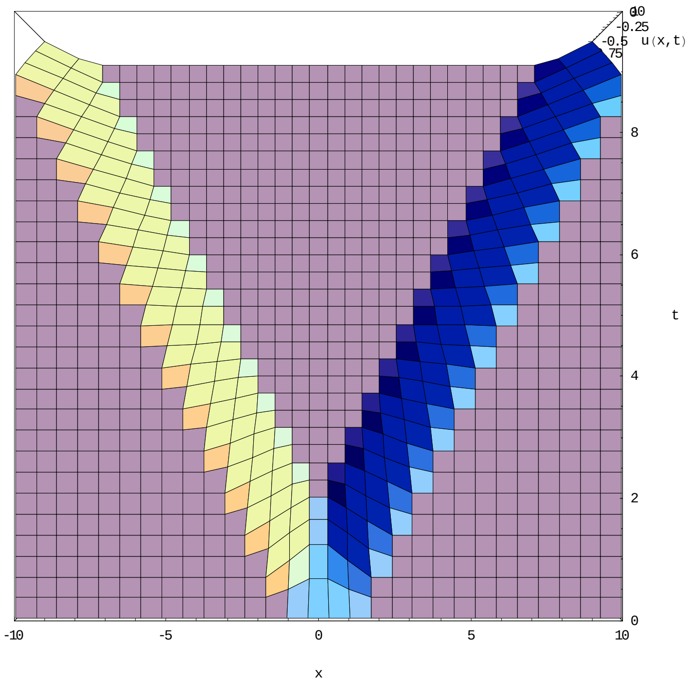
$$u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(\tau)d\tau$$
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} -2\delta(\tau)d\tau = \frac{-1}{c}H(\tau)|_{x-ct}^{x+ct}$$
$$= \frac{-1}{c}[H(x+ct) - H(x-ct)]$$

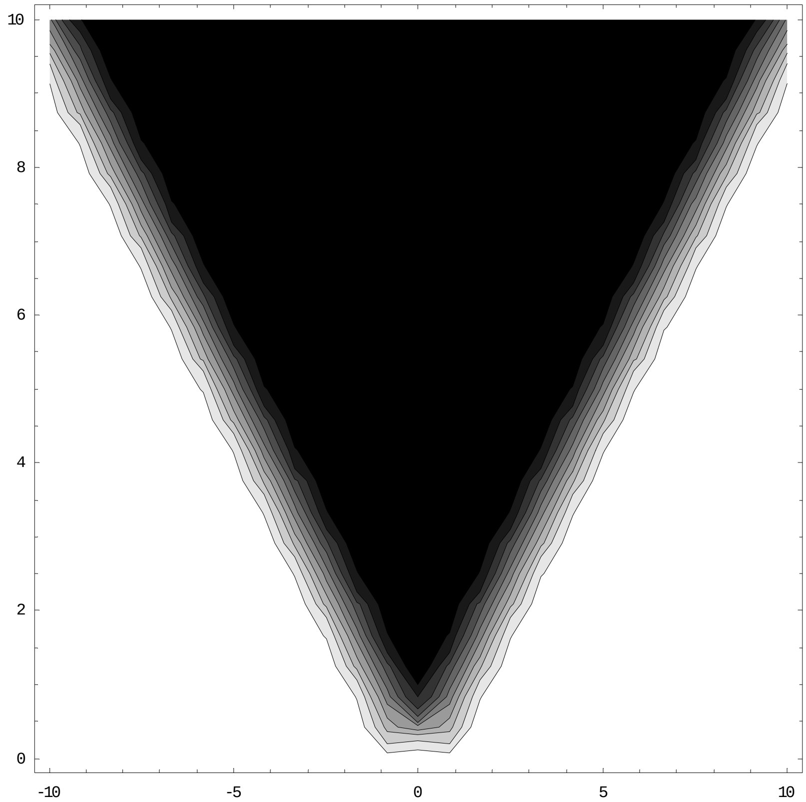
$$u(x,t) = \lim_{a\to 0}\{\frac{1}{2ac}[(x+ct-a)H(x+ct-a) - (x+ct+a)H(x+ct+a) -$$
$$(x-ct-a)H(x-ct-a) + (x-ct+a)H(x-ct+a)]\}$$

$$u(x,t)=\lim_{a\to 0}\{\frac{1}{2c}[-H(x+ct-a)-(x+ct-a)\delta(x+ct-a)-H(x+ct+a)-$$
$$(x+ct+a)\delta(x+ct+a)+H(x-ct-a)+(x-ct-a)\delta(x-ct-a)+$$
$$H(x-ct+a)+(x-ct+a)\delta(x-ct+a)]\}$$
$$=\frac{1}{2c}[-H(x+ct)-(x+ct)\delta(x+ct)-H(x+ct)-(x+ct)\delta(x+ct)+$$
$$H(x-ct)+(x-ct)\delta(x-ct)+H(x-ct)+(x-ct)\delta(x-ct)]$$
$$=\frac{1}{2c}[-2H(x+ct)+2H(x-ct)-2(x+ct)\delta(x+ct)+2(x-ct)\delta(x-ct)]$$
$$=\frac{-1}{c}[H(x+ct)-H(x-ct)]$$

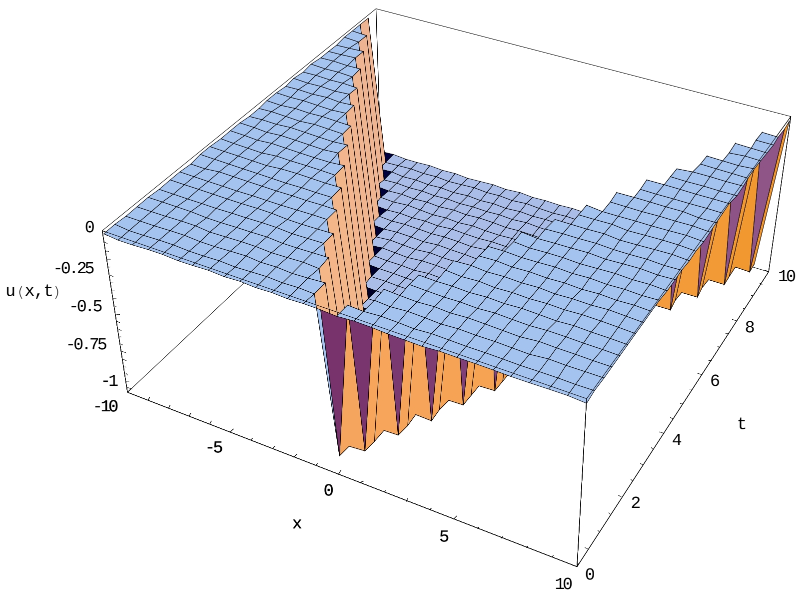
$a=1$

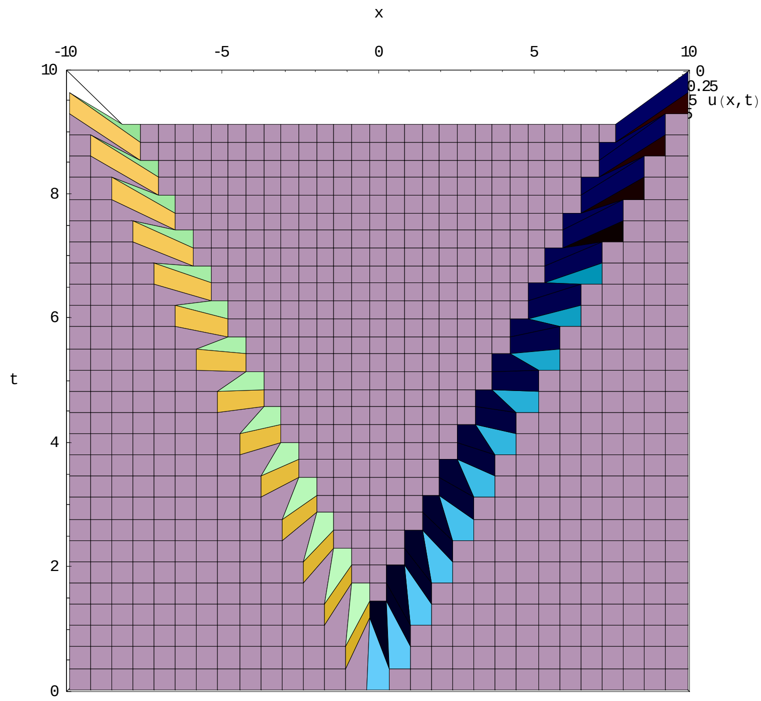


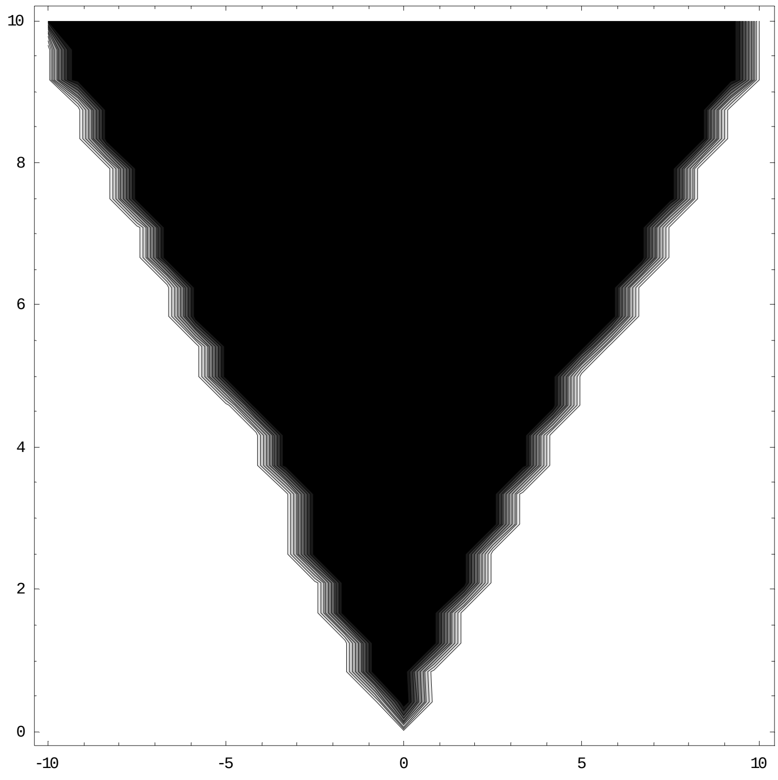




$a \to 0$




x
-10
-5
0
5
10
10
8
6
4
2
0
t
0
0.25
5 u(x,t)


10
8
6
4
2
0
-10
-5
0
5
10