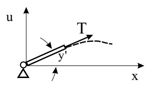
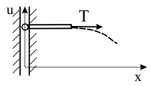
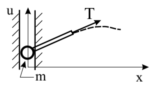
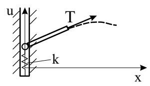
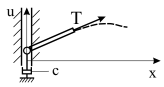
Table 1: Types of boundaries

Type	Diagram	Equation
Fixed		$u(0,t) = 0$
Free		$\frac{\partial u(0,t)}{\partial x} = 0$
Mass		$m\ddot{u}(0,t) = T\frac{\partial u(0,t)}{\partial x}$
Spring		$\gamma u(0,t) = T\frac{\partial u(0,t)}{\partial x}$
Dashpot		$c\dot{u}(0,t) = T\frac{\partial u(0,t)}{\partial x}$

1.2 Reflection and transmission at boundaries

1.2.1 Types of boundaries

There are several types of boundary conditions as shown in Table 1.2.1.

1.2.2 Reflection from a fixed boundary

Consider a displacement in a semi-infinite string of $x \geq 0$ with a fixed boundary at $x = 0$. As shown in Section 1.1.3, a general solution in time domain for the wave equation is written as

$$u(x,t) = f(t - x/c_0) + g(t + x/c_0) \tag{39}$$

where g and f denote the incident wave and the reflected wave propagating in the negative and positive x directions, respectively. From the boundary condition $u(0,t) = 0$, we have

$$f(t) = -g(t). \tag{40}$$

Thus the displacement in the semi-infinite string is expressed as

$$u(x,t) = -g(t - x/c_0) + g(t + x/c_0) \tag{41}$$

Exercise 1.2.2-a Determine the reflected wave from a free boundary subjected to the indicent wave $g(c_0 t + x)$.

1.2.3 Reflection of time harmonic waves from a fixed boundary

As shown in Section 1.1.2, a time harmonic general solution can be written as

$$\tilde{u}(x,t) = \bar{u}(x)e^{-i\omega t} \equiv F(\omega)e^{i(k_0 x - \omega t)} + G(\omega)e^{-i(k_0 x + \omega t)} \tag{42}$$

where $F(\omega)$ and $G(\omega)$ are amplitudes of harmonic waves.

If the fixed boundary condition $\tilde{u}(0,t) = 0$ is given at the end of a semi-infinite string, we have $F(\omega) = -G(\omega)$. It then follows that

$$\tilde{u}(x,t) = \bar{u}(x)e^{-i\omega t} = \{-G(\omega)e^{ik_0x} + G(\omega)e^{-ik_0x}\}e^{-i\omega t}. \tag{43}$$

Here the Fourier transform of a general function $p(t+\alpha)$ is defined by

$$\int_{-\infty}^{\infty} p(t+\alpha)e^{i\omega t}dt = \int p(s)e^{i\omega(s-\alpha)}ds = e^{-i\omega\alpha}P(\omega), \tag{44}$$

where $P(\omega)$ is the Fourier spectrum of $p(t)$. Conversely, the inverse Fourier transform is written as

$$p(t+\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)e^{-i\omega\alpha}e^{-i\omega t}d\omega. \tag{45}$$

If we consider $G(\omega)$ in eq.(43) as the Fourier spectrum of a function $g(t)$ in time domain, then the inverse Fourier transform of eq.(43) gives

$$\begin{aligned} u(x,t) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(x)e^{-i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \{-G(\omega)e^{ik_0x} + G(\omega)e^{-ik_0x}\}e^{-i\omega t}d\omega = -g(t - x/c_0) + g(t + x/c_0) \end{aligned} \tag{46}$$

1.2.4 Reflection of harmonic waves from an elastic boundary

- harmonic wave solution and boundary condition

$$\tilde{u}(x,t) = G(\omega)e^{i(k_0x+\omega t)} + F(\omega)e^{i(k_0x-\omega t)}, \quad \gamma\tilde{u}(0,t) = T\partial\tilde{u}(0,t)/\partial x. \tag{47}$$

- amplitude of the reflected wave

$$\gamma(G+F) = ik_0T(F-G) \tag{48}$$

$$\frac{F}{G} = -\frac{\gamma + ik_0T}{\gamma - ik_0T} = -1e^{i\phi(\omega)} \quad (\text{ no energy loss}) \tag{49}$$

If the spring has very large γ for the spring constant(stiff spring), then $F \approx -G$. In the limit of $\gamma \to 0$ for a very soft spring, we have $F \approx G$.

Exercise 1.2.4-a Assume that the transient pulse of the incident wave has positive and negative rectangular shapes, given by $g(t) = H(t) - H(t-a) - \{H(t-a) - H(t-2a)\}$, where $H(t)$ is the Heaviside function. Then $G(\omega)$ is obtained by

$$G(\omega) = \frac{1}{-i\omega}(1 - 2e^{-i\omega a} + e^{-2i\omega a}). \tag{50}$$

Calculate $F(\omega)$ numerically by choosing appropriate constants for γ, T c_0 and a from eq.(49), and take the inverse Fourier transform of $F(\omega)e^{ik_0x}|_{x=2a}$ to obtain the time variation of the reflected displacement at $x = 2a$.
Note that for the incident pulse given by $g(t) = H(t) - H(t-a)$, the transient displacement varies as shown in Fig. 2.

Exercise 1.2.4-b Determine the reflected wave due to a boundary with mass subjected to the harmonic indicent wave $G\exp[i(k_0x + \omega t)]$.

1.2.5 Reflection and transmission at interface

Consider reflection and transmission of the incident wave u^i from the interface between different springs as shown in Fig. 3. For the time harmonic incident wave given by $\tilde{u}^i = F^i(\omega)e^{i(k_1x-\omega t)}$, assume that reflection and transmission waves have the following form

$$\tilde{u}^r = G^r(\omega)e^{-i(k_1x+\omega t)} \text{ and } \tilde{u}^t = F^t(\omega)e^{i(k_2x-\omega t)} \tag{51}$$

where $k_i = \omega/c_i$, $c_i = \sqrt{T/\rho_i}$.

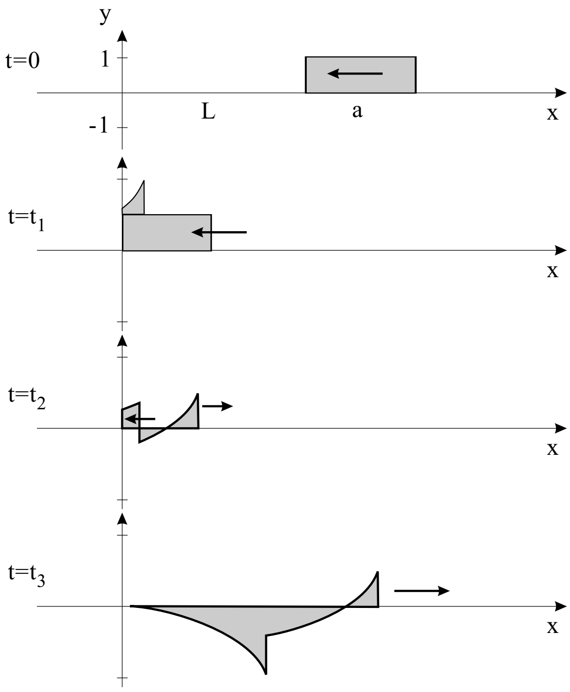


Figure 2: Reflection at the boundary supported by spring.

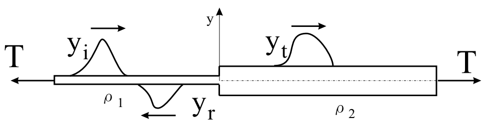


Figure 3: Incident, reflected, and transmitted waves at a discontinuity in the string

The continuity conditions at the interface are given by

$$\tilde{u}^i + \tilde{u}^r = \tilde{u}^t, \quad \frac{\partial \tilde{u}^i}{\partial x} + \frac{\partial \tilde{u}^r}{\partial x} = \frac{\partial \tilde{u}^t}{\partial x} \quad \text{at } x = 0. \tag{52}$$

Substitute eq.(51) into (52) yields

$$F^i + G^r = F^t, \;\; k_1 F^i - k_1 G^r = k_2 F^t. \tag{53}$$

Then we have

$$F^t = \frac{2k_1}{k_1 + k_2} F^i = \frac{2c_2}{c_1 + c_2} F^i, \quad G^r = \frac{k_1 - k_2}{k_1 + k_2} F^i = \frac{c_2 - c_1}{c_1 + c_2} F^i. \tag{54}$$

1.2.6 Propagator-matrix method for multi-connected media

Consider the problem to find a general solution in time harmonic state, which satisfies the following equation of motion.

$$\frac{d^2 u}{dx^2} + k^2 u = 0. \tag{55}$$

Assume the solutions as follows.

$$u(x) = u_1(x), \;\; \frac{du}{dx}(x) = u_2(x) \tag{56}$$

Substituting eq.(56) into eq.(55) yields

$$\frac{d^2 u_1}{dx^2} + k^2 u_1 = 0, \;\; \frac{du_2}{dx} + k^2 u_1 = 0. \tag{57}$$

Also we have

$$u_2 = \frac{du_1}{dx}. \tag{58}$$

From eqs.(57) and (58), it then follows

$$\frac{d}{dx}\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) = \left[\begin{array}{cc} 0 & 1 \\ -k^2 & 0 \end{array}\right]\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) \quad \text{or} \quad \frac{d}{dx}\boldsymbol{f}(x) = \boldsymbol{A}\boldsymbol{f}(x). \tag{59}$$

The solution of eq.(59) can be expressed as

$$\boldsymbol{f}(x) = \boldsymbol{C}\exp[\boldsymbol{A}x]. \tag{60}$$

Assume $\boldsymbol{f}(x) = \boldsymbol{f}(x_0)$ at $x = x_0$. Then $\boldsymbol{C} = \boldsymbol{f}(x_0)\exp[-\boldsymbol{A}x_0]$, and

$$\boldsymbol{f}(x) = \exp[\boldsymbol{A}(x - x_0)]\boldsymbol{f}(x_0). \tag{61}$$

To evaluate $\exp[\boldsymbol{A}(x - x_0)]$, we can use Sylvester's formula

$$F(\boldsymbol{A}) = \sum_{k=1}^{n} F(\lambda_k)\frac{\prod_{r\neq k}(\boldsymbol{A} - \lambda_k \boldsymbol{I})}{\prod_{r\neq k}(\lambda_k - \lambda_r)} \tag{62}$$

where $\lambda_k (k = 1, 2, \ldots, n)$ are distinct eigenvalues of a square matrix $\boldsymbol{A}$. The eigenvalues of the matrix $\boldsymbol{A}$ are $\lambda_{1,2} = \pm ik$. Putting these results in eq.(62), we find

$$\boldsymbol{P}(x, x_0, k) \equiv \exp[\boldsymbol{A}(x - x_0)] = \left[\begin{array}{cc} \cos k(x - x_0) & \frac{1}{k}\sin k(x - x_0) \\ -k\sin k(x - x_0) & \cos k(x - x_0) \end{array}\right] \tag{63}$$

where $\boldsymbol{P}(x, x_0, k)$ is called the propagator matrix. Hence we have

$$\left(\begin{array}{c} u_1(x) \\ u_2(x) \end{array}\right) = \underbrace{\left[\begin{array}{cc} \cos k(x - x_0) & \frac{1}{k}\sin k(x - x_0) \\ -k\sin k(x - x_0) & \cos k(x - x_0) \end{array}\right]}_{\boldsymbol{P}(x, x_0, k)}\left(\begin{array}{c} u_1(x_0) \\ u_2(x_0) \end{array}\right) \tag{64}$$

For multi-connected domains, the propagator matrix $\boldsymbol{P}(x, x_0, k)$ is $x_j > x > x_{j-1}$ is found from

$$\boldsymbol{f}(x) = \boldsymbol{P}(x, x_{j-1}, k_j)\boldsymbol{P}(x_{j-1}, x_{j-2}, k_{j-1})\ldots\boldsymbol{P}(x_1, x_0, k_1)\boldsymbol{f}(x_0). \tag{65}$$