工程數學 (四) – 偏微分方程

I. Mathematical model for Green's function

$$u_{tt} = c^2 u_{xx}, \quad for \ \ -\infty < x < \infty, \ \ t > 0$$

with initial conditions

$$u(x,0) = 0, \dot{u}(x,0) = \frac{1}{2a}[H(x-a) - H(x+a)]$$

Discuss the limiting case for $a \to 0$.

II. Mathematical model for Green's function

$$u_{tt} = c^2 u_{xx}, \quad for \ \ -\infty < x < \infty, \ \ t > 0$$

with initial conditions

$$u(x,0) = 0, \dot{u}(x,0) = \delta(x)$$

III. Mathematical model for Green's function

$$\frac{\partial^2 U(x,s;t,\tau)}{\partial t^2} - c^2 \frac{\partial^2 U(x,s;t,\tau)}{\partial x^2} = \delta(x-s)\delta(t-\tau), -\infty < x < \infty, t > 0$$

with initial conditions

$$\lim_{t \to \tau} U(x,s;t,\tau) = 0$$

$$\lim_{t \to \tau} \dot{U}(x,s;t,\tau) = 0$$

Table 1: Green's function for different one-dimensional PDEs

Equation	Governing Eq.	$U(x,s)$ or $U(x,s;t,\tau)$
Laplace	$\frac{\partial^2 U(x,s)}{\partial x^2} = \delta(x-s)$	$\frac{1}{2} \mid x-s \mid$
Heat	$\frac{\partial^2 U(x,s;t,\tau)}{\partial x^2} - \frac{\partial U(x,s;t,\tau)}{\partial t} = \delta(x-s)\delta(t-\tau)$	$\frac{-H(t-\tau)}{\sqrt{4\pi(t-\tau)}} e^{\frac{-(x-s)^2}{4(t-\tau)}}$
Wave	$\frac{\partial^2 U(x,s;t,\tau)}{\partial t^2} - c^2 \frac{\partial^2 U(x,s;t,\tau)}{\partial x^2} = \delta(x-s)\delta(t-\tau)$	$\frac{1}{2c} H(x-s+c(t-\tau)) - \frac{1}{2c} H(x-s-c(t-\tau))$
Helmholtz	$\frac{\partial^2 U(x,s)}{\partial x^2} + k^2 U(x,s) = \delta(x-s)$	$\frac{1}{2k} sink \mid x-s \mid$

$U(x,s;t,\tau)$: response at space x when time t due to a unit source at space s when time τ
As $t \to \tau$, $U(x,s;t,\tau) = -\delta(x-s)$ for heat conduction
As $t \to \tau$, $\dot{U}(x,s;t,\tau) = \delta(x-s)$ for wave propagation