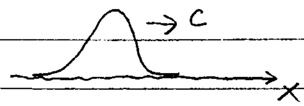
- $u = Ae^{i(\omega t - kx)}$ satisfies the kinematic equ

$$u_t + c u_x = 0 \qquad c = \frac{\omega}{k} = \text{cnt} = \text{"wave speed"}$$

this is an example of solutions (waves) that satisfy some equations.
In general,

$u = f(x-ct)$



bump / perturbation that propagates at speed c

Generalizations of the previous equation:

$u_t + c(u) u_x = 0$

- $c(u) = u \quad \Rightarrow \quad u_t + u u_x = 0$
- $u_t + u u_x = \nu u_{xx}$ Buger's equ.

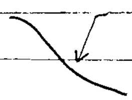
the common concept is a wave-like solution.

February 9, 2004 Lecture 2

Optional Reading Kevorkian § 5.1, 5.2
Debnath § 3.2-3.5

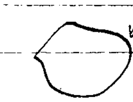
Review of lec. 1:

IVP data on $\mathcal{I}$



$\mathcal{I}$: smooth, open

BVP data on C



C: closed curve

Well-posed problem: ① Solution exists
and ② " is unique
and ③ " depends continuously on data

$\}$ ≠ Ill-posed problem

THEME : $\boxed{\text{PDE} \Rightarrow \text{ODE}}$ both for linear and nonlinear PDE

Useful concept: Wave Simplest PDE : $u_t + c u_x = 0$, $c = \frac{\omega}{k}$ (c — speed, ω — frequency, k — wave #)
(help understand the solution before actually find it)

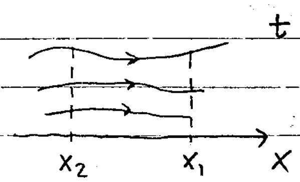
Modify ↓

$u_t + c(u) u_x = 0$

PDE's describe natural phenomena, or rather, our understanding of them.

-"- are based on principles (physical principles)

Ex. Conservation law (principle)
(of "mass")



material (fluid) moves
its mass should be conserved

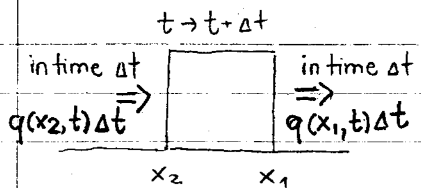
useful quantities:

- $\rho(x,t)$: density [mass/length]
- $q(x,t)$: flux [mass/time]

conservation

Principle of conservation of mass

we can relate these two quantities using the PCM.



(i) Total mass that remains in in time Δt:

$$[q(x_2,t) - q(x_1,t)]\,\Delta t$$

(ii) Mass inside the box at instant t : $-\int_{x_1}^{x_2} \rho(x,t)\,dx = \int_{x_2}^{x_1} \rho(x,t)\,dt$

↳ change with time for Δt: $\int_{x_2}^{x_1} dx \left[\rho(x,t+\Delta t) - \rho(x,t)\right]$

Conservation of mass: $\int_{x_2}^{x_1} dx \left[\rho(x,t+\Delta t) - \rho(x,t)\right] + \left[q(x_1,t) - q(x_2,t)\right]\Delta t = 0$

$$\int_{x_2}^{x_1} dx\, \frac{\rho(x,t+\Delta t) - \rho(x,t)}{\Delta t} + q(x_1,t) - q(x_2,t) = 0 \quad (\Delta t \to 0)$$

$$\Rightarrow \int_{x_2}^{x_1} dx\, \frac{\partial}{\partial t}\rho(x,t) + q(x_1,t) - q(x_2,t) = 0$$

under some conditions of integralibility (weak conditions)

$$\boxed{\frac{d}{dt}\int_{x_2}^{x_1} dx\, \rho(x,t) + q(x_1,t) - q(x_2,t) = 0}$$

integral form of conservation law

In order to get a PDE, $x_2 - x_1 \to 0$

$\Delta x = x_1 - x_2 > 0$, $\Delta x \to 0$: $\int_{x_2}^{x_1} dx\, \rho(x,t) \approx \rho(\bar{x},t)\,\Delta x \qquad x_2 < \bar{x} < x_1$

$\Rightarrow \Delta x \frac{\partial \rho}{\partial t} + q(x_1,t) - q(x_2,t) = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{q(x_1,t) - q(x_2,t)}{\Delta x} = 0 \quad (\Delta x \to 0)$

$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0}$ differential form of conservation law (suppose ρ, ρ_x, ρ_t are continuous)

- the integral form is more general (weaker conditions on ρ); allows for example jumps in ρ

PDE for ρ: We must relate q and ρ. This is what models are about. If the model is not adequate shocks may appear in the solution (non-physical phenomena)

Assumption: $q = Q(\rho)$ known function (constitutive relation)

Conservative law $\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + Q'(\rho) \cdot \frac{\partial \rho}{\partial x} = 0}$ PDE for ρ

(because of the assumption, ρ do not appear in the PDE, only ρ_t, ρ_x - sometimes this is not a good assumption)

$(\rho \leftrightarrow u)$ $\quad u_t + c(u)\, u_x = 0 \qquad c(u) \leftrightarrow Q'(\rho)$

"waves propagating with speed $Q'(\rho)$, waves in mass conserv."

↳ the deriv. of the function rel. q and ρ

Another way to bring this equation:

$\frac{d}{dt}\rho = 0, \quad \frac{dx}{dt} = Q'(\rho)$ total derivative of ρ is zero if you move at speed $Q'(\rho)$

$\frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} = 0 \leftrightarrow \frac{\partial \rho}{\partial t} + Q'(\rho)\frac{\partial \rho}{\partial x} = 0$

$x = x(t)$: characteristics (CHAR)

Theory of 1st order PDE

! can always be converted to ODE $\Rightarrow$ they are always solvable

→ what helps is the concept of CHAR

Case 1 $\quad \underset{\text{known}}{a(x,y)}\, u_x + \underset{\text{known}}{b(x,y)}\, u_y = 0, \quad u = u(x,y)$

linear, homogeneous PDE

($\equiv u \to \lambda u$ satisfies same PDE)

Suppose we vary x and y by Δx et Δy : $\begin{cases} x \to x + \Delta x \\ y \to y + \Delta y \end{cases}$

How does the function vary?

$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y) \approx u_x \Delta x + u_y \Delta y$

(u is differentiable)

We want to find specific $\Delta x, \Delta y$ such that $\Delta u = 0 \iff u = \text{const}$

for these variations

infinit. small const.

let $\begin{cases} \Delta x = a \cdot \varepsilon \\ \Delta y = b \cdot \varepsilon \end{cases} \Rightarrow \Delta u \approx (a u_x + b u_y)\, \varepsilon = 0$

by using the PDE

What this means?

$\Delta x \to dx$, $\Delta y \to dy$ $\quad \varepsilon = \boxed{\dfrac{dx}{a(x,y)} = \dfrac{dy}{b(x,y)}} \Rightarrow \Delta u = 0 \Rightarrow \boxed{u = \text{const}}$

when (x,y) satisfy this ODE, $u = \text{const}$ along this curve.

$\dfrac{dx}{a(x,y)} = \dfrac{dy}{b(x,y)} \Rightarrow$ solution $h(x,y) = \text{const}$ (a curve)

this is the CHAR

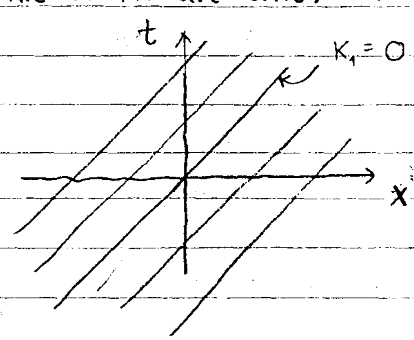
Ex.1 Kinematic eq. $\quad u_t + c u_x = 0 \quad$ c: constant

Find CHAR? $\quad u = u(x,t)$

$a(x,t) = 1$, $b(x,t) = c \Rightarrow \dfrac{dt}{1} = \dfrac{dx}{c} \Rightarrow x(t) = ct + \text{const}$

$x - ct = \text{const} = K_1$

$\Rightarrow$ the CHAR are lines $x - ct = K_1$



slope of the lines : $\frac{1}{c}$

Ex.2 Find the general solution of $u_t + c u_x = 0$

F: arbitrary

Along a specific CHAR, $K_1 \longrightarrow u = G_1$ - const $\quad$ i.e. $G_1 = F(K_1)$

to each K_1 we map a const value G_1

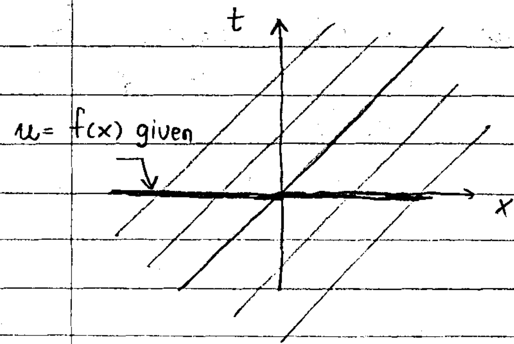
Along a CHAR, $u = F(K_1) = F(x - ct)$

arbitrary function of 1 var.

$\Rightarrow$ the general solution $\boxed{u = F(x-ct)}$

⚠ along these lines u is const (we find a general solution) but we haven't shown these are the unique lines on which u is a const. (add. work → see reading...)

How to find the arbitrary function?



- initial data → the CHAR propagate it for further times
- how to find the solution at a point? find the CHAR that passes through follow the CHAR to $t=0$

$\boxed{\text{IVP}}$ $u(x,0) = f(x) = F(x)$ → we find F

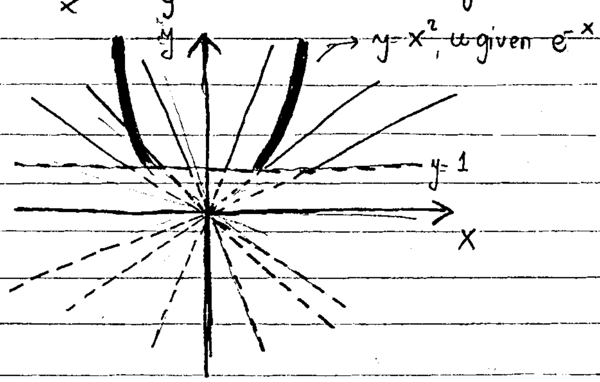
$\Rightarrow \boxed{u(x,t) = f(x-ct)}$

Suppose we are given data on some $x-ct = K_0$ (on a CHAR) → this contradicts the structure of the PDE

An IVP is well-posed if data is given on curves $\neq$ CHAR.

$\boxed{\text{Ex.3}}$ $\underbrace{x}_{a} u_x + \underbrace{y}_{b} u_y = 0$, $y>1$, $-\infty < x < \infty$

$\Rightarrow \dfrac{dx}{x} = \dfrac{dy}{y} \Rightarrow \ln x = \ln y + \bar{K}_1 \Rightarrow \boxed{\dfrac{y}{x} = K_1, \quad y = K_1 x}$ CHAR



Along CHAR, $u = C_1 = F(K_1)$

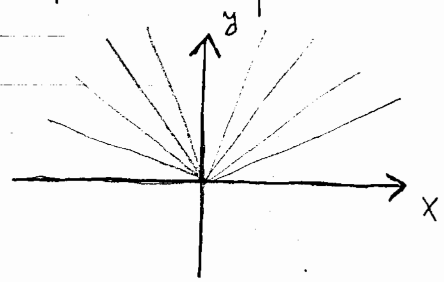
$\boxed{u = F(y/x)}$ generall solution

$u(x, y=x^2) = e^{-x} = F(x)$

data is given on the parabola $y = x^2$ for $y>1$ this curve cross each CHAR only 1 → good initial data

$\Rightarrow \boxed{u = e^{-y/x}}$

Suppose our equation is for $y \geq 0$



all CHAR cross at the origin
u: singular at $(0,0)$

(singular solution expresses as a weird behavior of the CHAR)

example with e^{-x} - not defined at origine
(identity crisis : value at origine depends on which CHAR we take.)

Generalization:
$a(x,y,z)\,u_x + b(x,y,z)\,u_y + d(x,y,z)\,u_z = 0$
$u = u(x,y,z)$ linear, homogeneous

CHAR : $\underbrace{\frac{dx}{a} = \frac{dy}{b}} = \frac{dz}{d}$ $\Rightarrow$ u = const along CHAR

$f(x,y,z) = K_1$ $\quad g(x,y,z) = K_2$
↳ surface $\cap$ ↳ surface = curve in 3D space

Case 2 $\quad a(x,y)\,u_x + b(x,y)\,u_y = d(x,y)$
linear, non-homogeneous

same concept : $\Delta x, \Delta y$ $\quad \begin{cases} \Delta x = a \cdot \varepsilon \\ \Delta y = b \cdot \varepsilon \end{cases}$
$\Rightarrow \Delta u \approx u_x \Delta x + u_y \Delta y = (a\,u_x + b\,u_y)\,\varepsilon = d\,\varepsilon$

$$\varepsilon = \boxed{\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{d(x,y)}}$$

February 11, 2004 Lecture 3

Review of ODEs : Friday 5-6 pm

Review of lecture 2 1st order PDE's

Case 1 $\quad a(x,y)\,u_x + b(x,y)\,u_y = 0$ linear, homogeneous