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EXACT SOLUTION OF DOUBLE FILLED HOLE OF AN INFINITE PLATE

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The plane stress linear elastic solution to the problem of a circular disk embedded in a ring fitted into a uniaxially loaded infinite plate is solved using Airy stress functions. This exact solution is validated by reduction to the benchmark solutions: plate without hole, plate with a circular hole, plate with a circular inclusion, and plate with a ring inclusion. Numerical results of stress distribution are presented for changing material properties.

1. Introduction

It is well known that for a uniaxially loaded infinite plate with a hole a stress concentration of three occurs at the point where the load direction is tangent to the hole boundary [Love 1944; Sokolnikoff 1956; Timoshenko and Goodier 1970; Little 1973; Ugural and Fenster 1994]. Savin [1961] extensively examined the problem of stress concentrations in plates including those related to disk and ring inclusions. The analytical solution for tension applied in one direction in an orthotropic plate with circular filled center is conducted by Lekhnitskii [1968]. Recently, the current researches dealing with the elastic inclusions problem are obtained numerically by Parhi and Das [1972], Greengard and Helsing [1998], Liu et al. [2000], Fanzhong et al. [2002], and Wang et al. [2005]. However, there is no previous literature relating to the exact stress in the double filled hole of an infinite plate (that is, a circular elastic inclusion embedded into a reinforced hole in an elastic plane, and a reinforced ring is used) by different materials to reduce the stress concentration around the hole of plate. This configuration is useful for designing of the filled hole of plate systems in many engineering applications.

In this plane stress study, a linearly elastic disk embedded in an elastic ring is fitted into a hole of an infinite plate, all with the same thickness. All surfaces are seamlessly bonded between three materials that may be different. A proposed general solution without body forces is considered in terms of Airy stress functions so that stresses, strains, and displacements may be calculated. For convenience, rectangular Cartesian coordinates together with polar coordinates are used interchangeably.

Nomenclature

a	constant
A	constant
b	constant
c	constant
d	constant

Keywords: stress function, stress concentration, plane stress, circular hole, inclusion.
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D	domain
E	modulus of elasticity, N/m^2
r	radius coordinate, m
s	uniform uniaxial stress tension, N/m^2
u	component of displacement in radius direction, m
v	component of displacement in tangential direction, m
x	x-coordinate, m
y	y-coordinate, m

Greek symbols

ε	component of strain
Φ	stress function, N
ν	Poisson's ratio
θ	angle, degree
σ	component of stress, N/m^2

2. Theoretical formulation

Figure 1 shows the plane stress problem of an infinite plate subjected to the uniform uniaxial tension s. A disk (material 1) and a ring (material 2) having radius R_1 and R_2, respectively, are seamlessly embedded into an infinite plate (material 3), all with the same thickness, and are assumed linear elastic, isotropic, and homogeneous.

Let u and v be components of displacements in the radial, r, and tangential, θ, directions. If the disk, ring, and plate, are labeled as 1, 2, and 3, respectively, then the regions in space occupied by them denoted by D_i; $i = 1, 2$, and 3 are

$$D_1 = ((r,\theta) : 0 \le r \le R_1, 0 \le \theta \le 2\pi), \qquad D_2 = ((r,\theta) : R_1 \le r \le R_2, 0 \le \theta \le 2\pi),$$
$$D_3 = ((r,\theta) : r \ge R_2, 0 \le \theta \le 2\pi). \tag{1}$$

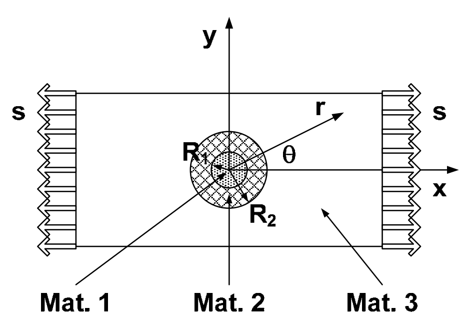

Figure 1. Boundary conditions of the double filled hole of plate.

The conditions at the interface of the disk, ring, and plate are

$$\begin{aligned} r = R_1; &\qquad u_1 = u_2, \qquad v_1 = v_2, \qquad \sigma_{rr1} = \sigma_{rr2}, \qquad \sigma_{r\theta 1} = \sigma_{r\theta 2}, \\ r = R_2; &\qquad u_2 = u_3, \qquad v_2 = v_3, \qquad \sigma_{rr2} = \sigma_{rr3}, \qquad \sigma_{r\theta 2} = \sigma_{r\theta 3}. \end{aligned} \tag{2}$$

The two-dimensional Cartesian ton polar stress transform equations are

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta, \qquad \sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta,$$
$$\sigma_{r\theta} = -(\sigma_{xx} - \sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta). \tag{3}$$

With corresponding equations of equilibrium in polar coordinates

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + F_r = 0, \qquad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + F_\theta = 0. \tag{4}$$

The plane stress stress-strain relations are

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}), \qquad \varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}), \qquad \varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{E}(1+\nu), \tag{5}$$

where the linear strain-displacement relations are given by

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}, \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right). \tag{6}$$

By following an Airy stress function (Φ) approach in which one assumes the body forces are negligible, the governing equations reduce to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) = 0, \tag{7}$$

where the stress components are defined by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \qquad \sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2}, \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right). \tag{8}$$

Consider the Airy's stress function in polar coordinates, written as

$$\Phi_i = a_{0i} + b_{0i}\ln r + c_{0i}r^2 + d_{0i}r^2\ln r + \left(a_{2i}r^2 + b_{2i}r^4 + c_{2i}r^{-2} + d_{2i}\right)\cos(2\theta). \tag{9}$$

From Equations (5)–(9), the general expressions for stresses and displacements can be obtained as follows:

$$
\begin{aligned}
\sigma_{rri} &= 2c_{0i} + d_{0i} + b_{0i} r^{-2} + 2d_{0i} \ln r - (6c_{2i} r^{-4} + 4d_{2i} r^{-2} + 2a_{2i}) \cos(2\theta), \\
\sigma_{\theta\theta i} &= 2c_{0i} + 3d_{0i} - b_{0i} r^{-2} + 2d_{0i} \ln r + (6c_{2i} r^{-4} + 2a_{2i} + 12b_{2i} r^{2}) \cos(2\theta), \\
\sigma_{r\theta i} &= (-6c_{2i} r^{-4} - 2d_{2i} r^{-2} + 2a_{2i} + 6b_{2i} r^{2}) \sin(2\theta),
\end{aligned}
$$

$$
\begin{aligned}
u_i = \frac{1}{E_i}\Big(&\Big(-(1+\nu_i) b_{0i} r^{-1} + 2(1-\nu_i) c_{0i} r - (1+\nu_i) d_{0i} r + 2(1-\nu_i) d_{0i} r \ln r\Big) \\
&+\Big(2(1+\nu_i) c_{2i} r^{-3} + 4d_{2i} r^{-1} - 2(1+\nu_i) a_{2i} r - 4\nu_i b_{2i} r^{3}\Big) \cos(2\theta)\Big) \\
&- A_{0i} \sin(\theta) + A_{1i} \cos(\theta), \\
v_i = \frac{1}{E_i}\Big(&4d_{0i} r\theta + \Big(2(1+\nu_i) c_{2i} r^{-3} - 2(1-\nu_i) d_{2i} r^{-1} + 2(1+\nu_i) a_{2i} r + 2(3+\nu_i) b_{2i} r^{3}\Big) \sin(2\theta)\Big) \\
&+ A_{0i} \cos(\theta) + A_{1i} \sin(\theta) + r A_{2i}.
\end{aligned} \tag{10}
$$

The constants a_{0i}, b_{0i}, c_{0i}, d_{0i}, a_{2i}, b_{2i}, c_{2i}, and d_{2i}; $i = 1, 2, 3$ and A_{0i}, A_{1i}, A_{2i}; $i = 1, 2, 3$ are determined using the interface, boundary, and mathematical conditions in Equation (2).

3. Mathematical implementation

The constants in the Airy stress function are obtained by the following considerations: substitution of Equation (10) into Equation (6) reveals that all strain components are free from A_{0i}, A_{1i}, and A_{2i}, meaning that these constants are related to rigid body motion. It is assumed that the translation and rotational rigid body motions are zero, and that the origin of xy-coordinates is the reference point for zero displacements, so that A_{0i}, A_{1i}, and $A_{2i} = 0$. For the disk, the displacement at $r = 0$ must be finite, so we must set b_{01}, c_{21}, and $d_{21} = 0$. Because polar coordinates are used, it is a requirement that at any r stresses and displacements must be equal if θ is replaced by $\theta + 360°$. Thus, d_{01}, d_{02}, and $d_{03} = 0$. Finally, the eighteen remaining constants: a_{01}, c_{01}, a_{21}, b_{21}, a_{02}, b_{02}, c_{02}, a_{22}, b_{22}, c_{22}, d_{22}, a_{03}, b_{03}, c_{03}, a_{23}, b_{23}, c_{23}, and d_{23} are determined by using the interface, boundary, and mathematical conditions.

The boundary conditions of the plate in polar coordinates are

$$
\sigma_{rr3} = \frac{s}{2}(1+\cos(2\theta)), \qquad \sigma_{\theta\theta 3} = \frac{s}{2}(1-\cos(2\theta)),
$$
$$
\sigma_{r\theta} = -\frac{s}{2}\sin(2\theta). \tag{11}
$$

At the plate, taking the limit $r \to \infty$, the results are expressed as $c_{03} = \frac{s}{4}$, $a_{23} = -\frac{s}{4}$, $b_{23} = 0$.

Therefore, at this stage, the stress and displacement components are reduced to

$$
\begin{aligned}
&\sigma_{rr1} = 2c_{01} - 2a_{21}\cos(2\theta), \qquad \sigma_{r\theta 1} = (2a_{21} + 6b_{21}r^2)\sin(2\theta),\\
&\sigma_{rr1} = 2c_{01} - 2a_{21}\cos(2\theta), \qquad u_1 = \frac{1}{E_1}\Big(2(1-\nu_1)c_{01}r - \big(2(1+\nu_1)a_{21}r + 4\nu_1 b_{21}r^3\big)\cos(2\theta)\Big),\\
&\qquad v_1 = \frac{1}{E_1}\big(2(1+\nu_1)a_{21}r + 2(3+\nu_1)b_{21}r^3\big)\sin(2\theta),\\
&\qquad \sigma_{rr2} = 2c_{02} + b_{02}r^{-2} - (6c_{22}r^{-4} + 4d_{22}r^{-2} + 2a_{22})\cos(2\theta),\\
&\qquad \sigma_{r\theta 2} = \big(-6c_{22}r^{-4} - 2d_{22}r^{-2} + 2a_{22} + 6b_{22}r^2\big)\sin(2\theta),\\
&\qquad u_2 = \frac{1}{E_2}\Big(\big(-(1+\nu_2)b_{02}r^{-1} + 2(1-\nu_2)c_{02}r\big)\\
&\qquad\qquad + \big(2(1+\nu_2)c_{22}r^{-3} + 4d_{22}r^{-1} - 2(1+\nu_2)a_{22}r - 4\nu_2 b_{22}r^3\big)\cos(2\theta)\Big),\\
&v_2 = \frac{1}{E_2}\Big(2(1+\nu_2)c_{22}r^{-3} - 2(1-\nu_2)d_{22}r^{-1} + 2(1+\nu_2)a_{22}r + 2(3+\nu_2)b_{22}r^3\Big)\sin(2\theta),\\
&\sigma_{rr3} = \frac{s}{2} + b_{03}r^{-2} + \Big(\frac{s}{2} - 6c_{23}r^{-4} - 4d_{23}r^{-2}\Big)\cos(2\theta),\\
&\sigma_{r\theta 3} = \Big(-6c_{23}r^{-4} - 2d_{23}r^{-2} - \frac{s}{2}\Big)\sin(2\theta),\\
&u_3 = \frac{1}{E_3}\bigg(\Big(-(1+\nu_3)b_{03}r^{-1} + (1-\nu_3)\frac{s}{2}r\Big) + \Big(2(1+\nu_3)c_{23}r^{-3} + 4d_{23}r^{-1} + (1+\nu_3)\frac{s}{2}r\Big)\cos(2\theta)\bigg),\\
&v_3 = \frac{1}{E_3}\Big(2(1+\nu_3)c_{23}r^{-3} - 2(1-\nu_3)d_{23}r^{-1} - (1+\nu_3)\frac{s}{2}r\Big)\sin(2\theta).
\end{aligned}
\tag{12}
$$

Substitution of Equation (12) into the interface conditions (Equation (2)) yields

$$
\begin{aligned}
2c_{01} &= 2c_{02} + b_{02}R_1^{-2},\\
2a_{21} &= 6c_{22}R_1^{-4} + 4d_{22}R_1^{-2} + 2a_{22},\\
2a_{21} + 6b_{21}R_1^2 &= -6c_{22}R_1^{-4} - 2d_{22}R_1^{-2} + 2a_{22} + 6b_{22}R_1^2,\\
\frac{1}{E_1}(2(1-\nu_1)c_{01}R_1) &= \frac{1}{E_2}\big(-(1+\nu_2)b_{02}R_1^{-1} + 2(1-\nu_2)c_{02}R_1\big),\\
\frac{1}{E_1}\big(2(1+\nu_1)a_{21}R_1 + 4\nu_1 b_{21}R_1^3\big) &= \frac{1}{E_2}\Big(2(1+\nu_2)c_{22}R_1^{-3} + 4d_{22}R_1^{-1}\\
&\qquad - 2(1+\nu_2)a_{22}R_1 - 4\nu_2 b_{22}R_1^3\Big),
\end{aligned}
$$

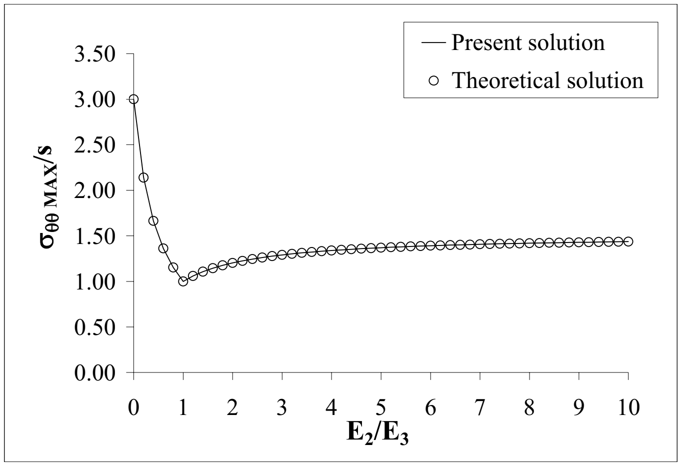

Figure 2. Stress concentration distribution for various modulus of elasticity ratios.

$$\frac{1}{E_1}\big(2(1+\nu_1)a_{21}R_1+2(3+\nu_1)b_{21}R_1^3\big)=\frac{1}{E_2}\Big(2(1+\nu_2)c_{22}R_1^{-3}-2(1-\nu_2)d_{22}R_1^{-1} +2(1+\nu_2)a_{22}R_1+2(3+\nu_2)b_{2i}R_1^3\Big),$$

$$2c_{02}+b_{02}R_2^{-2}=\frac{s}{2}+b_{03}R_2^{-2},$$

$$-6c_{22}R_2^{-4}-4d_{22}R_2^{-2}-2a_{22}=\frac{s}{2}-6c_{23}R_2^{-4}-4d_{23}R_2^{-2},$$

$$-6c_{22}R_2^{-4}-2d_{22}R_2^{-2}+2a_{22}+6b_{22}R_2^2=-6c_{23}R_2^{-4}-2d_{23}R_2^{-2}-\frac{s}{2},$$

$$\frac{1}{E_2}\big(-(1+\nu_2)b_{02}R_2^{-1}+2(1-\nu_2)c_{02}R_2\big)=\frac{1}{E_3}\Big(-(1+\nu_3)b_{03}R_2^{-1}+(1-\nu_3)\frac{s}{2}R_2\Big),$$

$$\frac{1}{E_2}\Big(2(1+\nu_2)c_{22}R_2^{-3}+4d_{22}R_2^{-1}-2(1+\nu_2)a_{22}R_2-4\nu_2b_{22}R_2^3\Big) =\frac{1}{E_3}\Big(2(1+\nu_3)c_{23}R_2^{-3}+4d_{23}R_2^{-1}+(1+\nu_3)\frac{s}{2}R_2\Big),$$

$$\frac{1}{E_2}(2(1+\nu_2)c_{22}R_2^{-3}-2(1-\nu_2)d_{22}R_2^{-1}+2(1+\nu_2)a_{22}R_2+2(3+\nu_2)b_{2i}R_2^3) =\frac{1}{E_3}\Big(2(1+\nu_3)c_{23}R_2^{-3}-2(1-\nu_3)d_{23}R_2^{-1}-(1+\nu_3)\frac{s}{2}R_2\Big).$$

There are three constants a_{01}, a_{02}, and a_{03} left undetermined. It is fortunate that these constants are not used in the expressions of stresses and displacements. Hence, the problem is solved mathematically. The appendix (available as an online supplement to this paper) details the values of all constants, and the stress equations in polar coordinates for all materials.

4. Validations

To confirm that the obtained stress functions are acceptable, benchmark solutions are examined.

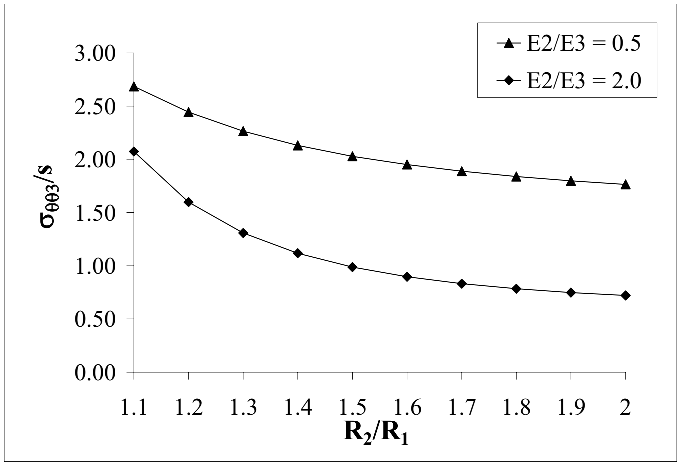

Figure 3. Circumferential stress at a hole in the function of radius ratio.

4.1. ***Plate without hole.*** For the case of a plate without a hole, the material properties of the disk, ring, and plate are set to the same values. For a uniaxial load in the x-direction the stress components are:

$$\sigma_{rr1} = \sigma_{rr2} = \sigma_{rr3} = \frac{s}{2}(1+\cos(2\theta)), \qquad \sigma_{\theta\theta 1} = \sigma_{\theta\theta 2} = \sigma_{\theta\theta 3} = \frac{s}{2}(1-\cos(2\theta)),$$
$$\sigma_{r\theta 1} = \sigma_{r\theta 2} = \sigma_{r\theta 3} = -\frac{s}{2}\sin(2\theta). \tag{13}$$

They are exactly the same expressions as those for a linear elastic, homogeneous, isotropic plate loaded by a uniform normal traction in the x-direction.

4.2. ***Plate with a hole.*** To investigate the case of plate with a hole, the material properties of the disk and ring are set to zero. The problem becomes that of a plate with a hole of radius R_2 subjected to a uniform normal load in the x-direction. The stresses reduce to:

$$\sigma_{rr3} = \frac{s}{2}\left(1-\left(\frac{R_2}{r}\right)^2 + \left(1-4\left(\frac{R_2}{r}\right)^2 + 3\left(\frac{R_2}{r}\right)^4\right)\cos(2\theta)\right),$$
$$\sigma_{\theta\theta 3} = \frac{s}{2}\left(1+\left(\frac{R_2}{r}\right)^2 - \left(1+3\left(\frac{R_2}{r}\right)^4\right)\cos(2\theta)\right),$$
$$\sigma_{r\theta 3} = \frac{s}{2}\left(-1-2\left(\frac{R_2}{r}\right)^2 + 3\left(\frac{R_2}{r}\right)^4\right)\sin(2\theta),$$
$$\sigma_{rr1} = \sigma_{\theta\theta 1} = \sigma_{r\theta 1} = 0, \qquad \sigma_{rr2} = \sigma_{\theta\theta 2} = \sigma_{r\theta 2} = 0. \tag{14}$$

Again reproducing known results of a plate with a hole.

4.3. ***Plate with a circular inclusion.*** Considering a circular inclusion in an infinite plate, this result of doubly embedded elastic materials is reduced into the simple embedded composite material in a hole of an infinite plate by assuming that the material properties of the disk and ring are the same, but different from that of an infinite plate. Figure 2 shows the stress concentration factor distribution for various moduli of elasticity ratios of inclusion and matrix materials. The stress concentration factor decreases quickly with

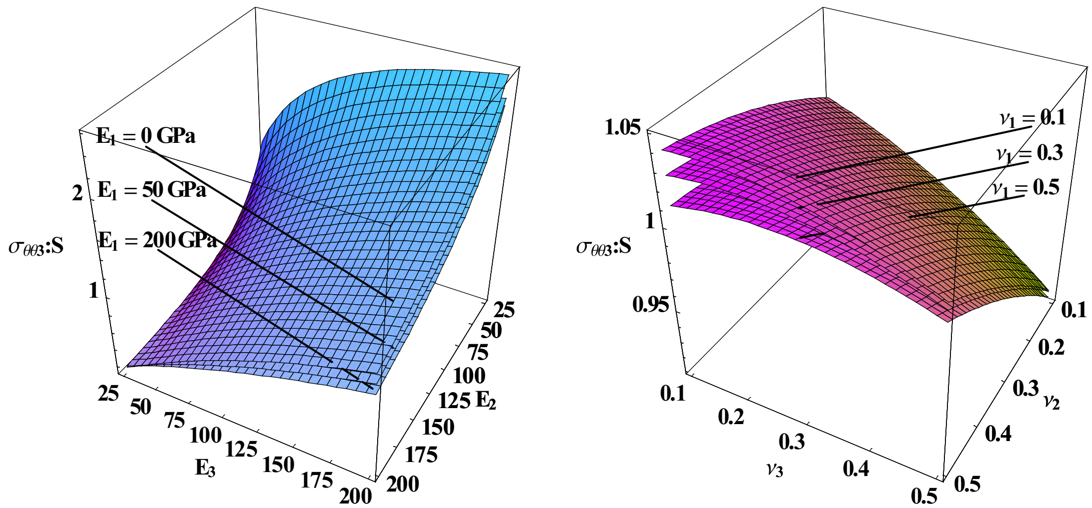

Figure 4. Left: effect of Poisson's ratio on the circumferential stress of plate at the interface ($\theta = 90°$, $R_2 = 2R_1$, $E_1 = E_2 = E_3 = 100\,\text{GPa}$). Right: effect of modulus of elasticity on the circumferential stress of plate at the interface ($\theta = 90°$, $R_2 = 2R_1$, $\nu_1 = \nu_2 = \nu_3 = 0.3$).

an increase in the inclusion-matrix modulus ratio when the inclusion is softer than the matrix. When the inclusion is harder than the matrix, the stress concentration factor increases again slowly. These results coincide with those published previously [Lekhnitskii 1968; Fanzhong et al. 2002; Wang et al. 2005].

4.4. ***Plate with a ring inclusion.*** Figure 3 shows the results for the case of plate with a ring inclusion. The circumferential stress at $\theta = 90°$ is found to decrease slowly. This tendency compares well with the results by Savin [1961] (except that Savin examined a plate in plain strain), and a numerical study by Parhi and Das [1972].

5. Results and discussion

Some numerical results of the circumferential stress distributions in the plate (see Equation (A15) of the online supplement) at the interface $r = R_2$ due to different combinations of material properties are presented in Figure 4 for various values of E and ν, and for different materials. The results reveal that the stresses $\sigma_{\theta\theta 3}$ are very sensitive to the material properties of E_1, E_2, E_3, ν_1, ν_2, and ν_3. For example, to reduce of stress concentrations in bimaterial or trimaterial plates a compromise needs to be found between the material properties of the disk, ring, and plate. Moreover, the ratio between R_1 and R_2 should also be considered because the maximum circumferential stress is significantly depended on the radius ratio when the material properties are fixed.

6. Conclusion

The exact elastic solution of a circular disk embedded in a ring fitted in an infinite plate by different materials is conducted in this study. The plane stress problem in elasticity is considered for a plate subjected to uniaxial uniform load. Boundary, interface, and mathematical conditions are analyzed to determine the solution of stress, strain, and displacement components. Thus, the Airy's stress functions are investigated for a disk, ring, and plate. Expressions in the solution are limited to the assumptions that the three materials are linearly elastic, isotropic, and homogeneous, and the problem is solved by the theory of infinitesimal linear elasticity.

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