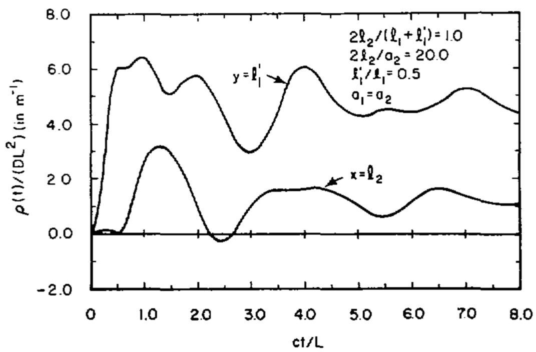

Fig. 16. Time history of charge per unit length at $y = l_1'$ and $x = l_2$ (using 5 poles).

flux density. On Fig. 15 the solid lines beginning at $ct/L = 7.0$ indicate the results of the time independent summation process as one includes one, three, and five poles in the sum. This indicates the rate of convergence of the method to the late time behavior obtained by the limiting process which in a sense should include all pole terms. Fig. 16 shows the time histories for the charge densities at the ends of the other two wire elements, $y = +l_1'$ and $x = l_2$.

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References

[1] C. E. Baum, "Electromagnetic transient interaction with objects with emphasis on finite size objects, and some aspects of transient pulse production," presented at the 1972 Spring URSI meeting, Washington, D.C.
[2] L. Marin and R. W. Latham, "Representation of transient scattered fields in terms of free oscillations of bodies," *Proc. IEEE*, vol. 60, pp. 640–641, May 1972.
[3] F. M. Tesche, "On the analysis of scattering and antenna problems using the singularity expansion technique," *IEEE Trans. Antennas Propagat.*, vol. AP-21, pp. 53–62, Jan. 1973.
[4] L. Marin, "Natural-mode representation of transient scattered fields," *IEEE Trans. Antennas Propagat.*, vol. AP-21, pp. 809–818, Nov. 1973.
[5] ——, "Natural-mode representation of transient scattering from rotationally symmetric bodies," *IEEE Trans. Antennas Propagat.*, vol. AP-22, pp. 266–274, Mar. 1974.
[6] S. A. Schelkunoff and H. T. Friis, *Antennas: Theory and Practice.* New York: Wiley, 1952, p. 370.
[7] C. D. Taylor and D. R. Wilton, "The extended boundary condition solution of the dipole antenna of revolution," *IEEE Trans. Antennas Propagat.* (Commun.), vol. AP-20, pp. 772–776, Nov. 1972.
[8] C. E. Baum, "On the singularity expansion method for the case of first order poles," Air Force Weapons Laboratory Interaction Note 129, Oct. 1972.
[9] T. T. Crow, B. D. Graves, and C. D. Taylor, "Numerical techniques useful in the singularity expansion method as applied to electromagnetic interaction problems," Air Force Weapons Laboratory Mathematics Notes, Note 27, Dec. 1972.

Extended Boundary Condition Integral Equations for Perfectly Conducting and Dielectric Bodies: Formulation and Uniqueness

K. A. AL-BADWAIHY AND J. L. YEN

Abstract—**The equivalence theorem is used to derive novel generalized boundary condition (GBC) integral equations for the tangential components of the electric and magnetic fields on the interfaces of a finite number of dielectric or conducting scatterers. Closed surface, plane, and line extended boundary conditions (EBC) equivalent to the GBC are introduced. The GBC integral equations can now be replaced by any of these EBC integral equations whose solutions are unique and easy to obtain numerically using the moment method. A perfectly conducting sphere and a dielectric sphere in the electrostatic field of two equal and opposite point charges are presented as simple examples of the general procedure.**

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Introduction

The integral equation approach is well suited for the numerical solution of scattering and radiation problems. However, the conventional form of an integral equation for such problems may have either or both of the following two drawbacks. First, the kernel of the integral equation is weakly singular when the source and observation points coincide and secondly, the equation does not have a unique solution at values of k for which resonant modes can exist in the interior, where k is the wave number [1].

Albert and Synge [2] used the reciprocity theorem to formulate the problem of scattering by a perfectly conducting body. Using a dipole field as the auxiliary field, they obtained a generalized boundary condition (GBC) integral equation by requiring the electric field to vanish everywhere inside the scatterer. Because of the analytic continuability of solutions of partial differential equations of elliptic type, Waterman [3] showed that one need only make the field vanish in any portion of the interior to ensure that the field vanishes everywhere inside the scatterer. This is referred to as Waterman's extended boundary condition (EBC).

So far the GBC and the EBC have mostly been applied to perfectly conducting bodies. A novel method for deriving GBC integral equations for dielectric and composite scatterers is presented in this paper. The EBC concept is redefined as the requirement that a set of observables vanishes over an observation domain in the zero-field region. It is shown that the solution is unique if the observation domain is restricted to a closed surface, a portion of a plane, or a portion of a straight line, all completely in the zero-field region, provided the proper observables vanish in each case. This class of EBC, when applied explicitly, is simpler than analytic continuation methods [5] and the conventional field integral equations.

Formulation and Uniqueness of Generalized Boundary Condition Integral Equations

We proceed to derive GBC integral equations for a scattering problem using the equivalence theorem [6]. The unknown functions are the components of the electric and magnetic fields tangential to the interfaces between different media. The number of equations is twice the number of interfaces between dielectrics plus the number of interfaces between perfect conductors and dielectrics. The dielectric bodies may be lossy or lossless.

Single Scatterer

Fig. 1(a) shows a homogeneous body with linear constitutive parameters (ε_1,μ_1) and impressed sources $(\boldsymbol{J}_1,\boldsymbol{M}_1)$ bounded by a closed surface S and embedded in an infinite homogeneous host medium (ε_0,μ_0) with impressed sources $(\boldsymbol{J}_0,\boldsymbol{M}_0)$. It is required to find the tangential components of the electric and magnetic fields on S.

From the equivalence theorem [6], the field outside S can be maintained by surface currents $\boldsymbol{j}_1 = \boldsymbol{n} \times \boldsymbol{H}$ and $\boldsymbol{m}_1 = \boldsymbol{E} \times \boldsymbol{n}$ on S, where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields on S, respectively, and $\boldsymbol{n}$ along the outward normal. As shown in Fig. 1(b) these currents together with the external sources $(\boldsymbol{J}_0,\boldsymbol{M}_0)$ produce the same exterior field as the original problem but zero

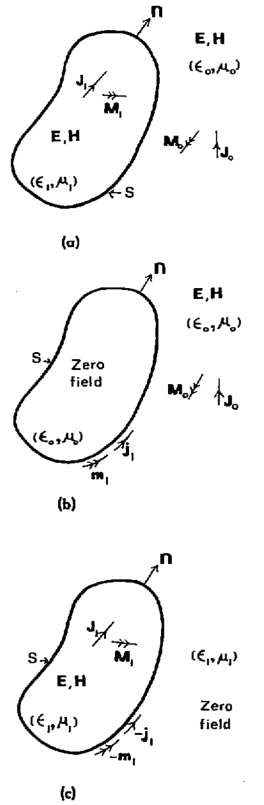

Fig. 1. Single scatterer and generalized boundary condition component problems.

field in V which can now be assumed filled with the host medium. Since the currents are now radiating in a homogeneous unbounded space, the field everywhere can be simply obtained using the unbounded Green's function. A similar argument can be used to show that the surface currents $(-j_1, -m_1)$ along with (J_1, M_1) radiating in a homogeneous unbounded space of constitutive parameters (ε_1, μ_1) produce the same interior field as the original problem but zero field outside S, as is shown in Fig. 1(c). We have thus derived two "component" scattering problems from the original problem, each of which involves a single medium. The conditions that the fields vanish in appropriate regions in the component problems constitute the GBC's for the original problem.

Two coupled integral equations for (j_1, m_1) can be easily written by imposing the zero-field conditions in Figs. 1(b) and 1(c). These are the GBC integral equations for the original scattering problem. Use can be made of the following integral representation [7]:

$$E(r) = TE^{\text{inc}}(r) - \frac{T}{4\pi} ⨍_S \{j\omega\mu(n' \times H)\phi - (n' \times E) \times \nabla'\phi - (n' \cdot E)\nabla'\phi\}\, ds'$$

$$H(r) = TH^{\text{inc}}(r) + \frac{T}{4\pi} ⨍_S \{j\omega\varepsilon(n' \times E)\phi + (n' \times H) \times \nabla'\phi + (n' \cdot H)\nabla'\phi\}\, ds' \quad (1)$$

where r is the observation point position vector. $⨍$ is used to denote the principal value integral over S [7], E^{inc} and H^{inc} are the contributions due to the impressed sources in each case. The vector operations indicated are performed in the primed source coordinates and

$$\phi = \frac{e^{-jk|r-r'|}}{|r - r'|}, \qquad r' \in S.$$

The factor T equals 1 or 2 if the observation point lies outside or on a smooth portion of S, respectively. If the observation point lies on an edge, then

$$T = 1\Big/\left(1 - \frac{\Omega}{4\pi}\right)$$

where Ω depends on the edge solid angle and is defined in [7, p. 163]. In (1) S has a tangent that may not be an analytic function of position at all points on S, but the field is nonetheless required to have a finite mean value. Since many equivalent sources outside a given region can produce the same field in that region, it is necessary to show that these two equations uniquely define (j_1, m_1).

Uniqueness

Let (j_1, m_1) and (j_1', m_1') both satisfy the GBC in Figs. 1(b) and 1(c). It then follows that $(\Delta j = j_1 - j_1', \Delta m = m_1 - m_1')$ produce zero field both outside and inside S. In particular, the tangential components $E_t^+, H_t^+, E_t^-, H_t^-$ vanish on S where the first two result when S is approached from outside and the latter result when S is approached from inside. This is independent of the two media involved. Since $\Delta j = n \times (H_t^+ - H_t^-)$ and $\Delta m = (E_t^+ - E_t^-) \times n$, it follows that $\Delta j = \Delta m = 0$ everywhere on S and uniqueness is proved. Because the tangential components of both E and H are thus uniquely specified on S, the field outside or inside S is uniquely determined because of the general uniqueness theorem of the electromagnetic field [6].

Composite Scatterers

The preceding formulation can now be extended to multiple composite scatterers. Consider a finite number N of regions V_i bounded by closed surfaces S_i, each occupied by a medium whose constants are (ε_i, μ_i) and having impressed sources (J_i, M_i), $i \in [1, N]$, all embedded in an unbounded host medium (ε_0, μ_0) with impressed sources (J_0, M_0). Let $j_i = n_i \times H_i$ and $m_i = E_i \times n_i$ where E_i and H_i are the electric and magnetic fields on the surface S_i and n_i is the outward normal to S_i. The equivalence theorem can be used as in the case of a single scatterer to arrive at a coupled set of integral equations for the unknowns (j_i, m_i), each of which is governed by a component scattering problem involving a single medium.

The procedure is best explained by considering the case of two scatterers depicted in Fig. 2(a). One GBC integral equation for (j_2, m_2) is obtained from the component problem of Fig. 2(b) by requiring the field to vanish outside S_2. Another equation for (j_3, m_3) is obtained from the component problem of Fig. 2(c) by requiring the field to vanish inside S_3. Finally, two equations are obtained from the third component problem of Fig. 2(d) by requiring the field to vanish in V_2 and outside S_3, respectively. To simplify computation in each component problem we assume the zero-field regions are filled with the same medium in which the actual field is retained. The uniqueness of the general many-body formulation can be proved easily following the same reasoning as in the single body case.

Extended Boundary Condition Integral Equations

If one restricts the observation point in the GBC integral equations to subdomains in the zero-field regions, EBC integral equations are obtained. If such observation domains do not intersect the bounding surfaces of the zero-field regions, the resulting EBC integral equations will have regular kernels, a desirable feature for numerical solution. The line type EBC was used before to study the dipole antenna of revolution [2], [8], [9]. A boundary condition closely related to the closed surface type was also used [10]. However, the EBC has only been used

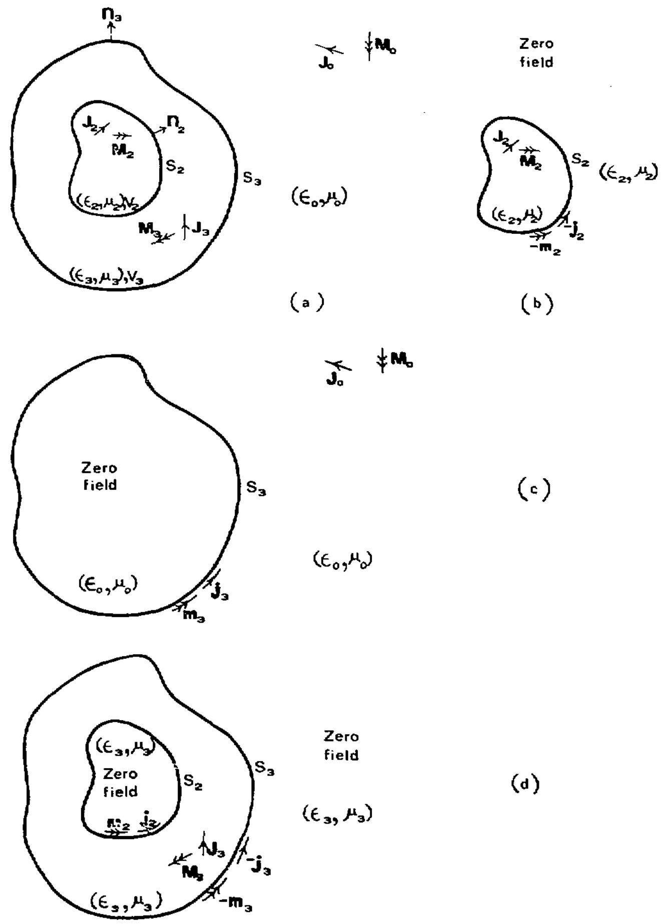

Fig. 2. Multiple composite scatterer and generalized boundary condition component problems.

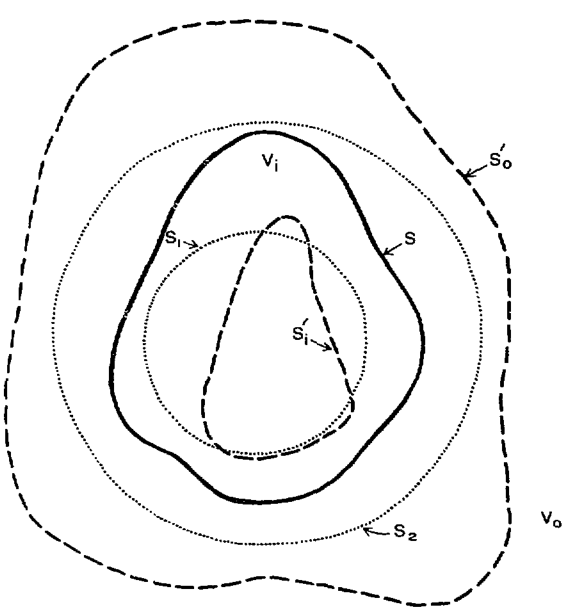

Fig. 3. Extended boundary condition for interior and exterior problems.

for axially symmetric cases and its uniqueness has not been thoroughly investigated. In the following we consider the EBC under general asymmetric fields and show their uniqueness.

Let closed surface S divide space into two regions V_i and V_0 inside and outside, respectively, as is shown in Fig. 3. Let V stand for the zero-field region (V can be either V_i or V_0) and let z be an arbitrary direction. We classify EBC as follows.

a) Waterman's EBC: The observation domain is a subvolume $V' \in V$ and the observables are two scalars defining the electromagnetic field in V.

b) Closed Surface EBC: The observation domain is a closed surface $S' \in V$ and the observables are either the tangential component of the electric or the magnetic field or both or a linear combination of both. S' may coincide with S in a limiting process.

c) Plane EBC: The observation domain is a portion of the XY plane totally contained in V and the observables are E_z, H_z, $(\partial E_z/\partial z)$, and $(\partial H_z/\partial z)$.

d) Line EBC or Axis Boundary Condition ABC: The observation domain is a portion of the Z axis contained in V and the observables are

$$\int_0^{2\pi} \left(\frac{\partial}{\partial \rho}\right)^n \begin{pmatrix} E_z \\ H_z \end{pmatrix} \begin{pmatrix} \cos n\phi \\ \sin n\phi \end{pmatrix} d\phi = 0, \qquad \rho = 0, \quad n \in [0,\infty] \tag{2}$$

where (ρ,ϕ,z) is a cylindrical coordinate system.

Uniqueness

Although the EBC is not unique in general, the particular choices given here are unique as shown in the following.

a) Waterman's EBC: This condition can be analytically continued to show that the field vanishes everywhere in V and is thus unique for both interior and exterior problems [3]. For the interior problem the field is required to vanish in the open volume bounded internally by the spherical surface S_2 (see Fig. 3). S_2 encloses S and can touch it at any number of points.

b) Closed Surface EBC: There are three possible forms of the extended boundary condition where S' stands for either S_i' or S_0' for exterior and interior problems, respectively.

1) Either $\boldsymbol{E}_t$ or $\boldsymbol{H}_t$ vanishes over S': Since Poynting's vector vanishes over S' and noting that there are no sources outside S_0' or inside S_i' and using the radiation condition for the interior problem, it follows from Poynting's theorem that if the medium is lossy, then the field vanishes everywhere in V' and hence in V by the same argument of analytic continuation of Waterman's EBC. However, for lossless media Poynting's theorem gives

$$\operatorname{Im} \int_{S'} (\boldsymbol{E}_t \times \boldsymbol{H}_t^*) \cdot d\boldsymbol{s} = 2\omega \int_{V'} \left(\frac{\mu H^2}{2} - \frac{\varepsilon E^2}{2}\right) dv \tag{3}$$

and hence this condition is not unique for both interior and exterior problems at values of the wave number k for which the electric and magnetic energies stored in V' are equal.

2) Both $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ vanish over S_i': The electromagnetic field in V' due to sources outside V' is the same as that produced by a surface distribution of electric and magnetic currents with densities equal to the components of the magnetic and electric fields tangential to S', respectively. Since these components are zero on S', it follows that the field vanishes everywhere in V' and hence in V by the same argument of analytic continuation in Waterman's EBC. This condition is thus unique for lossy or lossless media.

3) $(\alpha \boldsymbol{E}_t + \boldsymbol{n} \times \boldsymbol{H}_t) = 0$ over S_i' for exterior problems: Re $\{\alpha\} > 0$ and α can be arbitrarily specified over S_i' subject to the preceding constraint. Let us apply Poynting's theorem to V_i' bounded by S_i' under the preceding condition, noting that no sources exist in V_i':

$$\operatorname{Re} \int_{S_i'} (\boldsymbol{E}_t \times \boldsymbol{H}_t^*) \cdot d\boldsymbol{s} = 0 = \int_{S_i'} \operatorname{Re} \{\alpha\} |\boldsymbol{E}_t|^2 \, ds. \tag{4}$$

Since Re $\{\alpha\} > 0$, then $\boldsymbol{E}_t = \boldsymbol{H}_t = 0$ on S_i'; hence EBC 3 is equivalent to EBC 2 and both are unique. Let us now consider

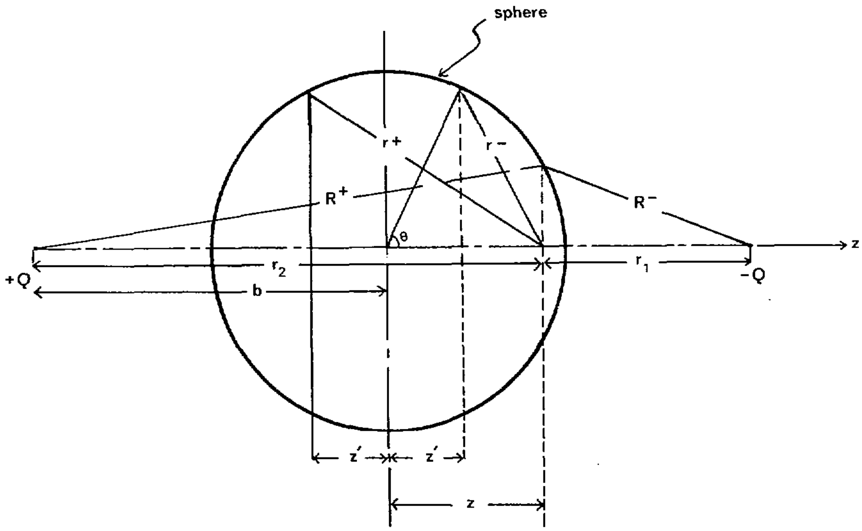

Fig. 4. Sphere in electrostatic field of two point charges.

the limiting case where S_i' is allowed to coincide with S. Condition 3 thus gives

$$j + \alpha(n \times m) = T(n \times H_t^{\mathrm{inc}} + n \times H_t^s + \alpha E_t^{\mathrm{inc}} + \alpha E_t^s). \tag{5}$$

c) Plane EBC: It is shown in the Appendix that if $E_z = (\partial E_z/\partial z) = 0$ on a portion of the XY plane in V, then E_z vanishes everywhere in V. The same applies to H_z. Assuming no TEM modes in V, all other field components can be related to E_z and H_z. Thus, all the field components vanish everywhere in V. This is true for both interior and exterior problems and hence this condition is unique.

d) Line EBC: It is shown in the Appendix that if E_z and H_z both satisfy (2), then they vanish everywhere in V. Assuming no TEM modes, then all other components vanish in V and this condition is thus unique. If a scatterer is axially symmetric then a very convenient choice of the Z axis is the axis of symmetry. Moreover, if a Fourier series representation for the angular dependence of the surface currents is used then the integration in (2) can be evaluated analytically, yielding direct integral equations for the expansion coefficients.

In a moment method solution with a suitable choice of the basis functions, the aforementioned class of EBC yields good results with a considerable reduction in computer time and storage requirements as compared to conventional integral equations. The preceding proof of the uniqueness of this class of EBC integral equations removes doubts about them and should stimulate efforts in finding efficient computer algorithms for their solution.

Examples

Two electrostatic examples are considered here. The first is a perfectly conducting sphere and the second is a dielectric sphere where each sphere is placed in the electrostatic field of two equal and opposite point charges located at equal and opposite distances from the center of the sphere. General electromagnetic problems can be dealt with in a similar manner.

a) Perfectly Conducting Sphere: The configuration is shown in Fig. 4. The line EBC integral equation for the charge density on the sphere is

$$\int_0^1 q(z')\left(\frac{1}{r^-} - \frac{1}{r^+}\right) dz' = Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \qquad |z| < 1 \tag{6}$$

where $r^{\pm} = \sqrt{1 + z^2 \pm 2zz'}$, $r_1 = b - z$, $r_2 = b + z$, $z' = \cos(\theta)$ and $q(\cos\theta) = 2\pi\sigma(\theta)$, where σ is the surface charge density. The potential of the sphere is zero and its radius is 1 m. The exact solution of this integral equation is

$$q(z) = \tfrac{1}{2}Q(b^2 - 1)\left[\left(\frac{1}{R^-}\right)^3 - \left(\frac{1}{R^+}\right)^3\right] \tag{7}$$

where $R^{\pm} = \sqrt{1 + b^2 \pm 2bz}$.

Equation (6) was solved numerically using the moment method with Chebyshev polynomials as basis functions and delta function weights. The matching points were equally spaced along the axis. The various integrals were obtained numerically with a relative error less than 10^{-4} using an iterative Simpson's rule. The expansion coefficients obtained in this way are shown in Table I along with the exact coefficients. The solution convergence is also shown graphically in Fig. 5.

b) Dielectric Sphere: The configuration is the same as in Fig. 4 where the sphere is made of a homogeneous dielectric whose relative permittivity is ε_r. The aforementioned formulation yields the following two coupled integral equations:

$$\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \int_0^1 \mu(z')\left(\frac{1}{r^-} - \frac{1}{r^+}\right) dz' - \int_0^1 \phi(z') \cdot \left[\frac{1 - zz'}{(r^-)^3} - \frac{1 + zz'}{(r^+)^3}\right] dz', \qquad |z| < 1$$

$$\int_0^1 \mu(z')\left(\frac{1}{r^-} - \frac{1}{r^+}\right) dz' = \varepsilon_r \int_0^1 \phi(z')\left[\frac{1 - zz'}{(r^-)^3} - \frac{1 + zz'}{(r^+)^3}\right] dz', \qquad |z| > 1 \tag{8}$$

where $\mu(z') = 2\pi\varepsilon_0 E_r$, E_r is the radial component of the electric field on the surface of the sphere and $\phi(z') = \varepsilon_0 V(\theta)$, where $V(\theta)$ is the potential on the surface of the sphere.

A moment method solution was applied to the preceding coupled integral equations using Chebyshev polynomials as basis functions for both $\phi(z')$ and $\mu(z')$ and delta function weights. The M matching points inside the sphere were evenly spaced along the axis $z_i = (2i - 1)/2M$, $i \in [1,M]$. The N matching points outside the sphere were inversely spaced along the axis $z_i = 2N/(2i - 1)$, $i \in [1,N]$. The expansion of $\mu(z')$ is truncated to M terms while that of $\phi(z')$ is truncated to N terms. The two simultaneous matrix equations obtained can easily be solved

TABLE I

$$a(z) = \sum_{i=1}^{M} a_{2i-1} T_{2i-1}(z)$$

M	a_1	a_3	a_5	a_7	a_9	a_{11}	a_{13}	a_{15}
2	0.941986	0.323228						
3	0.963847	0.330100	0.110000					
4	0.967131	0.334227	0.099718	0.035360				
5	0.967549	0.334572	0.101710	0.028028	0.010453			
6	0.967475	0.334503	0.101656	0.027125	0.012697	-0.001946		
7	0.967578	0.334601	0.102020	0.020996	0.048666	-0.074360	0.047047	
Exact	0.967688	0.334721	0.101638	0.029211	0.008142	0.002226	0.000601	0.00016

TABLE II

$$\mu(z) = \sum_{i=1}^{M} a_{2i-1} T_{2i-1}(z)$$

M	a_1	a_3	a_5	a_7	a_9	a_{11}	a_{13}
2	0.4904	0.1940					
3	0.5046	0.1996	0.0688				
4	0.5067	0.2023	0.0625	0.0230			
5	0.5070	0.2025	0.0637	0.0182	0.0066		
6	0.5071	0.2025	0.0640	0.0171	0.0078	0.000035	
7	0.5072	0.2026	0.0642	0.0131	0.0331	-0.051900	0.03477
Exact	0.50713	0.20255	0.06375	0.01862	0.00524	0.00144	0.00039

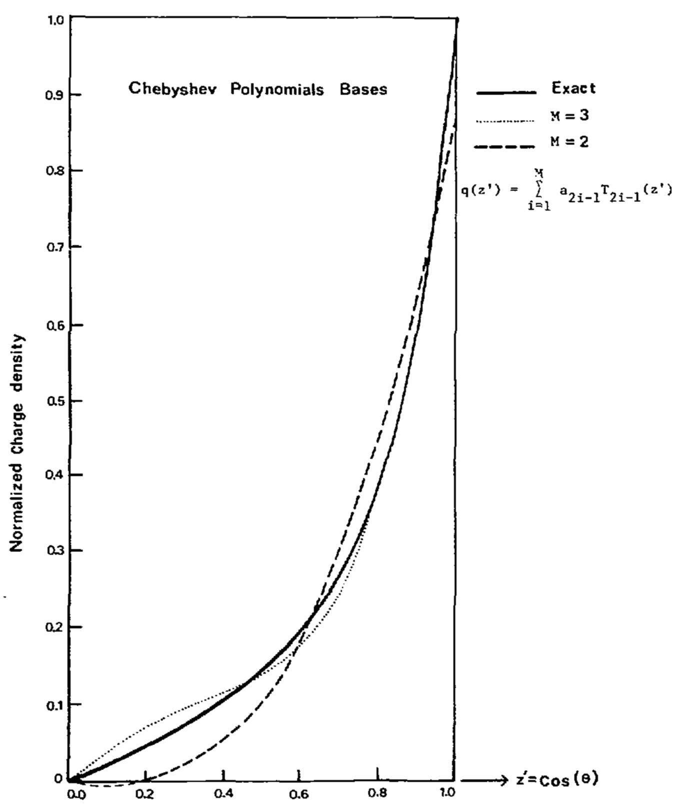

Fig. 5. Numerical solution convergence for perfectly conducting sphere.

to give the expansion coefficients of both $\mu(z')$ and $\phi(z')$. Table II shows the resulting coefficients for $\mu(z')$ and $N = M$ along with the exact coefficients.

Inspection of Tables I and II shows that the present method enables one to obtain accurate low-order solutions. The computer time and storage space needed are considerably reduced compared to standard boundary condition integral equations.

Error Propagation

In a numerical solution algorithm the EBC is only approximately satisfied. The residual error field in the observation domain propagates to the surface S according to the field expansions in the Appendix. The propagation process results in larger errors in the surface quantities than actually exist in the observation domain. Because of simplicity, the line EBC can be carefully handled to provide accurate low-order solutions. If more accurate solutions are needed, the closed surface type EBC seems very promising especially if S' is kept close enough to S.

Conclusions

Integral equations based on generalized boundary conditions are derived for multiple composite scatterers. The formulation applied to both perfectly conducting and dielectric scatterers. The EBC concept is generalized and particular choices of the EBC equivalent to the GBC are given. A combined electric and magnetic field integral equation is also derived. Two examples are given to demonstrate the use of the EBC integral equations.

Appendix

Field Expansion in Cartesian Coordinates

Let ψ be a continuously differentiable solution of the Helmholtz equation in a finite region V. Then ψ can be represented by [6]

$$\psi(x,y,z) = \int F_0(k_1,k_2) h(k_1 x) h(k_2 y) \sin(k_3 z)\, dk_1\, dk_2 + \int F_e(k_1,k_2) h(k_1 x) h(k_2 y) \cos(k_3 z)\, dk_1\, dk_2 \tag{A.1}$$

where F_0, F_e and the integration contour in the complex $k_1 k_2$ plane define ψ in V,

$$\frac{d^2 h(\tilde{k}\xi)}{d\xi^2} = -\tilde{k}^2 h(\tilde{k}\xi)$$

and $k_3^2 = k^2 - k_1^2 - k_2^2$.

Expanding both $\sin(k_3 z)$ and $\cos(k_3 z)$ in powers of $(k_3 z)$ and carrying out the integrations term by term, we obtain

$$\psi(x,y,z) = f_e \cos(kz) + f_0 \sin(kz) + \sum_1^\infty \frac{(-)^m}{m!} \frac{(kz)^{m+1}}{2^m} \cdot (j_m(kz) f_0^m + j_{m-1}(kz) f_e^m) \quad (A.2)$$

where $j_m(\tilde{z})$ is a spherical Bessel function of first kind and

$$f_e = \psi(x,y,0) \qquad f_0 = \frac{1}{k}\frac{\partial \psi}{\partial z}\bigg|_{z=0}$$

$$f_{\substack{e\\0}}^m = \left\{\frac{1}{k^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}^m f_{\substack{e\\0}}.$$

An analytic continuation process can be used to continue (A.2) to points where it may diverge by moving the origin as in [5]. It thus follows that the following holds.

Theorem 1: An analytic solution ψ satisfying the Helmholtz equation vanishes everywhere in V if

$$\psi(x,y,0) = \frac{\partial \psi(x,y,z)}{\partial z}\bigg|_{z=0} = 0$$

over a portion of the XY plane.

Field Expansion in Cylindrical Coordinates

In a cylindrical coordinate system (ρ,ϕ,z), a continuous differentiable solution ψ of the Helmholtz equation in a finite region V is given by [6];

$$\psi(\rho,\phi,z) = \sum_0^\infty \alpha_n(\rho,z)\cos(n\phi) + \sum_1^\infty \beta_n(\rho,z)\sin(n\phi) \quad (A.3)$$

where α_n has the following representation:

$$\alpha_n(\rho,z) = \int F_{en}(\lambda) J_n(\rho\xi)\cos\lambda z \, d\lambda + \int F_{0n}(\lambda) J_n(\rho\xi)\sin(\lambda z)\, d\lambda \quad (A.4)$$

where $\xi = \sqrt{k^2 - \lambda^2}$, Re $(\xi) > 0$, and $J_n(\tilde{z})$ is a Bessel function of first order. F_{en}, F_{0n}, and the integration contour in the complex λ plane define α in V. A similar representation holds for β_n.

Expanding $J_n(\rho\xi)$ in powers of $(\rho\xi)$ and integrating term by term, we get

$$\alpha_n(\rho,z) = \sum_0^\infty \frac{(-)^m}{m!}\left(\frac{k\rho}{2}\right)^m J_{n+m}(k\rho) f_n^m$$

where

$$f_n^m = \left(\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)^m f_n(z) \quad \text{and} \quad f_n(z) = 2^n \left(\frac{1}{k}\frac{\partial}{\partial \rho}\right)^n \alpha_n(\rho,z)\bigg|_{\rho=0}. \quad (A.5)$$

Equation (A.3) can be written as

$$\psi(\rho,\phi,z) = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-)^m}{m!}\left(\frac{k\rho}{2}\right)^m J_{n+m}(k\rho) \cdot [\cos(n\phi) a_n^m(z) + \sin(n\phi) b_n^m(z)] \quad (A.6)$$

where

$$\begin{pmatrix} a_n^m(z) \\ b_n^m(z)\end{pmatrix} = \left(\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)^m \begin{pmatrix} a_n(z) \\ b_n(z)\end{pmatrix}$$

$$\begin{pmatrix} a_n(z) \\ b_n(z)\end{pmatrix} = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{1}{k}\frac{\partial}{\partial\rho}\right)^n \psi\bigg|_{\rho=0} \begin{pmatrix}\cos(n\phi) \\ \sin(n\phi)\end{pmatrix} d\phi,$$

$$a_0(z) = \psi(0,\phi,z). \quad (A.7)$$

The electrostatic version of (A.6) is widely used in electrostatic electron lens calculations and is considered in [11].

An analytic continuation process can be used to continue (A.6) to points where it may diverge by moving the origin as in [5]. It thus follows that the following holds.

Theorem 2: An analytic solution ψ of the Helmholtz equation vanishes everywhere in V if all the a and b given by (A.7) vanish over a portion of a straight line in V.

References

[1] J. C. Bolomey and W. Tabbara, "Numerical aspects on coupling between complementary boundary value problems," *IEEE Trans. Antennas Propagat.*, vol. AP-21, pp. 356–363, May 1973.

[2] G. E. Albert and J. L. Synge, "The general problem of antenna radiation and the fundamental integral equation with application to an antenna of revolution—Part I," *Quart. Appl. Math.*, vol. 6, pp. 117–131, Apr. 1948.

[3] P. C. Waterman, "Matrix formulation of electromagnetic scattering," *Proc. IEEE*, vol. 53, pp. 805–812, Aug. 1965.

[4] ——, "Symmetry, unitarity, and geometry in electromagnetic scattering," *Phys. Rev.*, vol. D3, pp. 825–839, 1971.

[5] D. R. Wilton and R. Mittra, "A new numerical approach to the calculation of electromagnetic scattering properties of two-dimensional bodies of arbitrary cross section," *IEEE Trans. Antennas Propagat.*, vol. AP-20, pp. 310–317, May 1972.

[6] R. F. Harrington, *Time Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961.

[7] R. Mittra, Ed., *Computer Techniques for Electromagnetics.* New York: Pergamon, 1973.

[8] E. Hallén, "Theoretical investigation into transmitting and receiving antennae," *Nova Acta Regiae Soc. Sci. Upsal.*, Ser. 4, vol. 2, p. 1, 1938.

[9] C. D. Taylor and D. R. Wilton, "The extended boundary condition solution of the dipole antenna of revolution," *IEEE Trans. Antennas Propagat.* (Commun.), vol. AP-20, pp. 772–776, Nov. 1972.

[10] Ye. N. Vasil'yev and A. R. Seregina, "A method of calculation of the near field of a dipole antenna," *Radio Eng. Electron. Phys. (USSR)*, vol. 16, no. 6, pp. 930–935.

[11] W. R. Smythe, "Charged right circular cylinder," *J. Appl. Phys.*, vol. 27, pp. 917–920, Aug. 1956.

A Technique to Combine the Geometrical Theory of Diffraction and the Moment Method

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Abstract—A technique is presented in which the moment method (MM) is combined with the geometrical theory of diffraction (GTD). Since diffraction solutions exist for only relatively few structures, it is very desirable to have a means of obtaining the diffracted field for additional structures. Solutions for many structures can be obtained from this combination of techniques, and thus one is able to handle a wide variety of new problems which could not have been solved previously. The approach is developed and applied to a variety of structures in order to illustrate the approach and its validity.

Introduction

For many years, the geometrical theory of diffraction (GTD) has been applied to antenna and scattering problems in which the structure is large in terms of the wavelength. However, diffraction

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