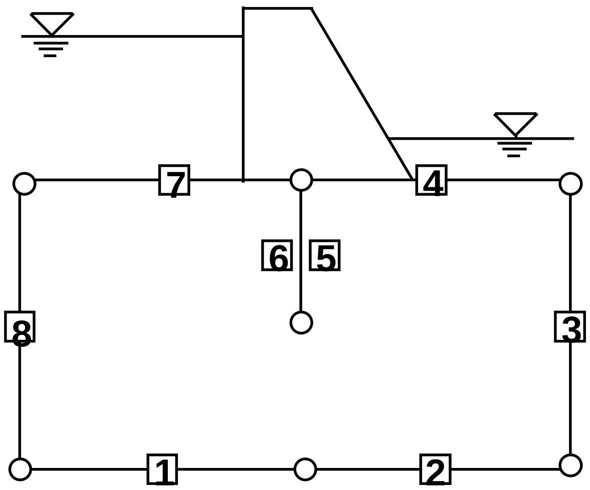
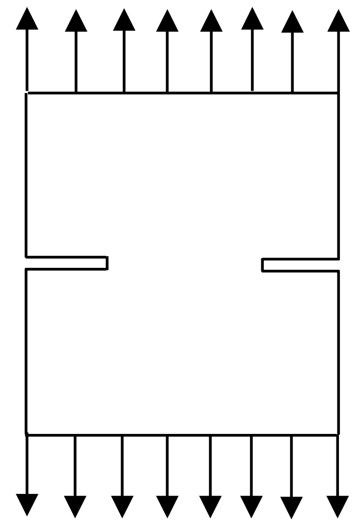
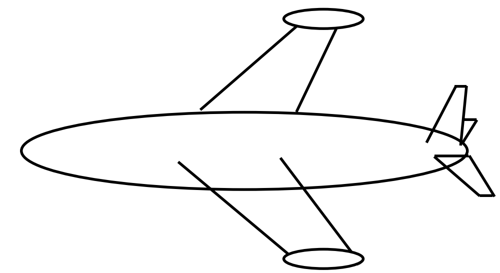
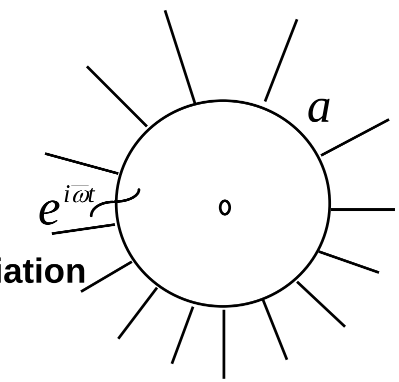
On the Dual Representation Model and Its Applications









J.T. Chen

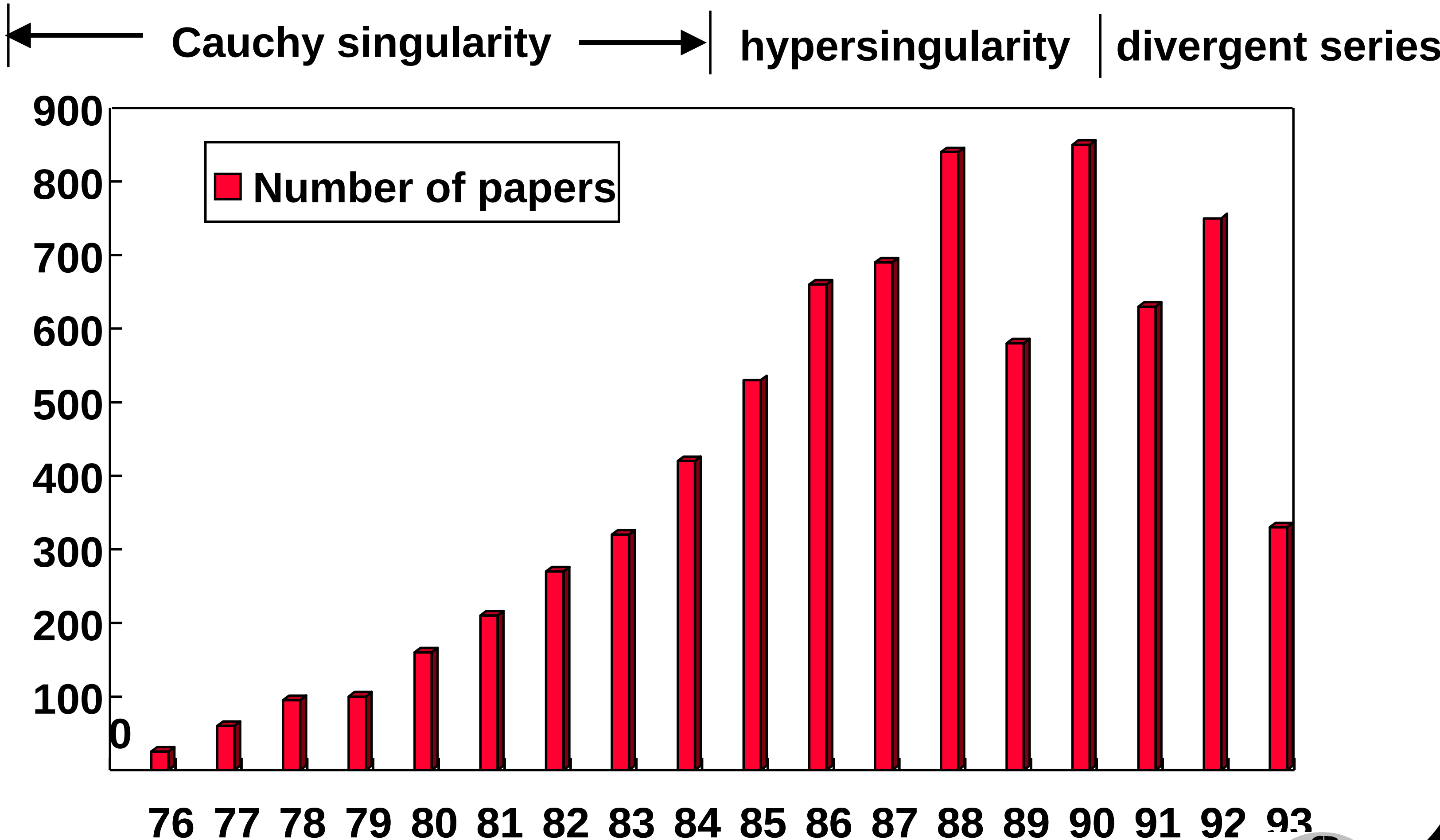
Department of Civil Engineerin, National Taiwan University

Presentation for Institute of Applied Mechanics, May 13, 1994

Outlines

- **Introduction to BEM**
- **Introduction to dual BEM**
- **Theory of dual integral equations**
- **Theory of dual series representations**
- **The roles of dual integral equations**
- **Regularization methods for hypersingularity**
- **Regularization methods for divergent series**
- **Applications of dual integral equations**
- **Applications of dual series representations**
- **Conclusions**

Growth Rate of BEM Papers



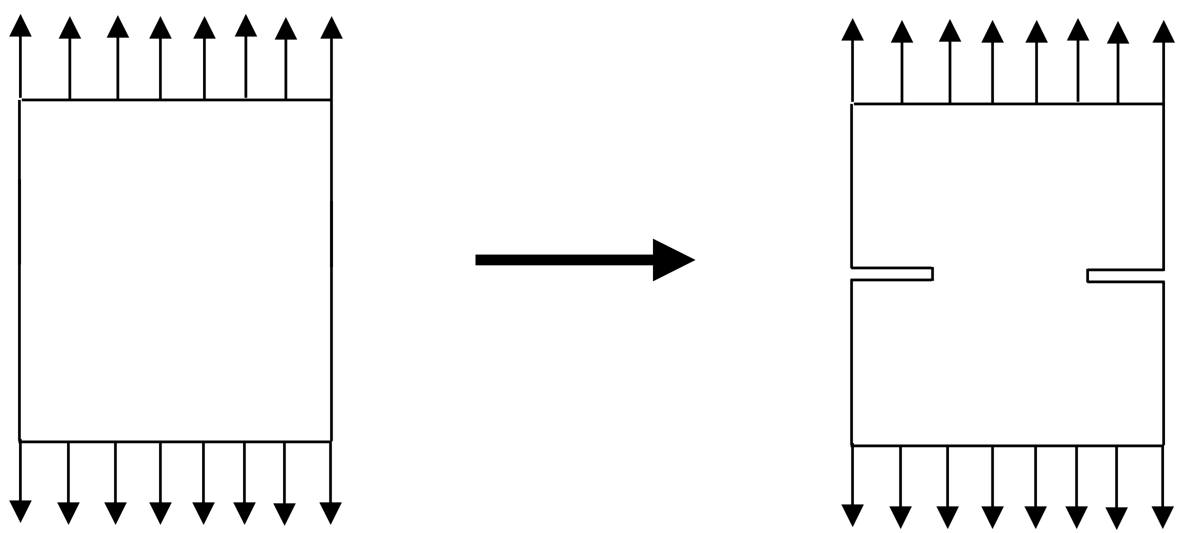
Dual Integral Equations by Hong and Chen(1984-1986)

Singular integral equation → **Hypersingular integral equation**

Cauchy principal value → **Hadamard principal value**

Boundary element method → **Dual boundary element method**

normal boundary

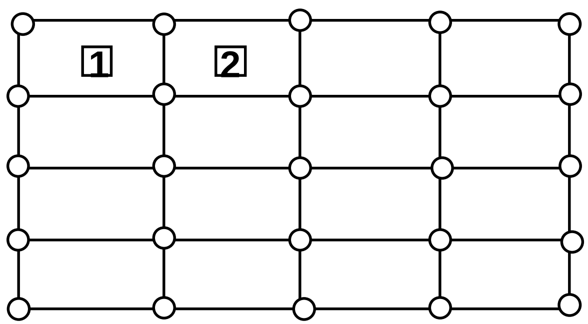


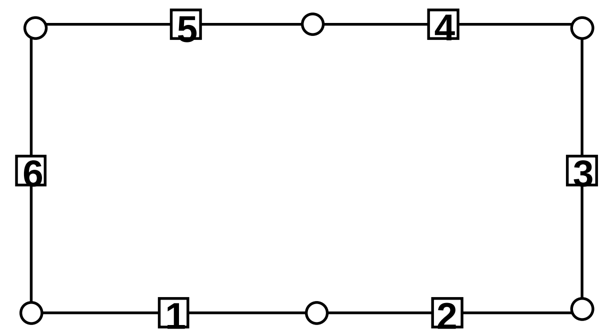
degenerate boundary

What Is Boundary Element Method ?

- **Finite element method**

Boundary element method



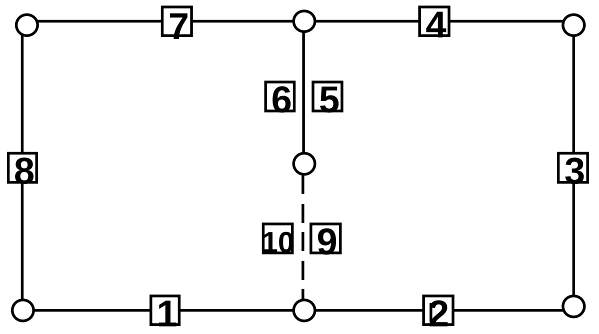


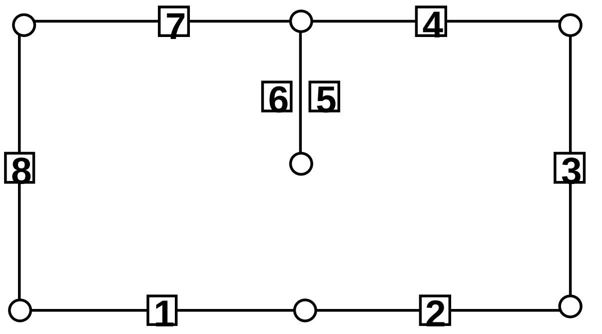
○ **geometry node**

N **the Nth constant or linear element**

What Is Dual Boundary Element Method ?

- **Boundary element method** — **Dual boundary element method**

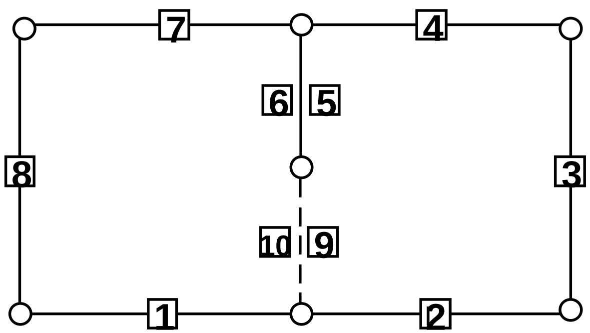




Artifical boundary introduced ! — **Dual integral equations needed !**

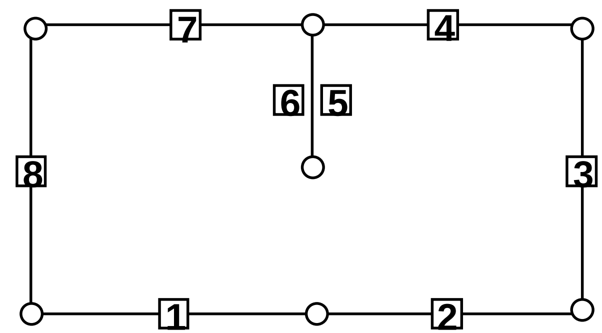
Theory of Dual Integral Equations
Dual Boundary Element Method

- **Boundary element method**



Only U, T equation is used

Dual boundary element method



Both equations are used

$$2\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D$$

$$2\pi t(x) = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)t(s)dB(s), \quad x \in D$$

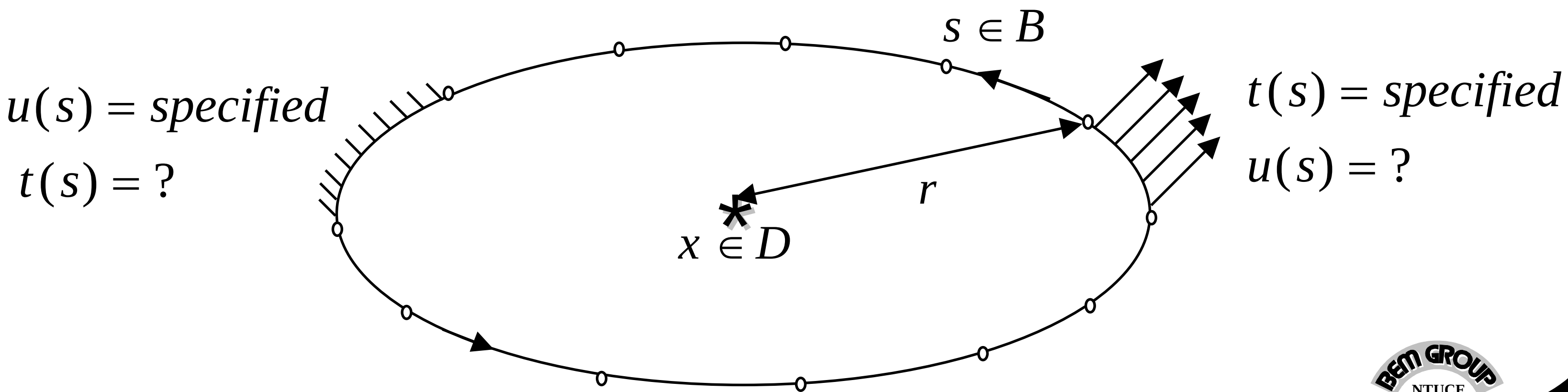
NTUCE BEM GROUP

Theory of Dual Integral Equations

- **Dual integral equations for domain point**

$$2\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s), \quad x \in D$$

$$2\pi t(x) = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)t(s)dB(s), \quad x \in D$$



Theory of Dual Integral Equations

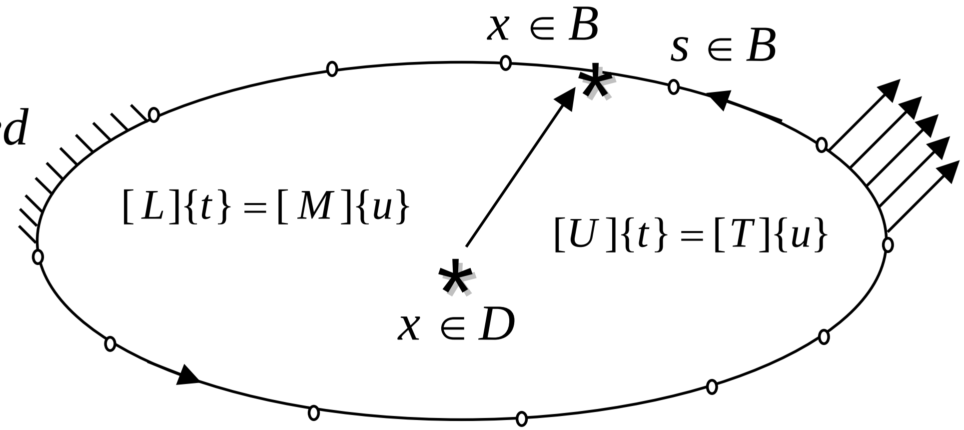
- **Dual integral equations for boundary point**

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), \quad x \in B$$

$$\pi t(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), \quad x \in B$$

$u(s) = specified$

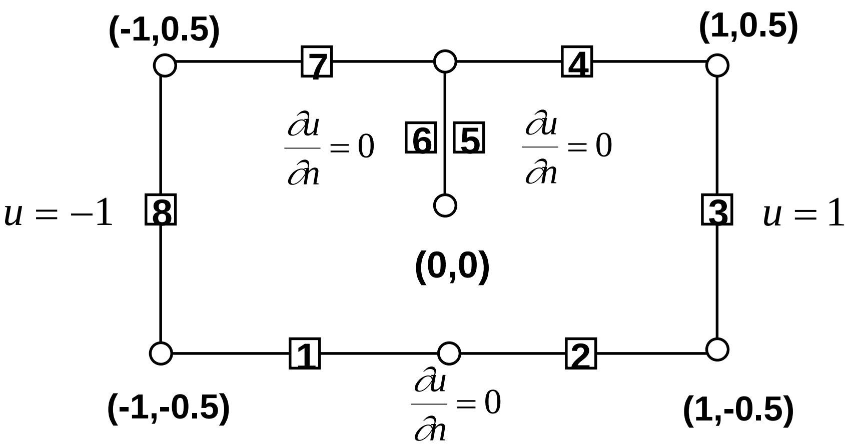
$t(s) = ?$



$t(s) = specified$

$u(s) = ?$

Degeneracy of the Degenerate Boundary



○ **geometry node**

[N] **the Nth constant or linear element**

$$[U]\{t\}=[T]\{u\}$$

$$[L]\{t\}=[M]\{u\}$$

$[U]=$

				5(+)	6(+)			
−1.693	−0.045	0.471	0.347	−0.054	−0.054	0.039	-0.335	
−0.045	−1.693	−0.335	0.039	−0.054	−0.054	0.347	0.471	
0.445	−0.335	−1.693	−0.335	0.019	0.019	0.445	0.703	
0.347	0.039	−0.335	−1.693	−0.281	−0.281	-0.045	0.471	
−0.081	−0.081	0.063	−0.638	−1.193	−1.193	-0.638	0.063	5(+)
−0.081	−0.081	0.063	−0.638	−1.193	−1.193	-0.638	0.063	6(+)
0.039	0.347	0.471	−0.045	−0.281	−0.281	-1.693	-0.334	
−0.335	0.445	0.703	0.445	0.019	0.019	-0.335	-1.693	

$n(s)$

$[T]=$

				5(+)	6(-)			
$-\pi$	0.000	0.588	0.519	−0.321	0.321	0.927	1.107	
0.000	$-\pi$	1.107	0.927	0.321	−0.321	0.519	0.588	
0.219	1.107	$-\pi$	1.107	0.464	−0.464	0.219	0.490	
0.519	0.927	1.107	$-\pi$	0.785	−0.785	0.000	0.588	
0.927	0.927	0.888	1.326	$-\pi$	$-\pi$	1.326	0.888	5(+)
0.927	0.927	0.888	1.326	$-\pi$	$-\pi$	1.326	0.888	6(+)
0.927	0.519	0.588	0.000	−0.7854	0.785	$-\pi$	1.107	
1.107	0.219	0.490	0.219	−0.464	0.464	1.107	$-\pi$	

$n(x)$

$[L]=$

				5(+)	6(+)			
π	0.000	0.184	0.519	0.458	0.458	0.927	0.805	
0.000	π	0.805	0.927	0.458	0.458	0.519	0.184	
0.612	0.805	π	0.805	0.464	0.464	0.612	0.490	
0.519	0.927	0.805	π	0.347	0.347	0.000	0.184	
−0.511	0.511	0.888	1.417	π	$-\pi$	-1.417	-0.888	5(+)
0.511	−0.511	−0.888	−1.417	$-\pi$	π	1.417	0.888	6(-)
0.927	0.519	0.184	0.000	0.347	0.347	π	0.805	
0.805	0.612	0.490	0.612	0.464	0.464	0.805	π	

$n(s)$

$n(x)$

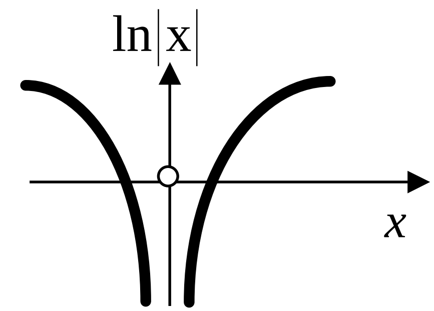
$[M]=$

				5(+)	6(-)			
4.000	−1.333	−0.205	−0.061	0.600	−0.600	-0.800	-1.600	
−1.333	4.000	−1.600	−0.800	−0.600	0.600	-0.061	-0.205	
−0.282	−1.600	4.000	−1.600	−0.400	0.400	-0.282	-0.236	
−0.061	−0.800	−1.600	4.000	−1.000	1.000	-1.333	-0.205	
0.853	−0.853	−0.715	−3.765	8.000	−8.000	3.765	0.715	5(+)
−0.853	0.853	0.715	3.765	−8.000	8.000	-3.765	-0.715	6(-)
−0.800	−0.062	−0.205	−1.333	1.000	−1.000	4.000	-1.600	
−1.600	−0.282	−0.235	−0.282	0.400	−0.400	-1.600	4.000	

Definitions of R.P.V., C.P.V. and H.P.V.

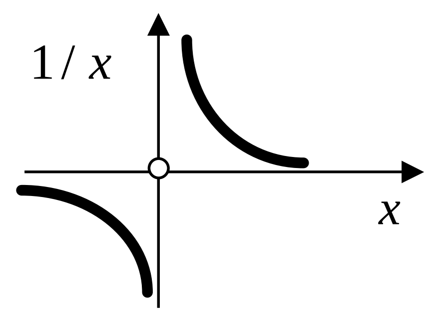
- **R.P.V.**

$$R.P.V.\int_{-1}^{1} \ln|x| \, dx = (x\ln|x| - x) \Big|_{x=-1}^{x=1} = -2$$



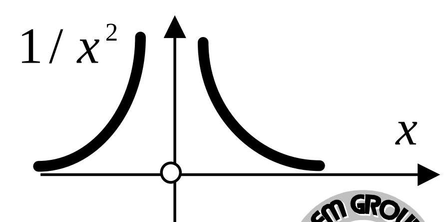
- **C.P.V.**

$$C.P.V.\int_{-1}^{1} \frac{1}{x} \, dx = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{1}{x} \, dx = 0$$



- **H.P.V.**

$$H.P.V.\int_{-1}^{1} \frac{1}{x^2} \, dx = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{1}{x^2} \, dx - \frac{2}{\varepsilon} = -2$$



Regularization methods for hypersingularity

use of simple solutions

1. constant potential
2. rigid body motion
3. complementary solution
   (a). nondegenerate boundary
   (b). degenerate boundary: enclosing technique

kernel function

1. static kernel subtraction
2. quasi-static part decom-position
3. integration by parts reduction one order singular-ity for kernel

density function

1. regularized u

   $u(s) \rightarrow u(s) - u(x)$

2. regularized u, t

   $u(s) \rightarrow u(s) - u(x) - u'(x) r_i \bar{s}_i - t(x) r_i \bar{n}_i$

   $t(s) \rightarrow t(s) - t(x)$

3. integration by parts

   $u(s) \rightarrow u'(s)$

trace to boundary

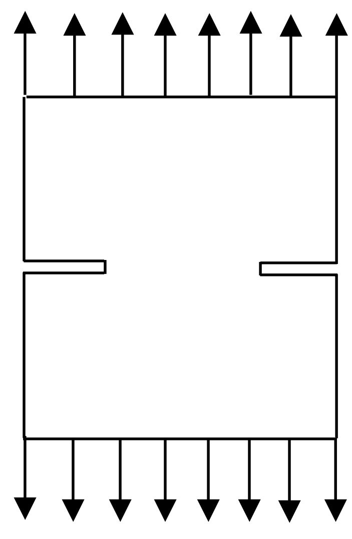
1. limiting process
2. jump function
3. L'Hospital rule
4. Cesaro sum

surrounding integral

1. CPV concept
2. HPV concept
3. free term
4. Introducing boundary terms
   (a). Stokes' theorem
   (b). integration by parts
   (c). Stokes' transformation
   (d). summation by parts
   (e). Leibnitz rule
5. quadrature rule

Applications of Dual Integral Equations

Crack problem



Work in CSIST (1986-1990)

Analysis, Design and Experiment for Solid Rocket Motor

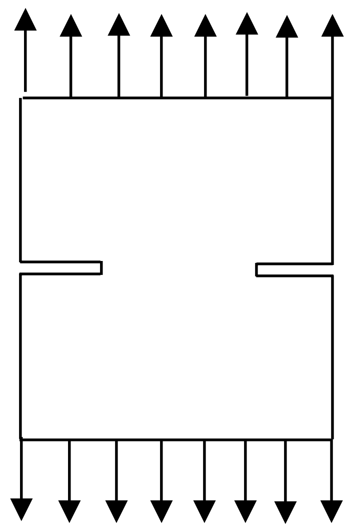
Research in CSIST (1986-1990)

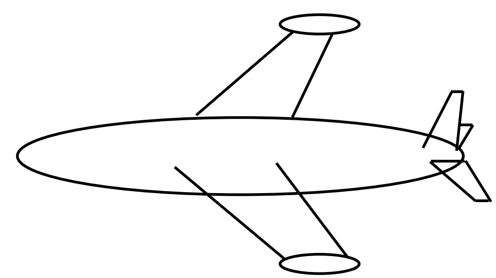
Solid mechanics → **Aerodynamics**

Hadamard principal value → **Mangler principal value**

Boundary element method → **Panel method**

Crack problem → **Thin-airfoil Aerodynamics**





Two Books in CSIST (1986-1990)

Book for academic research

Book for industry applications

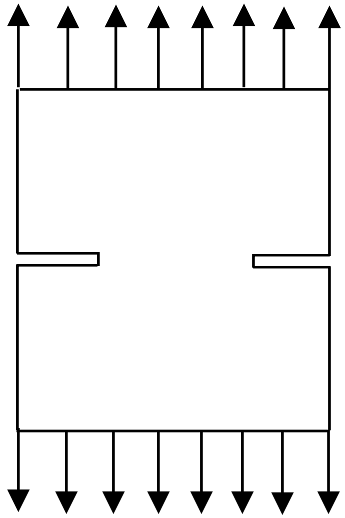
BEM

FEM

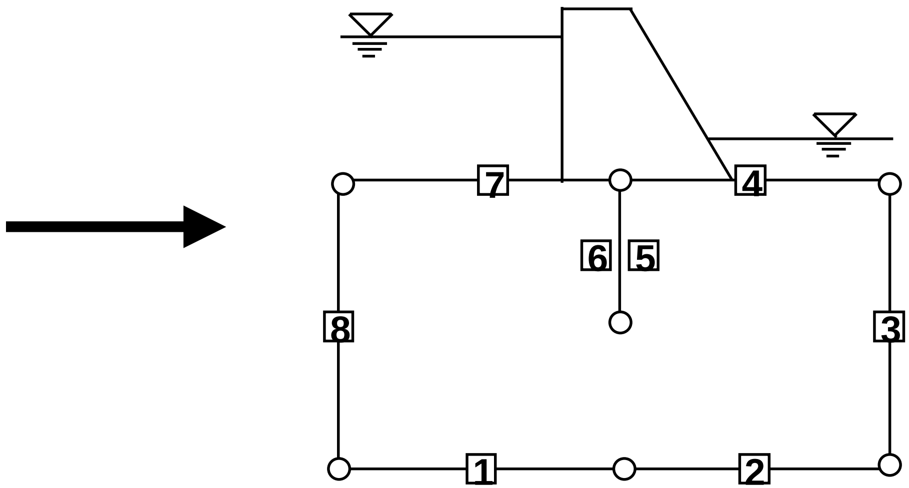
with MSC/NASTRAN

Research in NTUCE (1990-present)

Crack problem



Seepage with sheetpiles



Roles of hypersingularity in boundary element method

complementary constraints

higher order element
1. Hermite element

degenerate boundary
1. cutoff wall
2. sheet pile
3. crack
4. baffle
5. thin airfoil
6. antenna

(o) $\left\{ \begin{array}{l} [U]\{t\} = [T]\{u\} \\ [L^-]\{t\} = [M^-]\{u\} \end{array} \right\}$

(o) $\left\{ \begin{array}{l} [U]\{t\} = [T]\{u\} \\ [L^+]\{t\} = [M^+]\{u\} \end{array} \right\}$

(x) $\left\{ \begin{array}{l} [L^-]\{t\} = [M^-]\{u\} \\ [L^+]\{t\} = [M^+]\{u\} \end{array} \right\}$

corner problem
1. $\left\{ \begin{array}{l} [U]\{t\} = [T]\{u\} \\ [L^-]\{t\} = [M^-]\{u\} \end{array} \right\}$
2. $\left\{ \begin{array}{l} [U]\{t\} = [T]\{u\} \\ [L^+]\{t\} = [M^+]\{u\} \end{array} \right\}$
3. $\left\{ \begin{array}{l} [L^-]\{t\} = [M^-]\{u\} \\ [L^+]\{t\} = [M^+]\{u\} \end{array} \right\}$

fictitious eigenvalue
1. kernel function
2. region of singularity
3. boundary condition

adaptive BEM
1. error estimator

secondary field calculation
1. hoop stress on boundary
2. tangent flux along boundary
3. regularized version for stress near boundary

condition number
1. pseudo-differential operator

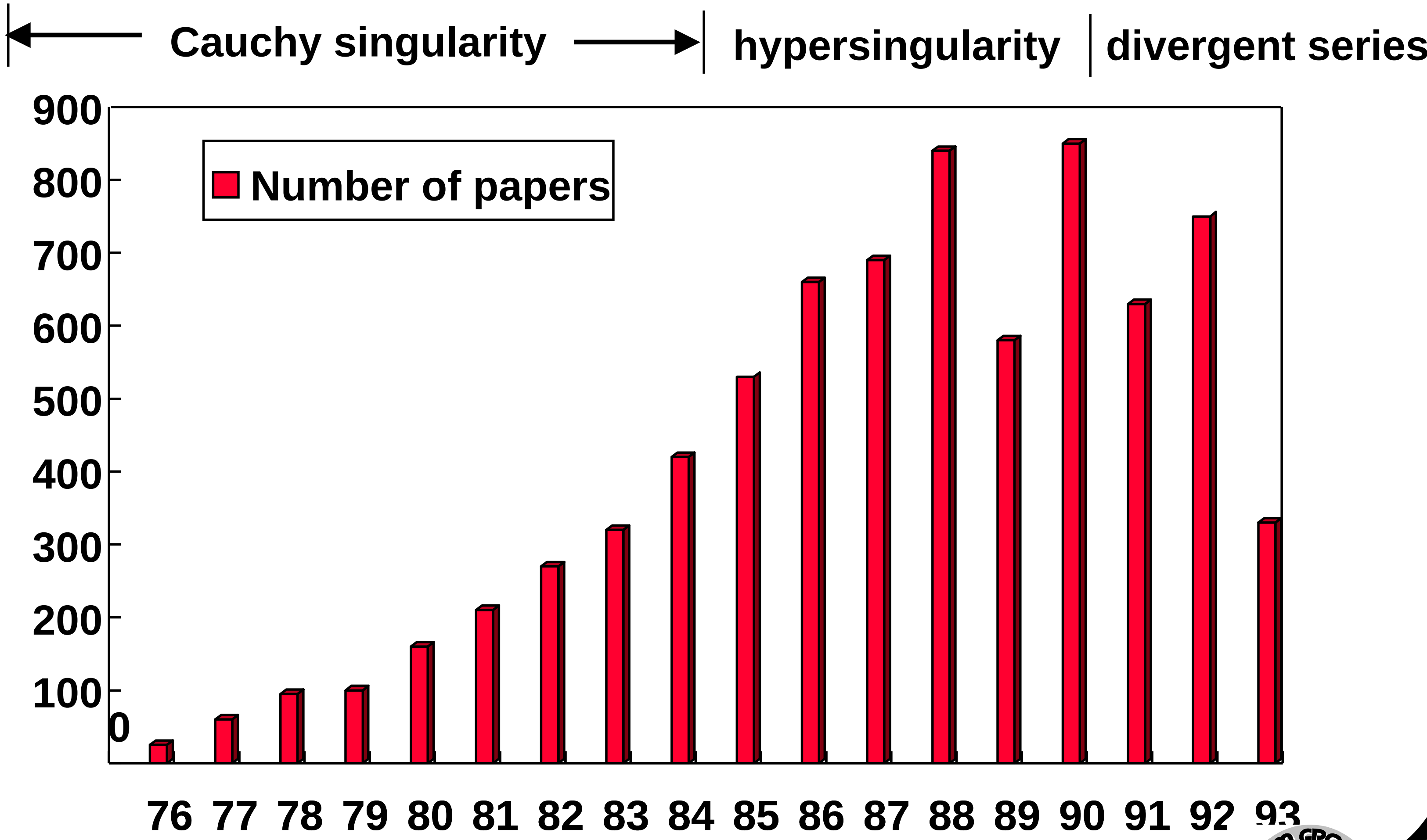
U (-1)	T (0)
L (0)	M (1)

2. T, L is more stable than U, M

symmetry formulation
1. double boundary integration

image system
1. normal vector of dipole or dislo-cation

Growth Rate of BEM Papers

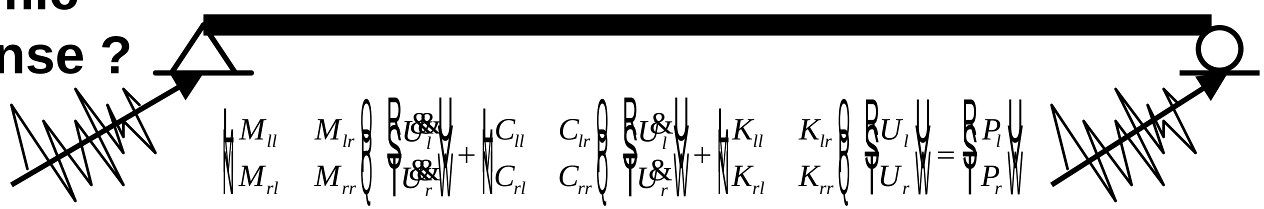


Why Dual Series Representations ?
Support Motion Problems

$$\rho \ddot{u}(x,t) + \left(2\alpha\rho - \beta G \frac{\partial^2}{\partial x^2}\right)\dot{u}(x,t) - G\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

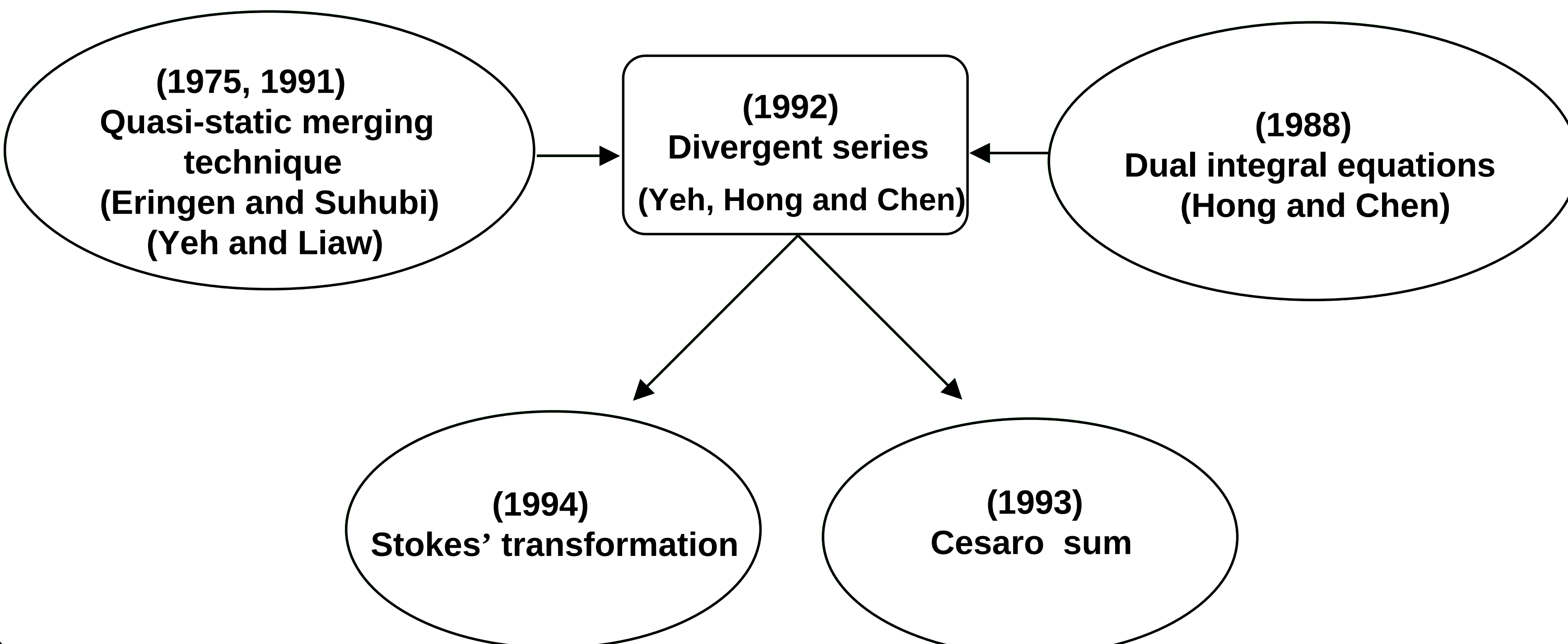
dynamic response ?

random response ?



$$\begin{bmatrix} M_{ll} & M_{lr} \\ M_{rl} & M_{rr} \end{bmatrix}\begin{Bmatrix} \ddot{U}_l \\ \ddot{U}_r \end{Bmatrix} + \begin{bmatrix} C_{ll} & C_{lr} \\ C_{rl} & C_{rr} \end{bmatrix}\begin{Bmatrix} \dot{U}_l \\ \dot{U}_r \end{Bmatrix} + \begin{bmatrix} K_{ll} & K_{lr} \\ K_{rl} & K_{rr} \end{bmatrix}\begin{Bmatrix} U_l \\ U_r \end{Bmatrix} = \begin{Bmatrix} P_l \\ P_r \end{Bmatrix}$$

History of The Research on Divergent Series



Methods of Solution

- **Quasi-static decomposition method (Mindlin and Goodman, 1950)**
  Accelerate the convergence rate
- **Series representation (Eringen and Suhubi, 1975, Yeh and Liaw, 1991)**
  Quasi-static solution is not necessary to be determined
- **Low convergence and divergence will occur using series representation proposed by Pilkey**

- **Two goals**

Omit the calculation of quasi-static solution

Accelerate the convergence rate

Dual Integral Equations and Dual Series Representation

- **Dual Integral Equations:**

$$u(x,t)=\int_0^t\int_B U(s,x;\tau,t)\,\hat{t}(s,\tau)\,dB(s)\,d\tau-\int_0^t\int_B T(s,x;\tau,t)\,\hat{u}(s,\tau)\,dB(s)\,d\tau$$
$$+\int_0^t\int_V U(s,x;\tau,t)\,f(s,\tau)\,dV(s)\,d\tau$$
$$t(x,t)=\int_0^t\int_B L(s,x;\tau,t)\,\hat{t}(s,\tau)\,dB(s)\,d\tau-\int_0^t\int_B M(s,x;\tau,t)\,\hat{u}(s,\tau)\,dB(s)\,d\tau$$
$$+\int_0^t\int_V L(s,x;\tau,t)\,f(s,\tau)\,dV(s)\,d\tau$$

- **Series :**

$$U(s,x;\tau,t) = \lim_{N\to\infty} C(N,1)\,\{\sum_{m=1}^{N} e^{-k\omega_m(t-\tau)}u_m(x)\,u_m(s)/N_m\}$$
$$T(s,x;\tau,t) = \lim_{N\to\infty} C(N,1)\,\{\sum_{m=1}^{N} e^{-k\omega_m(t-\tau)}u_m(x)\,t_m(s)/N_m\}$$
$$L(s,x;\tau,t) = \lim_{N\to\infty} C(N,2)\,\{\sum_{m=1}^{N} e^{-k\omega_m(t-\tau)}t_m(x)\,u_m(s)/N_m\}$$
$$M(s,x;\tau,t) = \lim_{N\to\infty} C(N,2)\,\{\sum_{m=1}^{N} e^{-k\omega_m(t-\tau)}t_m(x)\,t_m(s)/N_m\}$$

$C(N,r)$: Cesaro operator with order r

Cesaro Regularization Technique

- **Series Solution(Partial Sum)**

$$s_0 = a_0$$

$$s_1 = a_0 + a_1$$

$$s_2 = a_0 + a_1 + a_2$$

$$\vdots$$

$$s_{N-1} = a_0 + a_1 + a_2 + \cdots + a_{N-1}$$

$$(\textit{partial sum}) \quad s_N = a_0 + a_1 + a_2 + \cdots + a_{N-1} + a_N \quad (\textit{divergent}, \ N \to \infty)$$

$$\frac{s_0 + s_1 + \cdots + s_{N-1} + s_N}{N+1} = a_0 + \frac{N}{N+1} a_1 + \frac{N-1}{N+1} a_2 + \cdots + \frac{2}{N+1} a_{N-1} + \frac{1}{N+1} a_N \quad (\textit{convergent}, \ N \to \infty)$$

$$(\textit{Cesaro sum}) \quad S_N = \frac{1}{N+1} \sum_{k=0}^{N} (N-k+1) \ a_k \quad (\textit{moving average})$$

Stokes' Transformation --- Summation by Parts

- **Term by Term Differentiation Is Not Always Legal**
- **Boundary Term Is Present for Some Cases**

$$f'(x) = \frac{d}{dx}\left\{ f(x) \right\} = \frac{d}{dx}\left\{ \sum_{k=0}^{N} c_k\, u_k(x) \right\} = \sum_{k=0}^{N} c_k\, u_k'(x) + \underline{\underline{\sum_{k=0}^{N} b_k\, u_k'(x)}}$$

Boundary term

if $\sum_{k=0}^{N} b_k\, u_k'(x) \neq 0$

- **Term by Term Differentiation Is Legal**

if $\sum_{k=0}^{N} b_k\, u_k'(x) = 0$

Regularization Techniques for Derivative of Double Layer Potential Different Points of View

- **Divergent Integral (Hypersingular kernel) :**

$$H.P.V. \int M(s,x)\ u(s) dB(s)$$

- **Divergent Series (Dual series representation) :**

$$C(N,2)\ \left\{\sum_{m=0}^{N} \int_B t_n(s)\ u(s,t) dB(s)\ t_n(x)\right\}$$

- **Cesaro Sum (Arithmetic mean) :**

$$S_N(x,t) = C(N,1)\left\{\sum_{m=0}^{N} a_m(x,t)\right\} = \frac{s_0(x,t)+s_1(x,t)+\cdots+s_{N-1}(x,t)+s_N(x,t)}{N+1}$$

- **Reproducing Kernel (Fejer kernel) :**

$$F_{N+1}(x) = \frac{1}{2\pi(N+1)} \frac{\sin^2((N+1)x/2)}{\sin^2(x/2)}$$

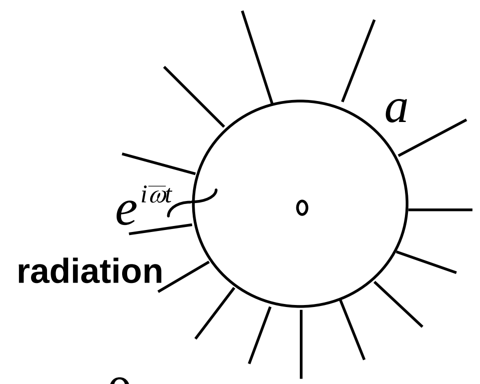
- **Moving Average (MA model) :**

$$S_N(x,t) = \frac{1}{N+1}\sum_{m=0}^{N} (N-m+1)\ a_m(x,t)$$

- **Stokes' Transformation (Summation by parts) :**

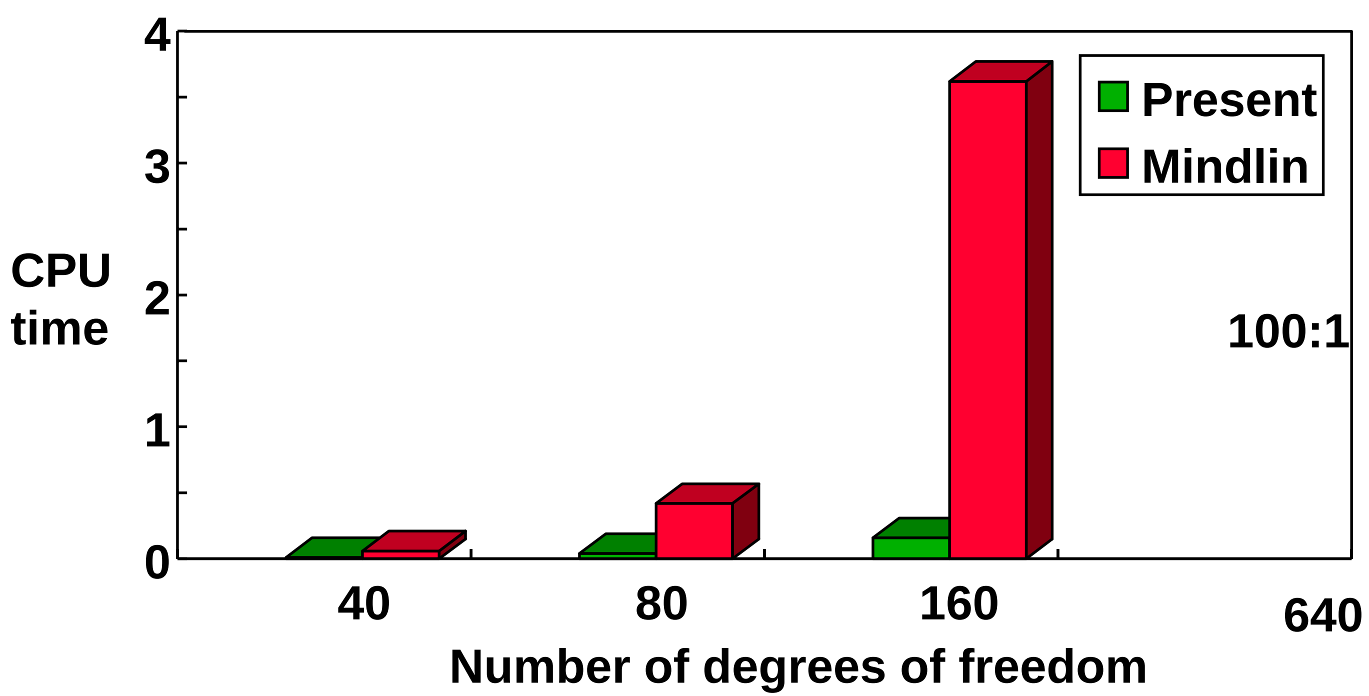
$$f'(x) = \frac{d}{dx}\left\{f(x)\right\} = \frac{d}{dx}\left\{\sum_{k=0}^{N} c_k\ u_k(x)\right\} = \sum_{k=0}^{N} c_k\ u'_k(x) + \sum_{k=0}^{N} b_k\ u'_k(x)$$

Applications of Dual Series Representations



$$u = \lim_{\bar{\omega} \to \omega} \frac{0}{0} \to \textit{finite}, \text{ if } \bar{\omega} \to \omega$$

Comparisons of CPU Time



Reproducing Techniques for Solutions

Solution $u(x,t)$ **?**

Modal dynamics | **Fourier integral**

$$u(x,t)=\sum_{k=0}^{k=n}\bar{q}_k(t)\,u_k(x)$$

?

$$u(x,t)=\int_{-\infty}^{\infty}U(x,\omega)\,e^{-i\omega t}\,d\omega$$

Reproducing

$$u(x,t)=C(N,r)\sum_{k=0}^{k=N}\bar{q}_k(t)\,u_k(x)$$

$$u(x,t)=\int_{-\pi}^{\pi}K_N^r(x-s)\,u(s,t)\,ds$$

$$u(x,t)=\int_{-\infty}^{\infty}U(x,\omega)\,\mathrm{W}(\omega)\,e^{-i\omega t}\,d\omega$$

$$u(x,t)=\sum_{k=-N}^{k=N}U(x,\omega_k)\,w_k^r\,e^{-i\omega t}\,\Delta\omega$$

$$K_N^1(x-s)=\frac{1}{(N+1)}\frac{\sin^2((N+1)(x-s)/2)}{\sin^2((x-s)/2)}$$

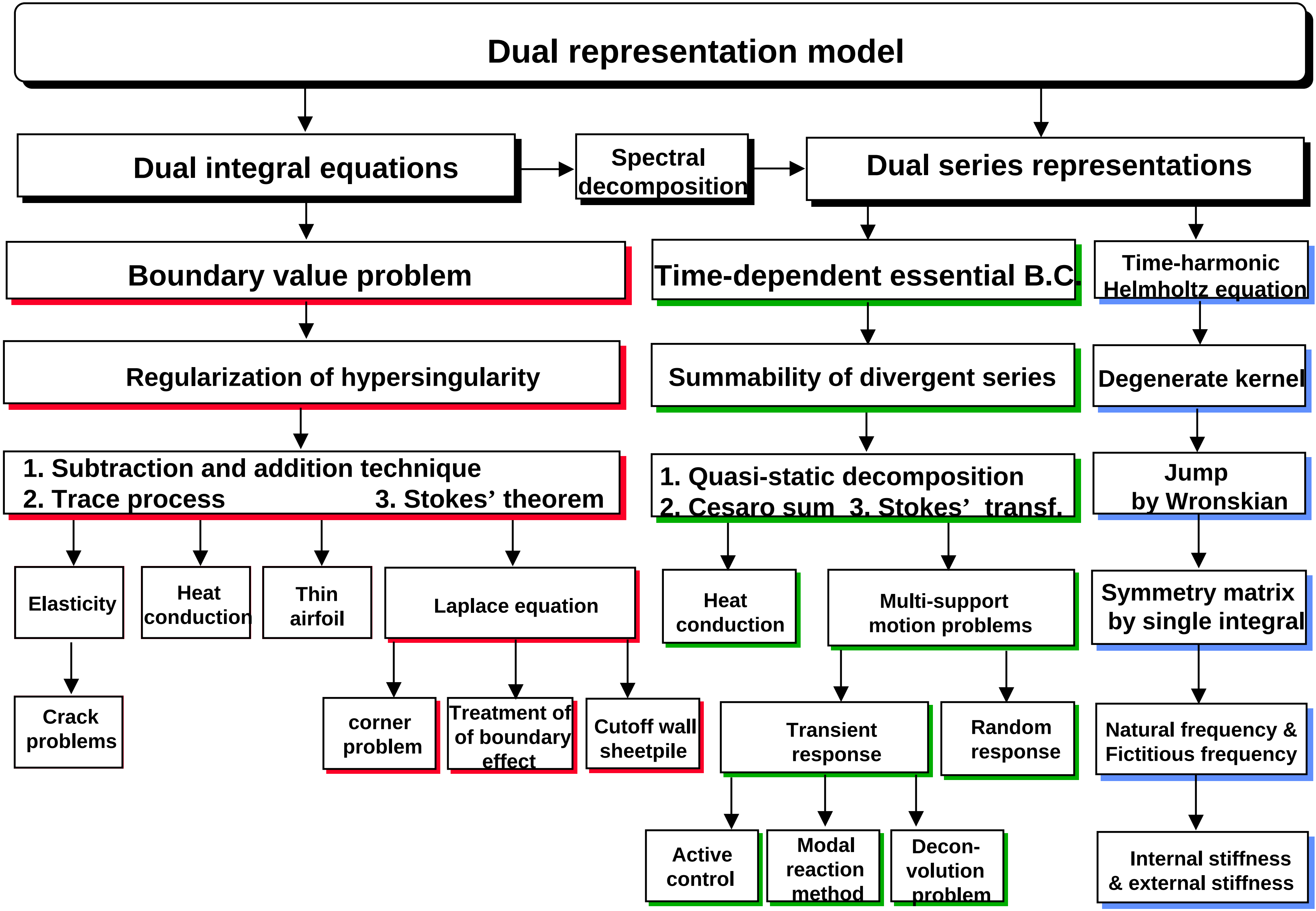
$$w_n^r=\frac{\Gamma(N+1)\,\Gamma(N+r-n+1)}{\Gamma(N-n)\,\Gamma(N+r+1)}$$

Conclusions

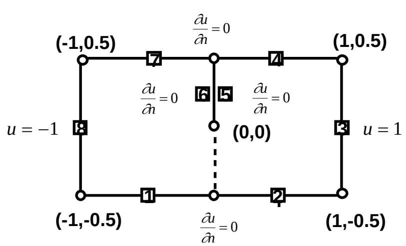
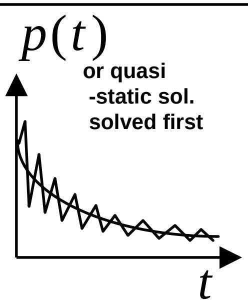
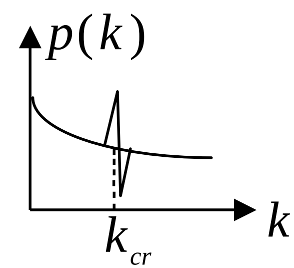
- **The theory of dual integral equation has been reviewed**
- **The role of hypersingularity is examined**
- **The dual series representations are introduced**
- **The applications to seepage flow with sheet piles, crack problem and thin airfoil aerodynamics have been demonstrated.**
- **The applications of dual series representations to multi-support motions is demonstrated.**

Hypersingularity → **Divergent series**

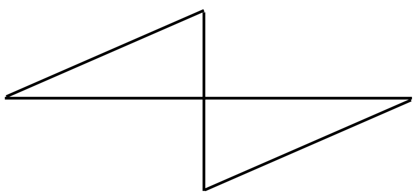
scop1.ppt

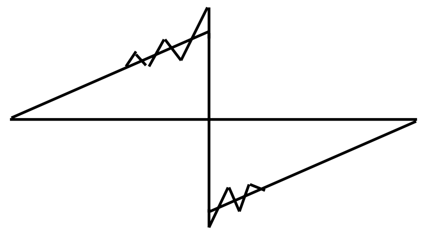

Dual representation model
Dual integral equations
Spectral decomposition
Dual series representations
Boundary value problem
Regularization of hypersingularity
1. Subtraction and addition technique
2. Trace process
3. Stokes' theorem
Elasticity
Crack problems
Heat conduction
Thin airfoil
Laplace equation
corner problem
Treatment of of boundary effect
Cutoff wall sheetpile
Time-dependent essential B.C.
Summability of divergent series
1. Quasi-static decomposition
2. Cesaro sum 3. Stokes' transf.
Heat conduction
Multi-support motion problems
Transient response
Random response
Active control
Modal reaction method
Decon-volution problem
Time-harmonic Helmholtz equation
Degenerate kernel
Jump by Wronskian
Symmetry matrix by single integral
Natural frequency & Fictitious frequency
Internal stiffness & external stiffness

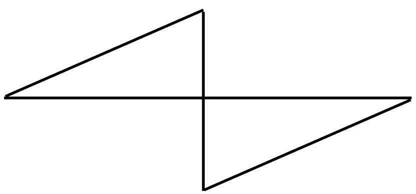
Why Dual Representation Model ?

	Boundary value problem	Initial-boundary value problem	Time harmonic problem
Physical problem	Flow with corners or sheetpiles	Multi-support motions	exterior radiation
Mathematical tools	H.P.V. hypersingularity	Cesaro sum Stokes' transformation	Wronskian Hilbert transform determinant
Numerical problem		 divergence	
Numerical improvement	avoid artifical boundary and. boundary effect	avoid quasi-static solution and accelerate convergence	Avoid fictitious eigenvalue

How to Simulate the Discontinuity ?







CPV + jump term

$$\pm\frac{1}{2}u(x)+CPV\int T(s,x)u(s)dB(s)$$

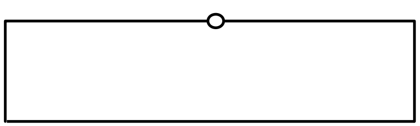
Gibbs phenomenon

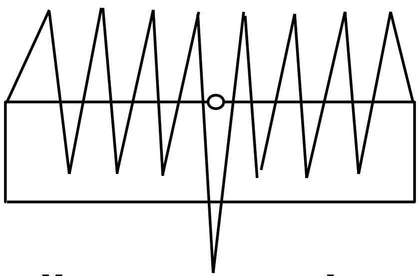
$$\sum c_i u_i(x)$$

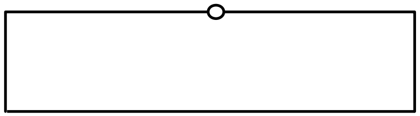
degenerate series

$$T^i(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_s C_m(ks)\cdot n(s)\}R_m(kx)$$

$$T^e(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}C_m(kx)\{\nabla_s R_m(ks)\cdot n(s)\}$$







HPV

$$HPV\int M(s,x)u(s)dB(s)$$

divergent series

$$\sum c_i t_i(x)$$

degenerate series

$$M^i(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_x C_m(ks)\cdot n(s)\}\{\nabla_s R_m(kx)\cdot n(x)\}$$

$$M^e(s,x)=\sum_{m=-\infty}^{m=\infty}\frac{i}{c_m}\{\nabla_x C_m(kx)\cdot n(x)\}\{\nabla_s R_m(ks)\cdot n(s)\}$$