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Journal Name	Archive of Applied Mechanics	
Corresponding Author	Family Name	**Provatidis**
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Schedule	Received	17 August 2007
	Revised	
	Accepted	9 January 2008
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ORIGINAL

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Time- and frequency-domain analysis using lumped mass global collocation

Received: 17 August 2007 / Accepted: 9 January 2008

Abstract Quite recently, a novel global collocation method for the eigenvalue analysis of freely vibrated elastic structures was proposed (Archive of Applied Mechanics: DOI: 10.1007/s00419-007-0159-4). This paper extends the latter methodology on several levels, in both the time and frequency domain. Firstly the formulation is updated so that it can also deal with rods of variable cross section. Then, the fully populated mass matrices of the previous formulation are properly replaced by lumped masses, thus saving still more computer effort. Subsequently, a new general formulation for the transient response analysis is proposed. Finally, a novel procedure for the coupling of two neighboring collinear rods is presented. The theory is supported by six test cases concerning elastic rods of constant and variable cross sections. Among these, transient analysis refers to the response of a single rod due to a Heaviside-type loading as well as to the impact between two collinear rods of different cross sections.

Keywords Collocation · Finite element · Eigenvalues · Transient response · Impact

1 Introduction

Static as well as eigenvalue and transient response analysis of one-, two-, and three-dimensional structures is usually performed using the well-known finite-element method in conjunction with approximating polynomials of low degree [1,15]. Besides, higher-order p-methods have been used since the 1970s [14]; in general, for a certain number of n subdivisions the p-version (polynomial of nth degree) has better performance than the h-version (n linear finite elements) [14]. An alternative way to create higher-order elements is based on the use of Coons–Gordon interpolation, which is well known in computer-aided design (CAD) theory, and allows for the automatic derivation of global shape functions for any discretization of the boundary and the interior of the structure [8–10]; the thus obtained finite elements have been called Coons macroelements. In the context of CAD-oriented techniques, Bézier and nonuniform rational B-spline (NURBS) interpolation [7] have been also applied in engineering analysis [3,5,6,13].

Although the aforementioned Coons macroelements can be applied in conjunction with piecewise-linear and piecewise-quadratic interpolation, thus achieving compact support, numerical experience has shown that they are more accurate and converge faster when applied in conjunction with higher-order Lagrange polynomials, for example up to the eighth or tenth degree [8–10]. In such a case, the obtained matrices become fully populated and the computer effort may be substantial [10].

As a remedy to the aforementioned shortcoming of Coons macroelements, it was recently proposed to preserve the same global shape functions and substitute Galerkin–Ritz by a novel global collocation scheme

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that can deal with any type of boundary conditions using only the displacement value (not its derivative); therefore, the cases of two Dirichlet as well as one Dirichlet and one Neumann condition were successfully treated [11]. In the latter work, not only Chebyshev [2] but also Lagrange and Bernstein polynomials were tested and the conclusion was that all of them lead to identical eigenvalues that coincide with those obtained through a Taylor series expansion [11]. Despite the clear advantage of that novel collocation method, it was however reported that, when fulfilling the governing stress equilibrium equation at equidistant internal points, unrealistic complex eigenvalues may appear; therefore, the additional potential advantage of dealing with a trivial *unitary* (or at least diagonal) mass matrix has not yet been accomplished [11] (a detailed numerical study on this issue including a comparison with the conventional finite elements was later reported in [12]).

In this paper, the aforementioned inability to built up robust lumped masses [11,12] is overcome, and a simple way to eliminate the appearance of complex eigenvalues is proposed. Since the eigenvalues depend only on the polynomial degree and the position of the collocation points, it seems reasonable to use Lagrange polynomials defined by the two ends of the interval $[0, L]$ plus those collocation points that led to the most accurate solution in [11]. Therefore, instead of using uniformly distributed internal nodes, as was the case in [11,12], the main novel feature of this work is to locate them at the roots of either Legendre or Chebyshev polynomials.

The paper is structured as follows. The frequency-domain formulation (Helmholtz equation) of [11] is reconstructed in the time domain so that it can deal with a rod of variable cross section subjected to a general time-dependent traction at its free end. New closed-form expressions are obtained for the lumped mass and stiffness matrices. In the case of a forced excitation, the time-dependent force vector is formulated for the first time. Concerning the numerical results, firstly eigenvalue and transient response analysis is performed to rods of constant and smoothly variable cross-sectional area by using one macroelement only. Then, the new formulation is applied to a rod with a sudden change of cross section as well as to the impact analysis of two rods of different cross-sectional -areas, where it becomes necessary to decompose the domain into two macroelements.

2 General formulation

2.1 Governing equations

In the case of an elastic, isotropic, homogeneous rod, in $0 \le x \le L$, the governing equation is:

$$\frac{\partial}{\partial x}\left(A(x)E\frac{\partial U(x,t)}{\partial x}\right) + b(x,t) - (A(x)\rho)\frac{\partial^2 U(x,t)}{\partial t^2} = 0, \tag{1}$$

where $U(x,t)$ is the axial displacement, x is the Cartesian coordinate, t is the time, $A(x)$ is the variable cross-sectional area, E is the elastic modulus, ρ is the mass density, and $b(x,t)$ is the body force towards the x-direction. Hereafter, the spatial and temporal derivatives will be denoted by the prime ($\prime$) and the dot ($\dot{}$) notation, respectively.

In order to find a particular solution of Eq. (1) within the one-dimensional domain $[0, L]$, it is necessary to specify initial conditions:

$$\begin{aligned} U(x,0) &= U_0(x), \\ V(x,0) &= \left.\frac{\partial U(x,t)}{\partial t}\right|_{t=0} = V_0(t). \end{aligned} \tag{2}$$

In the particular case in which kinematic and traction excitation occurs at the left and right end, respectively, at any time instant t the boundary conditions are written as:

$$\begin{aligned} U &= \bar{U}(t) \qquad \text{at } x = 0, \\ \sigma_L &= E\left(\partial U / \partial x\right)_{x=L} \quad \text{at } x = L. \end{aligned} \tag{3}$$

For simplicity, henceforth absence of body sources ($b = 0$) is considered.

2.2 Global collocation approach

The solution of Eq. (1) can be expressed as a series expansion:

$$U(x,t) = \sum_{j=1}^{n} \phi_j(x) U_j(t) \tag{4}$$

where $\phi_j(x)$ are the shape functions associated to the nodal potentials U_j at the positions $x = x_j$, $j = 1, \ldots, n$ ($x_1 = 0$, $x_n = L$). We recall that usually $\phi_j(x)$ are cardinal functions, i.e., $\phi_j(x_i) = \delta_{ij}$ (Kronecker's delta) and also partition unity, i.e., $\sum_{j=1}^{n} \phi_j(x) \equiv 1$, for all $x \in [0, L]$. A typical set of such cardinal functions consists of the Lagrange polynomials.

Instead of choosing the aforementioned n nodes through a uniform subdivision of the domain $[0, L]$, as was the case in Ref. [11], in this paper we take the two extreme nodes at their apparent position ($x_1 = 0$, $x_n = L$), but the interior ones are chosen to coincide with the roots of either Legendre or Chebyshev polynomials of the second kind (details are given in Appendix A).

With respect to the implementation of boundary conditions, we distinguish two cases (A and B) as follows.

A. One end fixed the other is excited. Here, we assume that the rod is fixed at the end $x = 0$ and it is subjected to a prescribed traction $\sigma_L(t)$ at $x = L$. In order to implement the boundary conditions, substituting Eq. (4) into Eq. (3) for $x = 0$ leads to:

$$U_1(t) = U(0,t) = \sum_{j=1}^{n} \phi_j(0) U_j(t) = 0. \tag{5}$$

Furthermore, the Neumann boundary condition at the other end ($x = L$) gives:

$$\sigma_L(t) = E \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=L} = E \sum_{j=1}^{n} \left[\frac{\partial \phi_j(x)}{\partial x}\right]_{x=L} U_j(t). \tag{6}$$

Combining Eqs. (5) and (6), the potentials of the first and last node (at $x = L$) are eliminated, thus leading to:

$$U_n(t) = \frac{\left[\frac{\sigma_L(t)}{E} - \sum_{j=2}^{n-1} \phi_j'(L) U_j(t)\right]}{\phi_n'(L)}. \tag{7}$$

Moreover, we demand that the governing stress equilibrium equation (1) is fulfilled at the same internal points $\bar{x}_i \in [0, L]$, $i = 1, \ldots, n_p = n - 2$ (where $\bar{x}_i = x_{i+1}$), which determine the Lagrange polynomials:

$$\frac{\partial}{\partial x}\left(A(\bar{x}_i) E \frac{\partial U(\bar{x}_i, t)}{\partial x}\right) - (A(\bar{x}_i)\rho) \frac{\partial^2 U(\bar{x}_i, t)}{\partial t^2} = 0\ , \quad i = 1, \ldots, n_p \tag{8}$$

Substituting Eqs. (4) and (7) into Eq. (8), on which the chain rule is applied, and by writing it for all the $n_p = n - 2$ intermediate nodes, one obtains:

$$m_{ii} \ddot{U}_{i+1}(t) + \sum_{j=1}^{n-2} k_{ij} U_{j+1}(t) = f_i(t), \qquad i = 1, \ldots, (n-2), \tag{9}$$

where

$$\begin{aligned} m_{ii} &= \rho A(x_{i+1}), \\ k_{ij} &= E A(x_{i+1}) \left(-\phi_{j+1}''(x_{i+1}) + \phi_n''(x_{i+1}) \frac{\phi_{j+1}'(L)}{\phi_n'(L)}\right) \\ &\quad + E A'(x_{i+1}) \left(-\phi_{j+1}'(x_{i+1}) + \phi_n'(x_{i+1}) \frac{\phi_{j+1}'(L)}{\phi_n'(L)}\right), \\ f_i(t) &= \sigma_L(t) \left[\frac{A(x_{i+1})\phi_n''(x_{i+1}) + A'(x_{i+1})\phi_n'(x_{i+1})}{\phi_n'(L)}\right]. \end{aligned} \tag{10}$$

Obviously, the formation of a lumped mass, **M**, instead of the fully populated matrix that was found in reference [11], is due to the fact that the shape functions (Lagrange polynomials) are cardinal ($\phi_j(x_i) = \delta_{ij}$, Kronecker's delta).

A.1 Variable cross section

Writing Eq. (9) for all $(n-2)$ values, the following matrix formulation is obtained:

$$\underbrace{\mathbf{M}}_{(n-2)\times(n-2)} \cdot \underbrace{\ddot{\mathbf{U}}_{\text{internal}}(t)}_{(n-2)\times 1} + \underbrace{\mathbf{K}}_{(n-2)\times(n-2)} \cdot \underbrace{\mathbf{U}_{\text{internal}}(t)}_{(n-2)\times 1} = \underbrace{\mathbf{f}(t)}_{(n-2)\times 1}, \tag{11}$$

where

$$\mathbf{U}_{\text{int}\,ernal}(t) = \left[U_2(t), \ldots, U_{n-1}(t)\right]^T. \tag{12}$$

Obviously, Eq. (11) can be solved using any known time-integration scheme such as explicit (central difference), implicit (Newmark, θ-Wilson, Houbolt) or modal analysis. By recalling [11], while in usual finite-element schemes the order of the mass and stiffness matrices is $n-1$, which is the total number of n nodes minus the restricted node at the boundary $x = 0$, in the proposed procedure the elimination of the degree of freedom (DOF) associated with the excited node at the free boundary $x = L$ leads to an order of $n-2$ (cf. Eq. 11). However, since at the end of each time step the vector $\mathbf{U}_{\text{int}\,ernal}(t)$ is known, Eq. (7) can be immediately used to determine the displacement $U_n(t)$. We also recall that $U_1(t) \equiv 0$.

Concerning the transient response, according to the usual recipe [15, p. 587], for the central-difference method the critical time step should be:

$$\Delta t_{\text{cr}} = \frac{T_n}{\pi} = \frac{2}{\omega_{\text{n}}}, \tag{13}$$

where $T_n = 2\pi/\omega_{\text{n}}$ is the smallest period of the finite-element assemblage with n degrees of freedom. In addition, the well-known Courant–Friedrich–Lewy (CFL) criterion offers an a priori estimation for Δt_{cr}, which is the time it takes the wave ($c = \sqrt{E/\rho}$) to travel across the smallest element of the mesh, Δl_{min}, that is:

$$\Delta \hat{t}_{\text{cr}} = \frac{\Delta l_{\text{min}}}{c} \tag{14}$$

Concerning the eigenvalue problem, the $(n-2)$ eigenvalues can be easily calculated by vanishing the determinant of the matrix ($\det(\mathbf{K} - \lambda\mathbf{M}) = 0$), for example, by using the QR algorithm.

A.2 Constant cross section. In case of a constant cross-sectional area, i.e., $A(x_{i+1})$ =const., its derivatives found in the stiffness and the force vector (cf. Eq. 10) vanish. Therefore, as $A(x_{i+1})$ becomes a common factor in m_{ii}, k_{ij} and $f_i(t)$, it can be further eliminated. Consequently, the mass matrix becomes a *unitary* matrix, **I**, times the mass density, ρ:

$$\underbrace{\mathbf{M}}_{(n-2)\times(n-2)} = \rho \cdot \underbrace{\mathbf{I}}_{(n-2)\times(n-2)}. \tag{15}$$

In this particular case, it is obvious that the eigenvalues of the problem can be found by dividing the eigenvalues of the stiffness matrix, **K**, by the mass density, ρ. Alternatively, if the cross-sectional area A appears in the right-hand side of Eq. (15), then it should also appear in the aforementioned stiffness matrix and the force vector.

B. Both ends fixed. This type of boundary conditions usually restricts analysis to the extraction of modes (eigenvalue problem). In this case, the formulation of reference [11] is modified as follows:

$$m_{ii}\ddot{U}_{i+1}(t) + \sum_{j=1}^{n-2} k_{ij} U_{j+1}(t) = 0, \quad i = 1, \ldots, (n-2) \tag{16}$$

where

$$\begin{aligned} m_{ii} &= \rho A(x_{i+1}), \\ k_{ij} &= -E\left[A(x_{i+1})\phi''_{j+1}(x_{i+1}) + A'(x_{i+1})\phi'_{j+1}(x_{i+1})\right]. \end{aligned} \tag{17}$$

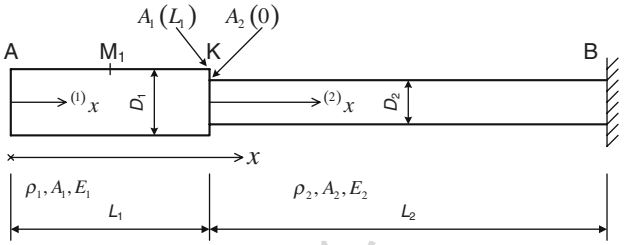

Fig. 1 The general case of a complex unified rod that consists of two parts (AK and KB) to be assembled. This also holds in case of impact between two different rods

3 Coupling of two adjacent rods

Let us assume that the assembled structure AB consists of two rods with dissimilar cross sections, which are joined at the point K, as shown in Fig. 1. For the purposes of variety, in this section we assume that the structure is fixed at the right node B. In order to couple the two rods, we should impose two conditions: zero stress at the traction-free end A and force balance at the junction K. Obviously, compatibility of displacement and velocity at the junction K is taken into consideration. Below, the shape functions and the corresponding displacements related to the first (AK) and the second (KB) rod refer to a local numbering and are denoted by the upper left indexes $^{(1)}$ and $^{(2)}$, respectively. Similarly, the local coordinates involved in the argument of the aforementioned shape functions are denoted by $^{(1)}x$ and $^{(2)}x$, respectively (Fig. 1); obviously, it holds that $^{(1)}x = x$ and $^{(2)}x = x - L_1$. In more detail, let us assume that the rod AK is discretized using n_1 nodes (two of them are its ends A and K and the rest are the roots of Legendre or Chebyshev polynomials); this choice is related to a series of Lagrange polynomials, $^{(1)}\phi_j(x),\ j = 1, \ldots, n_1$, each of degree $n_1 - 1$. Similarly, the rod KB is discretized using n_2 nodes (two of them are its ends K and B); again, this choice is related to another series of Lagrange polynomials, $^{(2)}\phi_j\left(^{(2)}x\right),\ j = 1, \ldots, n_2$, each of degree $n_2 - 1$. Thus, the total number of geometric nodes in the entire structure becomes $n_1 + n_2 - 1$. Without loss of generality, node numbering is performed from left to the right, so that the first node coincides with the point A while the $(n_1 + n_2 - 1)$th node coincides with the fixed point B.

The traction-free condition at the point A ($x = 0$) is equivalent to zero strain, and therefore gives the following relationship for the n_1th DOF of rod AK:

$$\sum_{j=1}^{n_1} {}^{(1)}\phi'_j(0) \cdot^{(1)} U_j = 0. \tag{18}$$

At this point it should be noted that if point A were excited by an external force instead of being traction free, then the above Eq. (18) has to be replaced by Eq. (6) in a similar way.

Also, force balance at the junction K ($x = L_1$) relates the displacements of rod AK with those of rod KB, and leads to:

$$E_1 \cdot A_1(L_1) \cdot \sum_{j=1}^{n_1} \left[{}^{(1)}\phi'_j(L_1) \cdot^{(1)} U_j \right] = E_2 \cdot A_2(0) \cdot \sum_{j=1}^{n_2} \left[{}^{(2)}\phi'_j(0) \cdot^{(2)} U_j \right]. \tag{19}$$

Equations (18) and (19) constitute a system that imposes two linear constraints between the $n_1 + n_2 - 2$ unrestrained nodal values (fixed point B being excluded). Therefore, two out of the $n_1 + n_2 - 2$ DOFs can be expressed in terms of the remaining n_m=n_1+$n_2 - 4$ DOFs. If, for instance, the displacements U_1 and U_{n_1} (global numbering), at the left end (A) and the junction (K), respectively, are considered as candidate for elimination, then collecting the corresponding columns in the two first positions, Eqs. (18) and (19) can be

written together as (below, displacements refer to a global numbering):

$$\begin{bmatrix} a_1 & a_{n_1} & a_2 & \dots & a_{n_1-1} & 0 & \dots & 0 \\ b_1 & b_{n_1} & b_2 & \dots & b_{n_1-1} & b_{n_1+1} & \dots & b_{n_1+n_2-2} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_{n_1} \\ U_2 \\ \vdots \\ U_{n_1-1} \\ U_{n_1+1} \\ \vdots \\ U_{n_1+n_2-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{20}$$

with

$$a_j =^{(1)} \phi'_j(0), \quad j = 1, \dots, n_1$$

and

$$b_j = \begin{cases} E_1 \cdot A_1(L_1) \cdot {}^{(1)}\phi'_j(L_1), & 1 \le j \le n_1 - 1 \\ E_1 \cdot A_1(L_1) \cdot {}^{(1)}\phi'_{n_1}(L_1) - E_2 \cdot A_2(0) \cdot {}^{(2)}\phi'_1(0), & j = n_1 \\ -E_2 \cdot A_2(0) \cdot {}^{(2)}\phi'_{j-n_1+1}(0), & n_1 + 1 \le j \le n_1 + n_2 - 2 \end{cases} \tag{21}$$

Therefore, eliminating U_1 and U_{n_1} in Eq. (20), the following expression is derived:

$$\underbrace{\begin{bmatrix} U_1 \\ U_{n_1} \end{bmatrix}}_{\mathbf{U}_s} = \underbrace{-\frac{1}{(a_1 b_{n_1} - a_{n_1} b_1)} \begin{bmatrix} b_{n_1} & -a_{n_1} \\ -b_1 & a_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & \dots & a_{n_1-1} & 0 & \dots & 0 \\ b_2 & \dots & b_{n_1-1} & b_{n_1+1} & \dots & b_{n_1+n_2-2} \end{bmatrix}}_{\mathbf{B}} \cdot \underbrace{\begin{bmatrix} U_2 \\ \vdots \\ U_{n_1-1} \\ U_{n_1+1} \\ \vdots \\ U_{n_1+n_2-2} \end{bmatrix}}_{\mathbf{U}_m}. \tag{22}$$

Equation (22) can also be written in brief, as follows:

$$\underbrace{\mathbf{U}_s}_{2\times 1} = \underbrace{\mathbf{B}}_{2\times n_m} \cdot \underbrace{\mathbf{U}_m}_{n_m \times 1}, \tag{23}$$

where $\mathbf{U}_s = \{U_1, U_{n_1}\}^T$ is the vector of the *slave* DOFs that are to be eliminated, $\mathbf{B}$ is a transformation matrix, and $\mathbf{U}_m$ is the vector of the remaining n_m *master* nodes (in other words, primary or active DOFs), i.e., the internal nodes between the segments AK and KB.

Therefore, separating the master from the slave DOFs, the matrix equation of motion for the internal nodes can be written as

$$\underbrace{[\mathbf{M}]}_{n_m \times n_m} \cdot \underbrace{\ddot{\mathbf{U}}_m}_{n_m \times 1} + \begin{bmatrix} \underbrace{\mathbf{K}_{ms}}_{n_m \times 2} & \underbrace{\mathbf{K}_{mm}}_{n_m \times n_m} \end{bmatrix} \cdot \begin{bmatrix} \underbrace{\mathbf{U}_s}_{2\times 1} \\ \underbrace{\mathbf{U}_m}_{n_m \times 1} \end{bmatrix} = \mathbf{0}. \tag{24}$$

Substituting Eq. (23) into Eq. (24) one obtains

$$\mathbf{M} \cdot \ddot{\mathbf{U}}_m + (\mathbf{K}_{ms}\mathbf{B} + \mathbf{K}_{mm}) \cdot \mathbf{U}_m = \mathbf{0}. \tag{25}$$

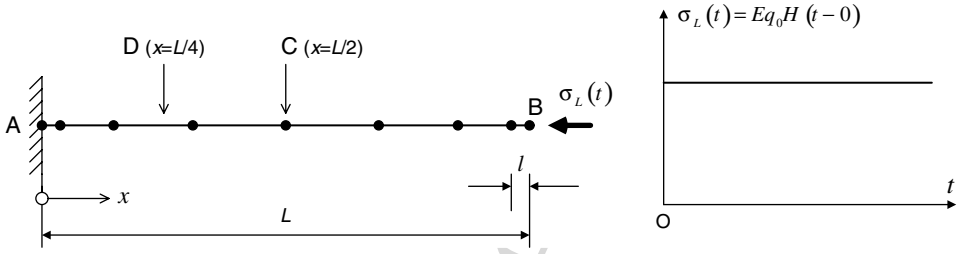

Fig. 2 Example 3. Discretization and loading time history of a rod with constant cross section

4 Numerical results

The first three examples refer to a rod with constant cross section, while the last three examples deal with rods of variable cross section.

Example 1 Rod with constant cross section and both ends restricted

When the nodes that determine Lagrange polynomials were taken at the roots of Legendre polynomials (Gauss points), the calculated eigenvalues were found to be identical with those previously obtained through Lagrange polynomials based on a uniform mesh and collocation of the stress equilibrium equation at the aforementioned roots (cf. Table 1 in Ref. [11]). Analogous findings for the roots of Chebyshev polynomials of the first kind (cf. Table 2 in Ref. [11]) and the second kind (cf. Table 3 in Ref. [11]) were obtained.

Example 2 Rod with constant cross section, one end restricted, the other free

Again, numerical coincidence of the proposed lumped mass formulation with Tables 4, 5, and 6 in Ref. [11] was found.

Example 3 Rod with constant cross section, fixed at one end and subjected to a Heaviside-type loading

An elastic rod is fixed at one of its extremities ($x = 0$) and is subjected to a Heaviside-type loading $\sigma_L = Eq_0H(t-0)$ (N/m^2) at the other one ($x = L$). For simplicity, all geometric and material data were assigned the unitary value. In order to elucidate the proposed method, the (explicit) central-difference scheme was applied.

For the particular discretization of Fig. 2 ($c = \sqrt{E/\rho} = 1$ m/s), if the minimum distance between two successive nodes was taken equal to that of a hypothetical uniform mesh ($\Delta\bar{l}_{\min} = 0.125$ m) Eq. (14) would give $\Delta\hat{t}_{\rm cr} = 0.125$ s; the latter would be useful for a either a fully populated mass matrix global collocation or a finite-element method (FEM) analysis. In contrast, taking the nodes at the roots of Chebyshev polynomial of the second kind, the actual minimum length becomes $l \cong 0.038$ m and corresponds to the largest root (shown in Fig. 2); therefore, the corresponding time step becomes $\Delta\hat{\hat{t}}_{\rm cr} \cong 0.038$ s, which normally should constitute an upper limit. Based on previous experience related to the value of maximum eigenvalue (Eq. (13)) [10], the time step in the Galerkin–Ritz formulation had been conservatively chosen as $\Delta t = 0.20\Delta\bar{l}_{min}/c = 0.025$ s, which in the present case is smaller than $\Delta\hat{\hat{t}}_{\rm cr}$, and therefore can be adopted. Implementing the central difference method using $\Delta t = 0.025$s, the displacement response at the points B ($x = L$), C ($x = L/4$), and D ($x = L/2$) is shown in Fig. 3, where a good-quality solution can be noticed. It should be mentioned that since the point D at $x = L/4$ does not coincide with any of the seven internal nodes, its value was determined in terms of the values at the eight unrestrained nodes, using Eq. (4). The significance of the points C and D is due to the fact that the corresponding peak values equal to the half and one-fourth of that at the point B, respectively; thus they have been often used in literature (e.g., [10]).

Example 4 Rod with linearly varying cross section

Rods of tapered section, or *horns*, find application in power ultrasonics. In this area, the interest is in the resonance characteristics [4, p. 114]. Within this context, this example considers a rod of unit length ($0 \le x \le 1$) with linear cross-section variation:

$$A(x) = A_0(1 + x) \tag{26}$$

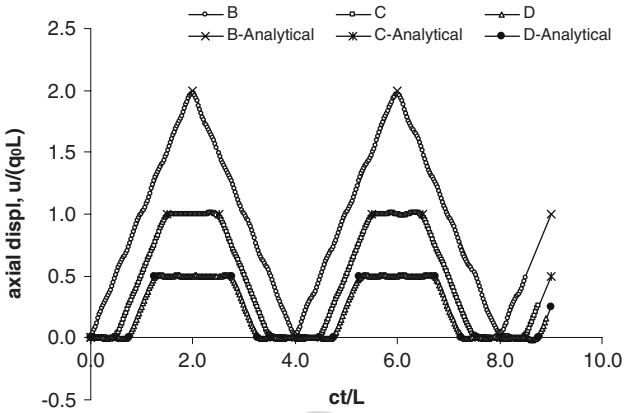

Fig. 3 Example 3. Axial displacements at points B ($x = L$), C ($x = L/2$), and D ($x = L/4$) using the proposed lumped mass formulation for the mesh shown in Fig. 2 and for time step $\Delta t = 0.2\Delta\bar{l}_{min}/c = 0.025\,\mathrm{s}$

Table 1 Example 4. Calculated eigenvalues using global collocation method in conjunction with polynomials of n_{pol}th degree

Mode	Exact $\omega^2(\mathrm{s}^{-2})$	Errors (in %)									
		Degree of polynomial (n_{pol})									
		2	3	4	5	6	7	8	9	10	11
1	9.7533	−17.98	7.03	−0.42	−0.01	0.00	0.00	0.00	0.00	0.00	0.00
2	39.3560	–	−18.50	20.78	−2.62	0.10	0.00	0.00	0.00	0.00	0.00
3	88.7026	–	–	−11.75	39.46	−6.03	0.85	−0.09	0.02	0.00	0.00
4	157.7893	–	–	–	1.44	63.40	−9.08	2.65	−0.50	0.11	−0.02
5	246.6157	–	–	–	–	21.13	92.45	−10.73	5.75	−1.44	0.34
6	355.1812	–	–	–	–	–	46.67	126.29	−10.52	10.26	−2.90

The governing differential equation is collocated at the $n_{\mathrm{pol}} - 1$ roots of Chebyshev polynomials of the second kind (the total number of nodes being $n = n_{pol} + 1$)

Both ends of the rod are assumed to be fixed.

The errors of the calculated eigenvalues (ω^2) were taken with respect to the exact analytical solution, which is extracted by solving the equation [4, p. 115]:

$$J_0(k)Y_0(2k) - J_0(2k)Y_0(k) = 0, \tag{27}$$

where J_0 and Y_0 are Bessel functions of the first and second kind and $k = \omega \Big/ \sqrt{E/\rho}$ is the wavenumber.

The quality of the results is shown in Table 1, where it can be noticed that, similarly to Example 1 (cf. also [11]), convergence is achieved for increasing degree of the polynomial. Also, a comparison for different choices in the position of nodes, as well as for the conventional FEM based on a uniform mesh, is shown in Fig. 4, where one can notice that the choice of Chebyshev roots of the second kind is rather the best one.

Example 5 Eigenvalues of a rod with a sudden change of cross section

This example concerns a complex rod AB made of two other rods (AK and KB as shown in Fig. 1) of the same material (steel) but dissimilar lengths L_1 and L_2, and dissimilar cross-sectional areas A_1 and A_2. The rod is fixed at end B while the other end A is free. Following [4, p. 98], the geometric and material data were taken as:

$$\begin{aligned} &\text{Rod lengths}: \quad L_1 = 0.457\,\mathrm{m},\ L_2 = 2L_1, \\ &\text{Rod diameters}: \quad D_1 = 1.5D_2,\ D_2 = 2.54 \quad \mathrm{cm} \\ &\text{Wave velocities}: c_1 = c_2 = c(= \sqrt{E/\rho}) = 5.08 \times 10^3\,\mathrm{ms}^{-1}, \end{aligned}$$

In order to calculate the eigenvalues of the unified rod AB, the shortest (AK) and the longest (KB) rods are discretized using n_1 and n_2 nodes, respectively, where $n_2 - 1 = 2(n_1 - 1)$. In other words, the displacement fields within AK and KB are described by Lagrange polynomials of degree $(n_1 - 1)$ and $2(n_1 - 1)$, respectively.

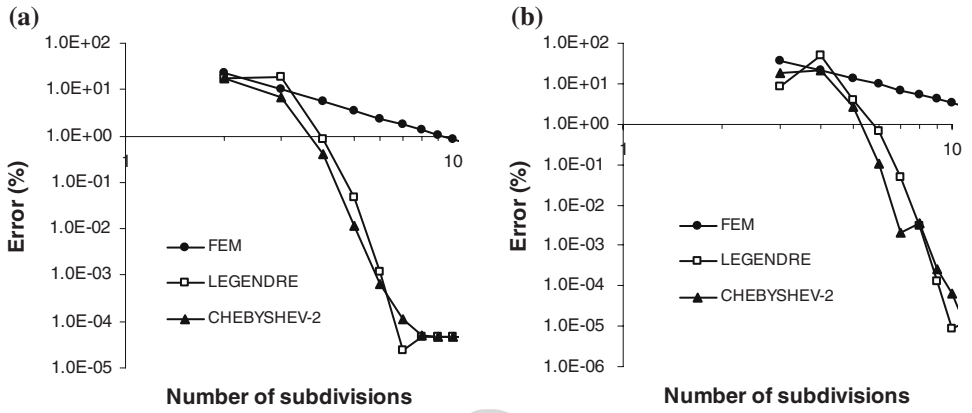

Fig. 4 Example 4. Convergence rate of the calculated eigenvalues for **a** the first and **b** the second mode, for different choices in the position of collocation nodes (roots of *Legendre* polynomial and roots of *Chebyshev* polynomial of the second kind), as well as for the conventional FEM based on a uniform mesh

Table 2 Example 5. Convergence quality of calculated eigenvalues for the rod AB (fixed at B) using the proposed global collocation method in conjunction with the coupling of the parts AK and KB shown in Fig. 1

Mode	'Exact' $\omega^2(\mathrm{s}^{-2})$	Errors (in %)				
		Degree of polynomial $(n_1 - 1)$ for the shortest rod AK				
		2	3	4	5	6
1	1.9945E+07	−0.49	−0.13	0.00	0.00	0.00
2	3.0293E+08	0.73	−0.37	−0.03	0.00	0.00
3	9.2075E+08	28.43	−3.56	−0.83	0.04	0.01
4	1.5426E+09	37.35	−6.15	−1.28	0.11	0.05
5	2.7264E+09	–	−5.74	−1.14	−0.09	0.04
6	4.2452E+09	–	46.58	−8.89	−1.20	0.23
7	5.4890E+09	–	57.49	−4.56	−0.79	0.36
8	7.5739E+09	–	–	−10.32	−3.39	−0.13

Taking the mass matrix $\mathbf{M}$ as diagonal, and calculating the stiffness matrix according to Eq. (25), i.e., like $\mathbf{K} = \mathbf{K}_{ms}\mathbf{B} + \mathbf{K}_{mm}$, the quality of the calculated eigenvalues is excellent, as shown in Table 2. In the absence of an analytical formula, 'exact' eigenvalues were taken as those obtained using a uniform fine mesh of 768 (=3×256) conventional finite elements, which appeared a relative error less than 0.02% (at the eighth mode) compared with the half mesh density ($3 \times 128 = 384$ elements).

Example 6 Longitudinal collinear impact of two rods

This example concerns the longitudinal impact of two flat-ended rods. At the instant of contact, the rod with the larger cross section (AK in Fig. 1) has a uniform velocity $V_1 = 6 \times 10\,\mathrm{ms}^{-1}$, while the other rod (KB) is stationary. All geometric and material data were considered identical with those of Example 5.

Based on the theoretical approach in [4, pp. 95–100], immediately after the contact ($t_0 = 0+$), the incident wave splits into two parts: the first is transmitted in the direction KB while the second is reflected in the direction KA. Since the length of KB is twice that of AK ($L_2 = 2L_1$), nothing of particular significance occurs here other than the reflection of the leftward-propagating wave from the left free end at the time instant $t_1 = L_1/c$; at this instance, the rightward-propagating wave has reached the middle of KB. The next change is due to the simultaneous reflection of the two aforementioned waves at the points B and K, respectively, and so on. In order to compare with the analytical solution in [4, pp. 98], the response is presented in terms of the normalized particle velocity (V/V_1) versus time; the latter is normalized with respect to the quantity $L_1/2c$. Obviously, the aforementioned time interval $L_1/2c$ is the time required for the reflected wave to reach the middle (point M_1 in Fig. 1) of the rod AK for the first time.

The proposed lumped mass methodology was applied using $n_1 = 6$ and $n_2 = 12$ nodes for the rods AK and KB, respectively. The calculated velocity of the end point A and the middle point M_1 are shown in Figs. 5 and 6, respectively. One can notice a good agreement with the analogous response obtained using a uniform mesh of conventional finite elements with the same number of DOFs, as well as with the analytical solution.

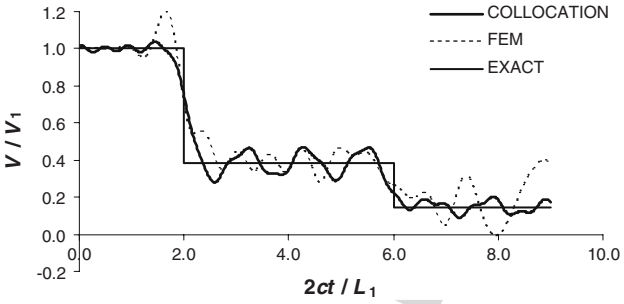

Fig. 5 Example 6. Calculated velocity of the end point A using the proposed lumped mass global collocation in conjunction with the roots of *Chebyshev* polynomial of the second kind, compared with the conventional finite-element method (FEM) and the exact analytical solution

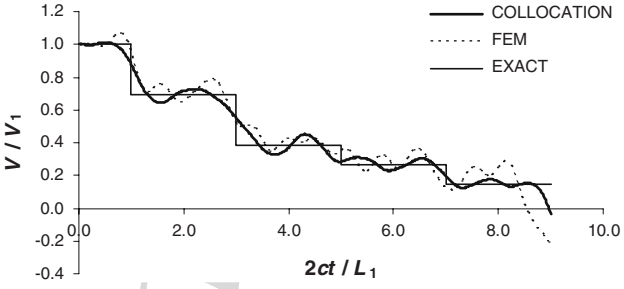

Fig. 6 Example 6. Calculated velocity of the middle point M_1 using the proposed lumped mass global collocation in conjunction with the roots of *Chebyshev* polynomial of second kind, compared with the conventional finite element method (*FEM*) and the exact analytical solution

5 Discussion

It was found that the global approximation of the displacement field in terms of Lagrange polynomials with nodal points at the well-known roots of Legendre or Chebyshev polynomials, in conjunction with a previously proposed global collocation method [11], leads to a diagonal mass matrix that has a robust behavior. In the particular case of a homogeneous rod with constant cross section, this mass matrix becomes the unitary matrix times the mass density; in this case, since the mass density can be incorporated as a factor in the stiffness matrix (**K**), the eigenvalues are calculated using only **K**.

In brief, it was found that the eigenvalues obtained using the aforementioned unitary-matrix mass formulation are identical with those previously obtained using the fully-populated ones in conjunction with a uniform mesh and collocation at the aforementioned roots of Legendre or Chebyshev polynomials [11]. In this regard, no complex eigenvalue appeared for all the test cases used in this paper. Concerning the calculated time response, it was found that the differences between using the proposed lumped mass and the fully populated mass matrix were negligible and could be hardly noticed on a graph. Moreover, it was noticed that, in the particular case of using a uniform mesh that happens to lead to complex eigenvalues, the corresponding time response solution diverged to extremely high values.

Concerning the time response results (Figs. 3, 5, and 6), this paper was limited to the central-difference method, for both the global collocation and the finite-element methods. It was also found that the use of implicit schemes, such as ϑ-Wilson and Newmark, leads to considerably smoother numerical solutions; however, a detailed investigation of this is beyond the scope of this paper.

With respect to a single rod with one fixed end and the other free, it was previously explained in [11] that the proposed global collocation method requires the elimination of one nodal DOF (such as U_n) through Eq. (6); this procedure leads to one fewer primary (in other words: active or master) DOFs than in the usual finite-element method. Moreover, in the case of two rods, two DOFs have to be eliminated through Eq. (18), thus leading to two fewer primary DOFs than in the usual FEM. Not only that, but in this way the governing equation is not fulfilled at the free end A (or/and the junction K). Concerning the proposed collocation method, it should be pointed out that for the success of the proposed method the aforementioned elimination is *crucial*.

In this context, in the case of a single rod, two attempts to recover the aforementioned decrease of the number of active (master) DOFs by collocating once more at the location of the slave node (the free end at $x = L$) failed. In more detail, if the number of unrestrained nodal points (the fixed end at $x = 0$ is excluded but the free one at $x = L$ is included) is $n - 1$, the resulting mass matrix, $\mathbf{M}$, and stiffness matrix, $\mathbf{K}$, are initially of dimension $(n - 1) \times (n - 1)$. The first attempt consisted of extracting the eigenvalues of the initial pair $(\mathbf{M}, \mathbf{K})$ without considering Eq. (6) at all, thus leading to a minimum eigenvalue equal to zero and to $(n - 2)$ *irrelevant* rest values. The second attempt consisted of considering the initial pair $(\mathbf{M}, \mathbf{K})$ in conjunction with Eq. (6), which led to a constraint of the form $U_n = \mathbf{B}\mathbf{U}_m$, where $\mathbf{B}$ $(1 \times (n - 2))$ is a transformation matrix and $\mathbf{U}_m((n - 2) \times 1)$ is the vector of the master DOFs. In this case, in order to eliminate the U_n in the initial equation $\underbrace{\mathbf{M}}_{(n-1)\times(n-1)} \cdot \underbrace{\ddot{\mathbf{U}}_{(n-1)\times 1}}_{} + \underbrace{\mathbf{K}}_{(n-1)\times(n-1)} \cdot \underbrace{\mathbf{U}}_{(n-1)\times 1} = \underbrace{\mathbf{0}}_{(n-1)\times 1}$, the total vector $\mathbf{U}((n - 1) \times 1)$ was written as $\mathbf{U} = \begin{bmatrix} \mathbf{U}_m \\ U_n \end{bmatrix} = \begin{bmatrix} \mathbf{U}_m \\ \mathbf{B}\mathbf{U}_m \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{B} \end{bmatrix} \cdot \mathbf{U}_m$, where $\mathbf{I}$ is a $(n - 2) \times (n - 2)$ identity matrix. Therefore, the aforementioned initial equation had to be multiplied from the left by $\begin{bmatrix} \mathbf{I} & \mathbf{B} \end{bmatrix}$ (*least-squares* procedure) and from the right by $\begin{bmatrix} \mathbf{I} \\ \mathbf{B} \end{bmatrix}$, thus leading to a system of $(n - 2)$ equations. Unfortunately, this procedure did not work satisfactorily, since for $n = 9$ and $n = 11$ nodes the error of the first eigenvalue was found to be -12.7% and -8.9%, respectively.

6 Conclusions

The results of this paper suggest that the global collocation method, which had been previously used to replace the Galerkin/Ritz formulation, can definitely be used equally well in conjunction with lumped masses, thus saving further computer effort. This fact is achieved by moving the degrees of freedom to the positions of the well-known roots of Legendre or Chebyshev polynomials. The formulation is capable of dealing with rods of variable cross section, and performs well in the time domain when a transient response analysis is required. Moreover, it was shown that, in the case of a rod with complex geometry, or in the case of impact, the coupling between two rods is a straightforward procedure. Ongoing research reveals that the conclusions obtained by one-dimensional structural problems are transferable to two- and three-dimensional problems including field problems.

Appendix A: Relationship between internal nodes and the roots of Legendre or Chebyshev polynomials

I. Legendre polynomials

When $m = 0, 1, 2, \ldots$, the solutions of the ordinary differential equation

$$\left(1 - x^2\right)y'' - 2xy' + m(m + 1)y = 0 \tag{A.1}$$

are called Legendre polynomials and are given by

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}\left(x^2 - 1\right)^m. \tag{A.2}$$

When $m \geq 1$, the roots of $P_m(x)$ are the well-known Gauss points, ξ_i, which lie in the interval $[-1, +1]$. While in the usual finite-element method the Gauss points are useful for the domain integration to evaluate the mass and stiffness matrices (see [1, p. 464] and [15, p. 200]), in the proposed collocation method they serve to determine the location of the internal nodes at $(x_1 \equiv 0 < x_{i+1} < x_n \equiv L)$:

$$x_{i+1} = \frac{(1 + \xi_i)}{2}L \quad , \quad i = 1, \ldots, m \equiv n - 2; \quad \xi_i \in [-1, +1]. \tag{A.3}$$

For example, in the case of $n = 3$ nodes, we consider that $m = 1$, and the unique internal node is taken as the root of $P_1(x) = x$, i.e., at $\xi_1 = 0$. Similarly, in the case of $n = 4$ nodes we consider that $m = 2$, and the two internal nodes are taken as the roots of $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$, i.e., at $\xi_1 = -1/\sqrt{3}$ and $\xi_2 = +1/\sqrt{3}$, and so on. In the general case of n nodes, we consider that $m = n - 2$.

II. Chebyshev polynomials

The Chebyshev polynomials of the second kind are defined as

$$U_m(x) = \frac{\sin\{(m+1)\cos^{-1}x\}}{\sin(\cos^{-1}x)} = \binom{m+1}{1}x^m - \binom{m+1}{3}x^{m-2}(1-x^2) + \binom{m+1}{5}x^{m-4}(1-x^2)^2 - \cdots \tag{A.4}$$

and their roots are given by

$$\hat{\xi}_i = \cos\frac{i\pi}{(m+1)}, \quad i = 1, \ldots, m. \tag{A.5}$$

In the proposed collocation method, for example, in the case of $n = 3$ nodes we consider that $m = 1$, and the unique internal node is taken as the root of $U_1(x)$, i.e., at $\hat{\xi}_1 = \cos\left(\frac{\pi}{2}\right) = 0$. Similarly, in the case of $n = 4$ nodes we consider that $m = 2$, and the two internal nodes are taken as the roots of $U_2(x)$, i.e., at $\hat{\xi}_1 = \cos\left(\frac{\pi}{3}\right) = 0.5$ and $\hat{\xi}_2 = \cos\left(\frac{2\pi}{3}\right) = -0.5$, and so on. In the general case of n nodes, we consider again that $m = n - 2$.

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