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A boundary element formulation with equilibrium satisfaction for thermo-mechanical problems considering transient and non-linear aspects

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Abstract

Boundary element formulations incorporating consistent transient potential theory, satisfying exact energy balance, and dynamic equilibrium satisfaction with respect to the co-ordinate axis directions and moments, including inertial forces, elastoplastic deformations and thermal loadings, are presented. The procedures are quite general and can be implemented into existing boundary element codes. The required expressions for the transient potential analysis and the dynamic formulation are discussed, and we include two examples that take into account linear and non-linear material behaviour to illustrate the potential of the proposed methodology.

Keywords: Boundary elements; Energy balance; Equilibrium satisfaction; Transient non-linear analysis; Thermo-mechanical problems

1. Introduction

As a typical mixed interpolation technique, the solutions produced by the boundary element method (BEM) do not inherently satisfy the equilibrium equations. Hence, early researchers, first faced with this problem, imposed approximate equilibrium via modified variational procedures such as Lagrange multipliers [1–3], mainly interested in a successful combination of BEM and the popular finite elements whose stiffness matrices already present this desirable condition built in. To a great extent, in these early implementations the ultimate goal was the computation of symmetric and rigid body motion satisfying BEM stiffness matrices, allowing for a consistent combination between the two techniques. But the recognition of equilibrium as a desirable feature in pure BEM solutions had already called the attention of other BEM researchers such as Kuhn et al. [4], who introduced this feature, in approximate fashion, through an over determined system of equations producing equilibrium in a least-square approximate fashion.

Notwithstanding the fact that for any starting BEM discretization, equilibrium is approached with mesh refinement or element improved interpolation, lack of proper balance can in some cases spoil the convergence rate otherwise obtained with suitable consistent modifications as discussed in a previous publication by Telles and de Paula [5], in which such general benefit in solution behaviour has been seen illustrated for problems involving steady-state potential and elastostatics. This formerly introduced original idea was later extended to accommodate inertial forces and plastic deformations as well by Soares, et al. [6] and has been the object of a number of connected publications, e.g. of Dumont [7] and He et al. [8].

In this paper, the BEM formulation with exact equilibrium satisfaction previously presented by the authors is extended to accommodate first transient temperature distribution within the scope of potential

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theory (i.e., diffusion equation) and then its implementation to thermal–elastoplastic dynamic problems. The starting motivation to follow this approach has been the extension of the benefits of equilibrium, previously verified for potential and elastostatics, to all possibly elastodynamic problems, including frequency and time domains with thermal, transient and steady-state loading.

2. Basic equations

The heat conduction equation for a homogenous isotropic medium is given by

$$T_{,ii} - (\rho c/K)\dot{T} + \dot{s} = 0, \tag{1}$$

where T stands for a relative temperature state; $\dot{s}$ is a source term and ρ, c and K are the mass density, specific heat per unit volume and thermal conductivity, respectively. Inferior commas and over dots indicate partial space and time derivatives, respectively.

The basic equations concerning thermo-dynamic modelling of solids read

$$\sigma_{ij,j} - \rho\ddot{u}_i + \rho b_i = 0, \tag{2}$$

$$\mathrm{d}\sigma_{ij} = D_{ijkl}(\mathrm{d}\varepsilon_{kl} - \mathrm{d}\varepsilon^0_{kl}) + \gamma_{ij}\,\mathrm{d}T, \tag{3}$$

where Eq. (2) is the momentum balance equation and Eq. (3) is the constitutive law (written in incremental form). The Cauchy stress, using the usual Cartesian index notation, is represented by σ_{ij}; u_i stands for the displacement component and b_i for the body force component. In addition, D_{ijkl} is a tangential tensor defined by suitable state variables and the direction of the increment. The incremental strain ($\mathrm{d}\varepsilon_{ij}$) components are defined in the usual way from the displacement, i.e., $\mathrm{d}\varepsilon_{ij} = (1/2)(\mathrm{d}u_{i,j} + \mathrm{d}u_{j,i})$ and ε^0_{ij} refers to a generic "initial" strain state (elastoplastic analysis). In Eq. (3), γ_{ij} is the thermoelasticity tensor defined by $\gamma_{ij} = \alpha(E/(1-2\nu))\delta_{ij}$ for 3D and 2D plane strain state problems or $\gamma_{ij} = \alpha(E/(1-\nu))\delta_{ij}$ for 2D plane stress problems, where E is Young's modulus, ν is Poisson's rate, α is the coefficient of thermal expansion and δ_{ij} is the Kronecker delta.

In addition to Eqs. (1)–(3), boundary and initial conditions have to be prescribed in order to completely define the problem.

3. Thermal analysis

Considering heat conduction problems, the direct BE formulation leads to the following integral equation (source terms are omitted for simplicity) [9]:

$$\begin{aligned} c(\xi)T(\xi,t) = {} & \int_\Gamma T^*(\xi,x)R(x,t)\,\mathrm{d}\Gamma(x) \\ & - \int_\Gamma R^*(\xi,x)T(x,t)\,\mathrm{d}\Gamma(x) \\ & - \int_\Omega T^*(\xi,x)(\rho c/K)\dot{T}(x,t)\,\mathrm{d}\Omega(x), \end{aligned} \tag{4}$$

where Γ and Ω stand for the boundary and the domain of the body, respectively; $c(\xi)$ depends on the boundary geometry and $T^*(\xi, x)$ and $R^*(\xi, x)$ (i.e., temperature and its normal derivative) represent the fundamental solution to the problem at x (field point) due to unit sources at ξ (source point).

To introduce energy equilibrium satisfaction into the thermal BEM formulation, the standard fundamental solution $T^*(\xi, x)$ is modified, adding to it a constant term k:

$$\bar{T}^*(\xi,x) = T^*(\xi,x) + k. \tag{5}$$

The implementation of $\bar{T}^*(\xi,x)$ generates an extra term in the right-hand side of Eq. (4). After discretization of boundary and domain and taking into account the fundamental solution (5), Eq. (4) leads to the following system of equations:

$$\mathbf{HT} = \mathbf{GR} - \mathbf{M}\dot{\mathbf{T}} + \mathbf{V}Q, \tag{6}$$

where $\mathbf{H}$ and $\mathbf{G}$ are the standard boundary element influence matrices and $\mathbf{M}$ is the standard domain influence matrix. $\mathbf{V}$ and Q are terms associated with the equilibrium satisfaction; they arise due to the constant term added in Eq. (5). Vector $\mathbf{V}$ is defined by

$$\mathbf{V} = [1 \quad 1 \quad \cdots \quad 1]^{\mathrm{T}} \tag{7}$$

and Q is given by

$$\begin{aligned} Q &= k\left[\int_\Gamma R(x,t)\,\mathrm{d}\Gamma(x)\right] - k\left[\int_\Omega (\rho c/K)\dot{T}(x,t)\,\mathrm{d}\Omega(x)\right] \\ &= k\mathbf{N}^{\mathrm{T}}\mathbf{R} - k\mathbf{\Pi}^{\mathrm{T}}\dot{\mathbf{T}}, \end{aligned} \tag{8}$$

where matrices $\mathbf{N}$ and $\mathbf{\Pi}$ are computed by simple integration of the interpolation functions along the boundary and the domain, respectively. For equilibrium satisfaction, one should have

$$\mathbf{N}^{\mathrm{T}}\mathbf{R} = \mathbf{\Pi}^{\mathrm{T}}\dot{\mathbf{T}}. \tag{9}$$

Writing Eqs. (6) and (9) together, in a less concise manner, one has

$$\begin{aligned} \begin{bmatrix} \mathbf{H}_{bb} & 0 \\ \mathbf{H}_{db} & \mathbf{I} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}_b \\ \mathbf{T}_d \end{bmatrix} = {} & \begin{bmatrix} \mathbf{G}_{bb} \\ \mathbf{G}_{db} \\ \mathbf{N}^{\mathrm{T}} \end{bmatrix} [\mathbf{R}_b] \\ & - \begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bd} \\ \mathbf{M}_{db} & \mathbf{M}_{dd} \\ \mathbf{\Pi}^{\mathrm{T}}_b & \mathbf{\Pi}^{\mathrm{T}}_d \end{bmatrix} \begin{bmatrix} \dot{\mathbf{T}}_b \\ \dot{\mathbf{T}}_d \end{bmatrix} + \begin{bmatrix} \mathbf{V}_b \\ \mathbf{V}_d \\ 0 \end{bmatrix} Q, \end{aligned} \tag{10}$$

in which the subscripts b and d denote boundary and domain associated coefficients and unknowns, respectively. Multiplying the first set of equations (10)by $\mathbf{N}^{\mathrm{T}}\mathbf{G}^{-1}_{bb}$, results in

$$\mathbf{N}^{\mathrm{T}}\mathbf{G}^{-1}_{bb}\mathbf{H}_{bb}\mathbf{T}_b = \mathbf{N}^{\mathrm{T}}\mathbf{R}_b - \mathbf{N}^{\mathrm{T}}\mathbf{G}^{-1}_{bb}\mathbf{M}_b\dot{\mathbf{T}} + \mathbf{N}^{\mathrm{T}}\mathbf{G}^{-1}_{bb}\mathbf{V}_bQ. \tag{11}$$

Taking into account Eq. (9), Q can be isolated in Eq. (11), as follows:

$$Q = \lambda^{-1}(\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{H}_{bb}\mathbf{T}_b + \mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{M}_b\dot{\mathbf{T}} - \mathbf{\Pi}^{\mathrm{T}}\dot{\mathbf{T}}), \tag{12}$$

where λ is the term originally multiplying Q in Eq. (11), using boundary only matrices.

Substituting Eq. (12) into Eq. (6) and rearranging, a reduced system of equations can finally be obtained:

$$\begin{bmatrix} \bar{\mathbf{H}}_{bb} & 0 \\ \bar{\mathbf{H}}_{db} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{T}_b \\ \mathbf{T}_d \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{bb} \\ \mathbf{G}_{db} \end{bmatrix} [\mathbf{R}_b] - \begin{bmatrix} \bar{\mathbf{M}}_b \\ \bar{\mathbf{M}}_d \end{bmatrix} [\dot{\mathbf{T}}] \tag{13a}$$

or, in a concise manner:

$$\bar{\mathbf{H}}\,\mathbf{T} = \mathbf{G}\mathbf{R} - \bar{\mathbf{M}}\,\dot{\mathbf{T}}, \tag{13b}$$

where the modified matrices $\bar{\mathbf{H}}$ and $\bar{\mathbf{M}}$ are given by

$$\bar{\mathbf{H}}_{bb} = \mathbf{H}_{bb} - \mathbf{V}_b\lambda^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{H}_{bb} \tag{14a}$$

$$\bar{\mathbf{H}}_{db} = \mathbf{H}_{db} - \mathbf{V}_d\lambda^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{H}_{bb} \tag{14b}$$

$$\bar{\mathbf{M}}_b = \mathbf{M}_b - \mathbf{V}_b\lambda^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{M}_b + \mathbf{V}_b\lambda^{-1}\mathbf{\Pi}^{\mathrm{T}} \tag{15a}$$

$$\bar{\mathbf{M}}_d = \mathbf{M}_d - \mathbf{V}_d\lambda^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{M}_b + \mathbf{V}_d\lambda^{-1}\mathbf{\Pi}^{\mathrm{T}} \tag{15b}$$

and $\lambda = \mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{V}_b.$ (16)

Considering heat conduction problems, Eq. (13a) represents the final system of equations (with energy equilibrium satisfaction) to be solved. An interesting feature of this solution is that it corresponds to the boundary element results with an infinite k value (Eq. (5)) added to the fundamental solution. For details regarding boundary elements with equilibrium satisfaction, applied to only steady-state potential problems, the reader is referred to [5].

4. Mechanical analysis

The basic displacement and stress integral equations associated with the initial stress formulation of dynamic elastoplasticity and thermal analyses are defined as follows (body force terms omitted for simplicity) [9–11]:

$$\begin{aligned} c(\xi)\mathbf{u}(\xi,t) = &\int_\Gamma \mathbf{u}^*(\xi,x)\mathbf{p}(x,t)\,\mathrm{d}\Gamma(x) \\ &- \int_\Gamma \mathbf{p}^*(\xi,x)\mathbf{u}(x,t)\,\mathrm{d}\Gamma(x) \\ &- \int_\Omega \mathbf{u}^*(\xi,x)\rho\ddot{\mathbf{u}}(x,t)\,\mathrm{d}\Omega(x) \\ &+ \int_\Omega \boldsymbol{\varepsilon}^*(\xi,x)(\boldsymbol{\sigma}^p(x,t) \\ &+ \mathbf{y}T(x,t))\,\mathrm{d}\Omega(x), \end{aligned} \tag{17}$$

$$\begin{aligned} \boldsymbol{\sigma}(\xi,t) = &\int_\Gamma \mathbf{u}'^*(\xi,x)\mathbf{p}(x,t)\,\mathrm{d}\Gamma(x) - \int_\Gamma \mathbf{p}'^*(\xi,x)\mathbf{u}(x,t)\,\mathrm{d}\Gamma(x) \\ &- \int_\Omega \mathbf{u}'^*(\xi,x)\rho\ddot{\mathbf{u}}(x,t)\,\mathrm{d}\Omega(x) + \int_\Omega \boldsymbol{\varepsilon}'^*(\xi,x)(\boldsymbol{\sigma}^p(x,t) \\ &+ \mathbf{y}T(x,t))\,\mathrm{d}\Omega(x) + \mathbf{g}(\boldsymbol{\sigma}^p(x,t) + yT(x,t)), \end{aligned} \tag{18}$$

where $\mathbf{u}^*(\xi, x)$, $\mathbf{p}^*(\xi, x)$, $\boldsymbol{\varepsilon}^*(\xi, x)$, $\mathbf{u}'^*(\xi, x)$, $\mathbf{p}'^*$ and $\boldsymbol{\varepsilon}'^*(\xi, x)$ are fundamental tensors. Vectors $\mathbf{u}(x, t)$ and $\mathbf{p}(x, t)$ describe the distribution of the displacement and traction components of the problem to be solved; $\boldsymbol{\sigma}^p(x, t)$ represents the "initial (plastic) stress" components and $T(x, t)$ stands for the relative temperature state as in Section 3. The free term $\mathbf{g}$ is due to the correct derivative of the initial stress domain integral [10] and $\mathbf{y}$ stands for a matrix representation of tensor γ_{ij}.

As before, in order to introduce the equilibrium into the BEM formulation, the standard fundamental solution $\mathbf{u}^*(\xi, x)$ is modified: now traction equilibrium is attained by introducing rigid body translations in the directions of the co-ordinate axes and moment equilibrium is achieved by superimposing rigid body rotations to the fundamental solution. The modified fundamental solution that arises is given by

$$\bar{\mathbf{u}}^*(\xi,x) = \mathbf{u}^*(\xi,x) + k(\mathbf{I} + r\mathbf{\Theta}), \tag{19}$$

where k is a constant amplitude, r is the distance between the source and field points, $\mathbf{I}$ is the identity matrix and $\mathbf{\Theta}$ is a rotational matrix [5,6].

The implementation of $\bar{\mathbf{u}}^*(\xi,x)$, as before, generates extra terms in the right-hand side of Eq. (17). After boundary and domain discretization, Eq. (17), taking into account the fundamental solution (19), can be written as the following system of equations:

$$\mathbf{HU} = \mathbf{GP} - \mathbf{M}\ddot{\mathbf{U}} + \mathbf{W}(\mathbf{O}^p + \mathbf{YT}) + \mathbf{VQ}, \tag{20}$$

where $\mathbf{H}$ and $\mathbf{G}$ are the standard boundary element influence matrices and $\mathbf{M}$ and $\mathbf{W}$ are the standard domain influence matrices (mass and initial stress matrices, respectively). $\mathbf{V}$ and $\mathbf{Q}$ are terms associated with the equilibrium satisfaction and they arise due to the extra terms added in Eq. (19). Matrix $\mathbf{V}$ is defined by

$$\mathbf{V} = \begin{bmatrix} 1 + X_2(\xi_1) & -X_1(\xi_1) & 1 \\ X_2(\xi_1) & 1 - X_1(\xi_1) & 1 \\ \cdots\cdots & \cdots\cdots & \cdots \\ 1 + X_2(\xi_2) & -X_1(\xi_2) & 1 \\ X_2(\xi_2) & 1 - X_1(\xi_2) & 1 \\ \vdots & \vdots & \vdots \end{bmatrix}, \tag{21}$$

where X_1 and X_2 stand for the co-ordinate axes (2D problems) and ξ_i corresponds to each nodal point i. $\mathbf{Q}$ is a vector whose components are associated with equilibrium in the X_1 and X_2 directions and moment equilibrium with respect to the origin of these global axes:

$$Q = k\begin{bmatrix} \int_\Gamma p_1(x,t)\,\mathrm{d}\Gamma(x) \\ \int_\Gamma p_2(x,t)\,\mathrm{d}\Gamma(x) \\ \int_\Gamma [X_1(x)p_2(x,t) - X_2(x)p_1(x,t)]\,\mathrm{d}\Gamma(x) \end{bmatrix} + \\ - k\begin{bmatrix} \int_\Omega \rho\ddot{u}_1(x,t)\,\mathrm{d}\Omega(x) \\ \int_\Omega \rho\ddot{u}_2(x,t)\,\mathrm{d}\Omega(x) \\ \int_\Omega \rho[X_1(x)\ddot{u}_2(x,t) - X_2(x)\ddot{u}_1(x,t)]\,\mathrm{d}\Omega(x) \end{bmatrix} \\ = k\mathbf{N}^{\mathrm{T}}\mathbf{P} - k\mathbf{\Pi}^{\mathrm{T}}\ddot{\mathbf{U}}, \quad (22)$$

where matrices $\mathbf{N}$ and $\mathbf{\Pi}$ are computed by simple integration of the interpolation functions and their products by co-ordinates along the boundary and the domain, respectively. For equilibrium satisfaction, one should have

$$\mathbf{N}^{\mathrm{T}}\mathbf{P} = \mathbf{\Pi}^{\mathrm{T}}\ddot{\mathbf{U}}. \quad (23)$$

As in the previous section, a reduced system of equations can be obtained from Eqs. (20) and (23), as follows: (i) multiply the first set of Eq. (20)by $\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}$ and isolate $\mathbf{Q}$ (one should note that $\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{G}_{bb}\mathbf{P} = \mathbf{\Pi}^{\mathrm{T}}\ddot{\mathbf{U}}$, according to Eq. (23)), (ii) back substitute the expression obtained for $\mathbf{Q}$ in Eq. (20) and rearrange.

As a result of the above-described steps, the following final reduced system of equations is achieved:

$$\bar{\mathbf{H}}\mathbf{U} = \mathbf{G}\mathbf{P} - \bar{\mathbf{M}}\ddot{\mathbf{U}} + \bar{\mathbf{W}}(\mathbf{O}^p + \mathbf{Y}\mathbf{T}), \quad (24)$$

where the modified matrices $\bar{\mathbf{H}}$, $\bar{\mathbf{M}}$ and $\bar{\mathbf{W}}$ are given by

$$\bar{\mathbf{H}}_{bb} = \mathbf{H}_{bb} - \mathbf{V}_b\boldsymbol{\lambda}^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{H}_{bb}, \quad (25a)$$

$$\bar{\mathbf{H}}_{db} = \mathbf{H}_{db} - \mathbf{V}_d\boldsymbol{\lambda}^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{H}_{bb}, \quad (25b)$$

$$\bar{\mathbf{M}} = \mathbf{M} - \mathbf{V}\boldsymbol{\lambda}^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{M}_b + \mathbf{V}\boldsymbol{\lambda}^{-1}\mathbf{\Pi}^{\mathrm{T}}, \quad (26)$$

$$\bar{\mathbf{W}} = \mathbf{W} - \mathbf{V}\boldsymbol{\lambda}^{-1}\mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{W}_b, \quad (27)$$

and

$$\boldsymbol{\lambda} = \mathbf{N}^{\mathrm{T}}\mathbf{G}_{bb}^{-1}\mathbf{V}_b. \quad (28)$$

One should observe that the equilibrium satisfaction is introduced by modifying the fundamental solution (Eq. (19)), adding to it rigid body movements. Once only rigid body movements are added, the stress integral Eq. (18) remains the same as that of the non-self-equilibrated formulation. After boundary and domain discretization, Eq. (18) can be written as the following standard system of equations:

$$\mathbf{O} = \mathbf{G}'\mathbf{P} - \mathbf{H}'\mathbf{U} - \mathbf{M}'\ddot{\mathbf{U}} + \mathbf{W}'(\mathbf{O}^p + \mathbf{Y}\mathbf{T}), \quad (29)$$

where matrices $\mathbf{H}'$ and $\mathbf{G}'$ correspond to the boundary integrals while matrices $\mathbf{M}'$ and $\mathbf{W}'$ correspond to the inertial and initial stress domain integrals (the free term depicted in Eq. (18) is included in $\mathbf{W}'$). Vectors $\mathbf{O}$ and $\mathbf{O}^p$ stand for the stress and "plastic stress" nodal values, respectively.

Eqs. (13), (24) and (29) must be taken into account in order to solve the thermo-mechanical problem considering a boundary element formulation with equilibrium satisfaction. Some details about the implementation and numerical solution of these equations, adopted in the present work, are discussed in the next section.

5. Numerical implementation and applications

In each time-step of the present solution process, the thermal analysis is initially carried out and once the relative temperature field is evaluated, it is used as an input parameter to the mechanical analysis. The mechanical problem is solved in the sequence taking into account an iterative process to evaluate the stress components. Once the real stress state is achieved, the displacements are evaluated and updated, regarding the "initial" stress state contribution.

In the present work, linear boundary elements are adopted to discretize the boundary and linear triangular cells are adopted to discretize the domain. The integrals related to the computation of matrices $\mathbf{N}$ and $\mathbf{\Pi}$ (both for the thermal and mechanical analyses) are evaluated analytically. The Houbolt scheme [12] is considered in order to deal with the time-domain solution of the thermal and mechanical transient equations (parabolic and hyperbolic problem, respectively). The Newton–Raphson method is employed to treat the non-linear problem solution and the stress equation is the only one used for the non-linear iterations [10]. The Moore–Penrose pseudo-inverse technique [13]is adopted to evaluate matrix $\mathbf{G}_{bb}^{-1}$ directly from matrix $\mathbf{G}_{bb}$ (one should observe that matrix $\mathbf{G}_{bb}$ can be singular, as for instance, when double nodes are considered). The inversion of matrix $\boldsymbol{\lambda}$ is trivial and is obtained analytically (for 2D mechanical problems, $\boldsymbol{\lambda}$ is 3×3 whereas for thermal problems it is a scalar).

If one compares the proposed BEM formulation to a classical BEM approach, it is noticed that prior to the start of the time marching scheme (which is the very same for both) there is an increase in the computer effort to obtain the required matrix equations (i.e, evaluating a $\bar{\mathbf{A}}$ matrix is more expensive than evaluating a $\mathbf{A}$ matrix). This increase, however, may be compensated by the fact that the built-in equilibrium satisfaction tends to improve the solution. Hence it requires less discretization (mesh refinement) for equivalent accuracy. Such behaviour is explored in the second example presented next.

5.1. Example 1

In this first example, a cantilever beam is analysed, taking into account a pure mechanical behaviour and a thermo-mechanical behaviour. A sketch of the model is depicted in Fig. 1. The properties of the beam are: $E = 200\,\mathrm{N/m^2}$; $\nu = 0.2$; $\rho = 0.1\,\mathrm{kg/m^3}$; $K = 4.0\,\mathrm{J/m/s/^\circ C}$; $c = 1.0\,\mathrm{J/kg/^\circ C}$; $\alpha = 0.1/^\circ\mathrm{C}$. Essential and natural boundary conditions are prescribed, for the isolated mechanical

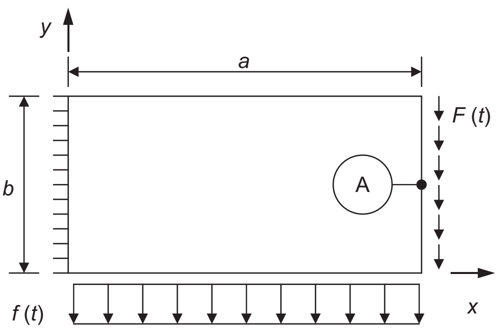

Fig. 1. Geometry and boundary conditions for the first example model.

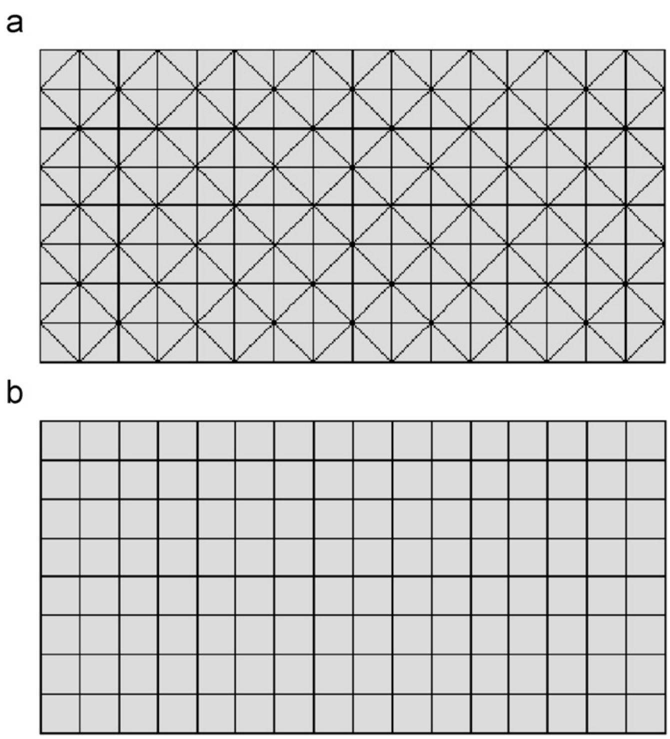

Fig. 2. Model discretization: (a) BEM mesh, (b) FEM mesh.

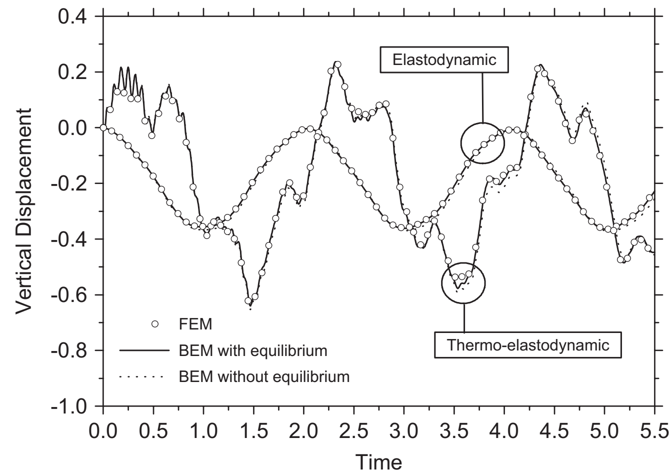

Fig. 3. Vertical displacement time history at point A considering elastodynamic and thermo-elastodynamic analyses.

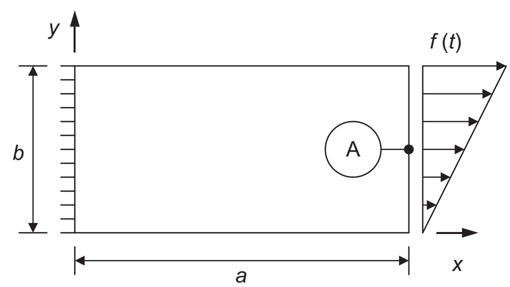

Fig. 4. Geometry and boundary conditions for the second example model.

problem, as $\bar{u}_i(x=0,y,t)=0$, $\bar{p}_i(x,y=0,t)=0$, $\bar{p}_i(x,y=b,t)=0$, for $i=1$, $2;\bar{p}_1(x=a,y,t)=0$ and $\bar{p}_2(x=a,y,t)=A_pF(t)$, where $A_p=-1.0\,\mathrm{N/m}$ and $F(t)$ is the Heaviside step function. As for the thermal problem, natural and essential boundary conditions are prescribed: $\bar{R}(x=0,y,t)=0$, $\bar{R}(x=a,y,t)=0$, $\bar{R}(x,y=b,t)=0$ and $\bar{T}(x,y=0,t)=A_Tf(t)$, where $A_T=5.0\,^\circ\mathrm{C}$ and $f(t)$ is the Heaviside step function. The geometry of the model is defined as $a=2\,\mathrm{m}$ and $b=1\,\mathrm{m}$.

The cantilever beam is analysed considering the BEM (with and without equilibrium satisfaction) and the FEM. The meshes adopted for each method are depicted in Fig. 2, 48 linear boundary elements (four double nodes) and 256 linear triangular cells are considered for the BEM (Fig. 2(a)) and 128 linear quadrangular finite elements are considered for the FEM (Fig. 2(b)). The same time discretization step is adopted for the BEM and for the FEM and the selected time-step is $\Delta t=0.00275\,\mathrm{s}$.

The time history solutions for the vertical displacements at point A ($x=a$, $y=b/2$) are shown in Fig. 3 for both boundary and finite element procedures. The results are depicted considering pure elastodynamic and thermo-elastodynamic analyses. As can be observed, improved results are obtained taking into account equilibrium satisfaction within the BEM formulation. Though the improvement of results may be small for some linear or sufficiently discretized models, it can become quite significant when non-linear behaviour or poorly discretized models are considered, as illustrated in the next example.

5.2. Example 2

In this example, a cantilever beam is analysed taking into account different spatial discretization and linear or non-linear thermo-mechanical behaviour. A sketch of the model is presented in Fig. 4. The properties of the beam are: $E=100\,\mathrm{N/m^2}$; $\nu=0.3$; $\rho=0.1\,\mathrm{kg/m^3}$; $K=10.0\,\mathrm{J/m/s/^\circ C}$; $c=1.0\,\mathrm{J/kg/^\circ C}$; $\alpha=0.25/^\circ\mathrm{C}$. For the mechanical problem, the essential and natural boundary conditions are $\bar{u}_i(x=0,y,t)=0$, $\bar{p}_i(x=a,y,t)=0$, $\bar{p}_i(x,y=0,t)=0$ and $\bar{p}_i(x,y=b,t)=0$, for $i=1$, 2. The natural and essential

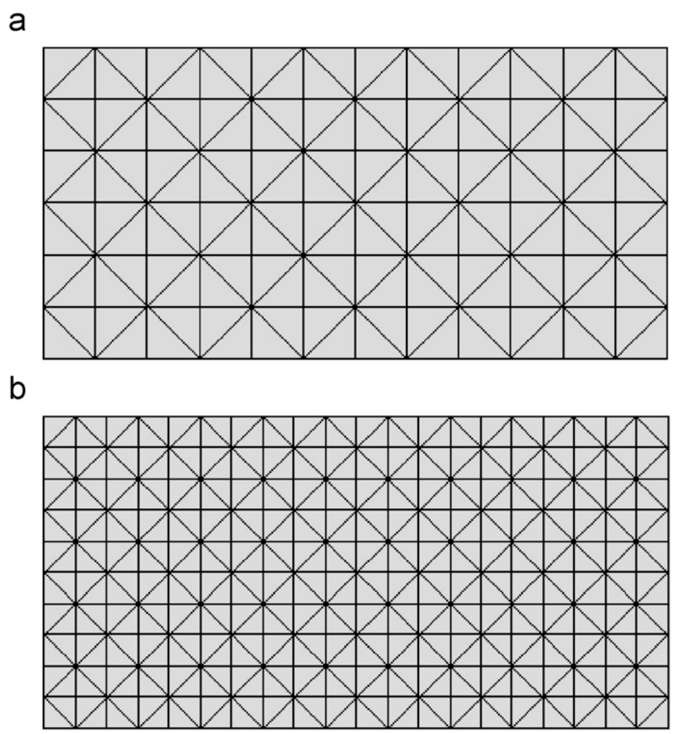

Fig. 5. Model discretization: (a) Mesh 1, (b) Mesh 2.

boundary conditions for the thermal problem are defined as $\bar{R}(x=0,y,t)=0$, $\bar{R}(x,y=0,t)=0$, $\bar{R}(x,y=b,t)=0$ and $\bar{T}(x=a,y,t)=A(y)f(t)$, where $A(y)=(2y/b)\,^{\circ}\mathrm{C}$ and $f(t)$ is the Heaviside step function. The geometry of the model is given by $a=12\,\mathrm{m}$ and $b=6\,\mathrm{m}$.

The cantilever beam is analysed considering the BEM (with and without equilibrium satisfaction) with two different spatial discretization refinement levels. The meshes adopted are depicted in Fig. 5: in the first discretization scheme (mesh 1), 36 linear boundary elements (four double nodes) and 144 linear triangular cells are considered (Fig. 5(a)); in the second discretization (mesh 2), 60 linear boundary elements (four double nodes) and 400 linear triangular cells are included (Fig. 5(b)). The time-steps adopted are $\Delta t_1=0.01\,\mathrm{s}$ (mesh 1) and $\Delta t_2=0.006\,\mathrm{s}$ (mesh 2).

The time history solutions for the vertical displacements at point A ($x=a$, $y=b/2$) are shown in Fig. 6, taking into account linear (elastic) and non-linear (elastoplastic) analyses. As one can observe, considering a relatively poor discretization, the BEM formulations with and without equilibrium satisfaction produce results with a good difference from each other. Nevertheless, the formulation with equilibrium is seen closer, in the average sense, to the results obtained with the refined discretization procedures. One should also observe that when non-linear behaviour is added, the differences produced by the two BEM formulations are amplified. In this case, the classical BEM formulation may give rise to inadequate results and the present proposed formulation should be adopted.

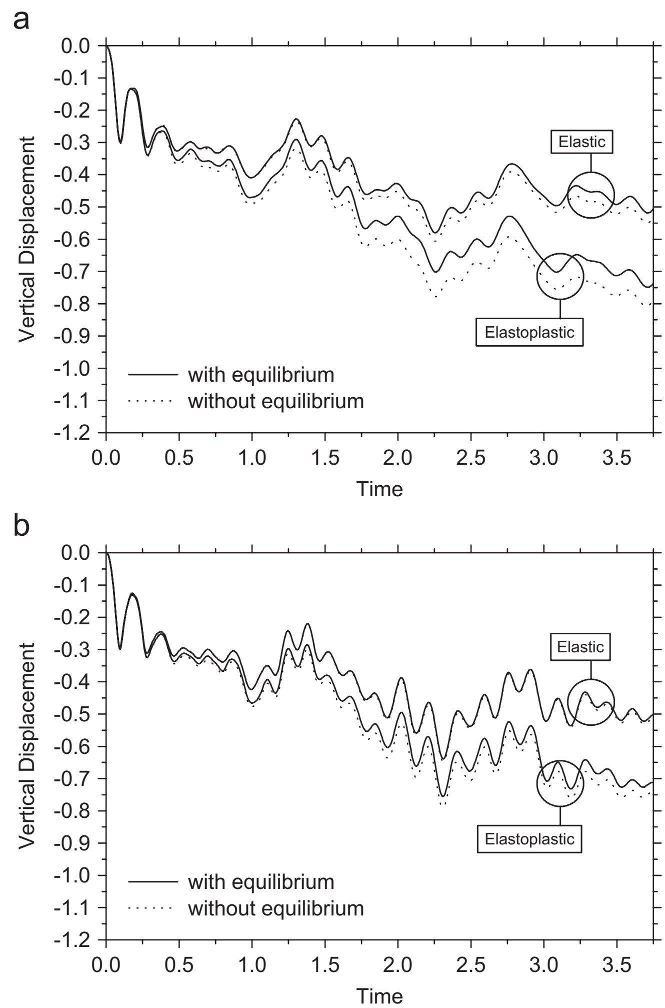

Fig. 6. Vertical displacement time history at point A considering elastic and elastoplastic analyses: (a) Mesh 1, (b) Mesh 2.

6. Conclusions

In the present work, a boundary element formulation with energy balance satisfaction has been introduced and the proposed methodology is applied to model heat conduction problems (diffusion equation). Taking into account dynamic applications, a boundary element formulation with equilibrium satisfaction with respect to the co-ordinate axis directions and moments, also considering inertia forces, has likewise been presented and both scalar and vector self-equilibrated formulations have been produced together, treating in consistent form thermo-mechanical problems, including plasticity.

The developed methodology improves the accuracy of the BEM, as illustrated by the numerical examples presented. The merit of the proposed procedure is highlighted when poor discretized or non-linear (elastoplastic) models are considered. The generality and ease of usage of the procedure, when implemented into existing boundary element codes, are also worth noticing.

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