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A new BEM formulation for transient axisymmetric poroelasticity via particular integrals

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Abstract

A simple particular integral formulation is presented for the first time in a purely axisymmetric poroelastic analysis. The axisymmetric elastostatic and steady-state potential flow equations are used as the complementary solution. The particular integrals for displacement, traction, pore pressure and flux are derived by integrating three-dimensional formulation along the circumferential direction leading to elliptic integrals.

Numerical results for three axisymmetric problems of soil consolidation are given and compared with their analytical solutions to demonstrate the accuracy of the present formulation. Generally, agreement among all of those results is satisfactory if one uses a few interior points, in addition to the regular boundary points.

Keywords: BEM; Particular integrals; Poroelasticity; Soil consolidation; Elastostatic; Potential flow

1. Introduction

The general theory of poroelasticity is governed by two coupled differential equations: the Navier equation with pore pressure body force and the pore fluid flow equation as (Banerjee, 1994)

$$(\lambda+\mu)u_{j,ji}+\mu u_{i,jj}-\beta p_{,i}+f_i=0 \tag{1}$$

$$\kappa p_{,jj}-\alpha\dot{p}-\beta\dot{u}_{j,j}+\psi=0 \tag{2}$$

where u_i is the displacement, p is the pore pressure, λ and μ are Lame's constants, κ is the effective permeability, $\alpha=\frac{\beta^2}{\lambda_u-\lambda}$, λ_u the undrained λ, $\beta=1-\frac{K}{K_s'}$, $K=\lambda+\frac{2\mu}{3}$ the drained bulk modulus, K_s' the empirical constant which in certain circumstances equals to bulk modulus of the solid constituents, f_i and ψ are the body force and source (if present) in the volume, and $i=1, 2(3)$ for two(three) dimensions. Indicial notation is employed. Thus, commas represent differentiation with respect to spatial coordinates, while a superposed dot denotes a

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time derivative. The constants β and α can also be expressed in terms of the undrained bulk modulus K_u as (Rice and Cleary, 1976)

$$\beta = \frac{1}{B}\left(1 - \frac{K}{K_u}\right) \tag{3}$$

$$\alpha = \frac{\beta}{K_u B} \tag{4}$$

where B is the well-known Skempton's coefficient of pore pressure.

Because of the pore pressure loading term in Navier equation (1) and/or the transient terms of pore pressure and displacement in the pore fluid flow equation (2), the direct application of the boundary element method (BEM) to the coupled poroelastic problems generates a domain integral in addition to the usual surface integrals (Banerjee and Butterfield, 1981). In order to eliminate this volume integration problem, the particular integral method has been proposed (Park and Banerjee, 2002a, 2006).

In the particular integral method a total solution is obtained as the sum of a complementary solution for the homogeneous part of the differential equation and a particular solution for the total governing inhomogeneous differential equation. Thus the first concern in the method is the selection of the combination of homogeneous and inhomogeneous parts from the governing differential equation. For 2D and 3D coupled poroelastic analysis, Park and Banerjee (2002a) first proposed the particular integral formulation by considering the following combination:

for homogeneous part,

$$(\lambda + \mu)u^c_{j,ji} + \mu u^c_{i,jj} - \beta p^c_{,i} = 0 \tag{5}$$

$$\kappa p^c_{,jj} = 0 \tag{6}$$

and for inhomogeneous part,

$$(\lambda + \mu)u^p_{j,ji} + \mu u^p_{i,jj} - \beta p^p_{,i} = 0 \tag{7}$$

$$\kappa p^p_{,jj} - \alpha \dot{p} - \beta \dot{u}_{j,j} = 0 \tag{8}$$

in the absence of the body force and source, where u^c_i, p^c and u^p_i, p^p are complementary functions and particular integrals for displacement and pore pressure, respectively, and superscripts c and p indicate complementary and particular solutions, respectively. In the above combination, the solution of the steady-state coupled poroelasticity equation is used as the complementary function. The required particular integrals for displacement, traction, pore pressure and flux are derived by using a set of global shape functions ($D_{ik} = \delta_{ik}(A - r)^2$ and $K_p = A - r$) to approximate the time derivative terms of displacement and pore pressure in the pore fluid flow equation.

However, one of the most important salient points in the particular integral method is that several types of combination of homogeneous and inhomogeneous parts are possible from the governing equation if the fundamental solution of the homogeneous equation is available and if the particular integral can be found for the inhomogeneous equation. With this idea, Park and Banerjee (2006) showed the success of the more efficient and simpler combination for the particular integral formulation of 2D coupled poroelastic analysis in which they used:

for homogeneous part,

$$(\lambda + \mu)u^c_{j,ji} + \mu u^c_{i,jj} = 0 \tag{9}$$

$$\kappa p^c_{,jj} = 0 \tag{10}$$

and for inhomogeneous part,

$$(\lambda + \mu)u^p_{j,ji} + \mu u^p_{i,jj} - \beta p_{,i} = 0 \tag{11}$$

$$\kappa p^p_{,jj} - \alpha \dot{p} - \beta \dot{u}_{j,j} = 0 \tag{12}$$

The differences from the previous one (Eqs. (5)–(8)) and advantages of the above one (Eqs. (9)–(12)) are:

1. Simpler homogeneous part. We use elastostatic and steady-state potential flow equations, instead of steady-state coupled poroelasticity.
2. Simpler particular integrals for displacement and traction. One can easily derive these particular integrals by separating the coupled equation and introducing one more global shape function for pore pressure loading term in the Navier equation.
3. Finally easier implementation into the computer program because of simpler homogeneous part, simpler particular integrals, and the elimination of significant matrix algebra involving some coupled terms.

Following the success of the above 2D formulation, this paper presents the simple particular integral formulation for the purely axisymmetric coupled poroelastic analysis on the basis of Eqs. (9)–(12). The axisymmetric elastostatic and steady-state potential flow equations are used as the complementary solution. The particular integrals for displacement, traction, pore pressure and flux are derived by integrating three-dimensional formulation along the circumferential direction leading to elliptic integrals. To the best of the authors' knowledge no such BEM formulation for axisymmetric coupled poroelastic analysis exists in the published literature.

In order to deal with axisymmetric problems, the three-dimensional particular integral formulation for coupled poroelastic analysis is briefly reviewed in the next section. Three examples of application for axisymmetric soil consolidation are presented along with their analytical solutions (AS) to test the present formulation.

2. Three-dimensional particular integral formulation

From the combination of homogeneous and inhomogeneous parts of governing equations (9)–(12), total solutions for displacement u_i, traction t_i, pore pressure p and flux q can be obtained as

$$u_i = u_i^{\mathrm{c}} + u_i^{\mathrm{p}} \tag{13a}$$
$$t_i = t_i^{\mathrm{c}} + t_i^{\mathrm{p}} \tag{13b}$$
$$p = p^{\mathrm{c}} + p^{\mathrm{p}} \tag{13c}$$
$$q = q^{\mathrm{c}} + q^{\mathrm{p}} \tag{13d}$$

where t_i^{p}, q^{p}, etc. are the particular integrals for traction and flux, etc.

Then the required particular integrals can be obtained separately from Eqs. (11) and (12). By approximating the pore pressure loading term in the Navier equation and the transient terms in the pore fluid flow equation with known global shape functions, $C(x,\xi_n)$, $D_{ik}(x,\xi_n)$ and $K_{\mathrm{p}}(x,\xi_n)$, and fictitious density functions, $\phi(\xi_n)$, $\dot{\phi}_k(\xi_n)$ and $\dot{\phi}_p(\xi_n)$, such that

$$p(x) = \sum_{n=1}^{\infty} C(x,\xi_n)\phi(\xi_n) \tag{14}$$
$$\dot{u}_i(x) = \sum_{n=1}^{\infty} D_{ik}(x,\xi_n)\,\dot{\phi}_k(\xi_n) \tag{15}$$
$$\dot{p}(x) = \sum_{n=1}^{\infty} K_p(x,\xi_n)\,\dot{\phi}_p(\xi_n) \tag{16}$$

the particular integrals which satisfy Eqs. (11) and (12) can be found as (Park and Banerjee, 2002a,b, 2006)

$$u_i^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} U_i(x, \xi_n)\phi(\xi_n) \tag{17}$$

$$\sigma_{ij}^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} S_{ij}(x, \xi_n)\phi(\xi_n) \tag{18}$$

$$t_i^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} T_i(x, \xi_n)\phi(\xi_n) \tag{19}$$

$$p^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} \left\{ P_k(x, \xi_n)\, \dot{\phi}_k(\xi_n) + P_p(x, \xi_n)\, \dot{\phi}_p(\xi_n) \right\} \tag{20}$$

$$q^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} \left\{ Q_k(x, \xi_n)\, \dot{\phi}_k(\xi_n) + Q_p(x, \xi_n)\, \dot{\phi}_p(\xi_n) \right\} \tag{21}$$

By introducing the following set of global shape function,

$$C(x, \xi_n) = C_1 A - C_2 r \tag{22}$$

$$D_{ik}(x, \xi_n) = \delta_{ik}(D_1 A - D_2 r)^2 \tag{23}$$

$$K_p(x, \xi_n) = K_1 A - K_2 r \tag{24}$$

the corresponding kernels can be derived as

$$U_i(x, \xi_n) = (U_1 A - U_2 r) y_i \tag{25}$$

$$S_{ij}(x, \xi_n) = \delta_{ij}(S_1 A - S_2 r) - S_3 \frac{y_i y_j}{r} \tag{26}$$

$$T_i(x, \xi_n) = S_{ij}(x, \xi_n) n_j(x) \tag{27}$$

$$P_k(x, \xi_n) = -(P_1 A - P_2 r) r y_k \tag{28}$$

$$P_p(x, \xi_n) = (P_3 A - P_4 r) r^2 \tag{29}$$

$$Q_k(x, \xi_n) = k\left\{ \delta_{ik}(P_1 A - P_2 r) r + (P_1 A - 2P_2 r)\frac{y_i y_k}{r} \right\} n_i \tag{30}$$

$$Q_p(x, \xi_n) = -k(2P_3 A - 3P_4 r) y_i n_i \tag{31}$$

where r is the distance between x and ξ_n, A is a constant chosen to be the largest dimension of the problem domain, and $n_j(x)$ is the unit normal at x in the jth direction.

Substituting Eqs. (14)–(31) into Eqs. (11), (12) one can obtain the following relationship among the coefficients

$$U_1 = \frac{\beta^*}{3} C_1, \quad U_2 = \frac{\beta^*}{4} C_2 \tag{32}$$

$$S_1 = -\frac{4}{3}\mu\beta^* C_1, \quad S_2 = -\frac{3}{2}\mu\beta^* C_2, \quad S_3 = \frac{1}{2}\mu\beta^* C_2 \tag{33}$$

$$P_1 = \frac{1}{2}\beta^{**} D_1 D_2, \quad P_2 = \frac{1}{5}\beta^{**} D_2^2, \quad P_3 = \frac{1}{6}\beta^{***} K_1, \quad P_4 = \frac{1}{12}\beta^{***} K_2 \tag{34}$$

where $\beta^* = \frac{\beta}{(\lambda+2\mu)}$, $\beta^{**} = \frac{\beta}{\kappa}$, and $\beta^{***} = \frac{\alpha}{\kappa}$.

Considering $U_1 = 2$, $U_3 = 3$ and $P_1 = P_2 = P_3 = P_4 = 1$ (Henry et al., 2002; Park and Banerjee, 2003), other coefficients can easily be obtained from Eqs. (32)–(34).

3. Axisymmetric particular integral formulation

If the body forces are known in an explicit algebraic form such as in case of bodies subjected to centrifugal forces or gravitational acceleration etc., the particular integrals can be constructed in the form of a simple polynomial. For axisymmetric problems, use of such polynomial functions as functions of r and z coordinates have been discussed in Henry et al. (1987). Unfortunately, for the present problem it is not possible to use the axisymmetric polynomial forms of Eqs. (22)–(31) and get reliable results. Instead we need to express these 3D equations into a general axisymmetric form (using r, θ, z coordinates) and complete a circumferential (θ) integration to get the resulting algebraic expressions which are then usable as particular integrals. It is of considerable interest to note that Wang and Banerjee (1988, 1990) in their developments of particular integrals in free-vibration analysis of axisymmetric solids also observed the same to be true.

Thus in order to obtain the corresponding axisymmetric BEM formulation, the three-dimensional particular integrals, given in Eqs. (14)–(21), are first rewritten in cylindrical coordinates (r,θ,z) by integrating along the circumferential (θ) direction, we get

$$p(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} C'(x,\xi_n)\mathrm{d}\theta\, \phi(\xi_n) \tag{35}$$

$$\dot{u}_\alpha(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} D'_{\alpha k}(x,\xi_n)\mathrm{d}\theta\, \dot{\phi}_k(\xi_n) \tag{36}$$

$$\dot{p}(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} K'_p(x,\xi_n)\mathrm{d}\theta\, \dot{\phi}_p(\xi_n) \tag{37}$$

$$u^{\mathrm{p}}_\alpha(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} U'_\alpha(x,\xi_n)\mathrm{d}\theta\, \phi(\xi_n) \tag{38}$$

$$t^{\mathrm{p}}_\alpha(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} T'_\alpha(x,\xi_n)\mathrm{d}\theta\, \phi(\xi_n) \tag{39}$$

$$p^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} \left\{ P'_k(x,\xi_n)\mathrm{d}\theta\, \dot{\phi}_k(\xi_n) + P'_p(x,\xi_n)\mathrm{d}\theta\, \dot{\phi}_p(\xi_n) \right\} \tag{40}$$

$$q^{\mathrm{p}}(x) = \sum_{n=1}^{\infty} \int_0^{2\pi} \left\{ Q'_k(x,\xi_n)\mathrm{d}\theta\, \dot{\phi}_k(\xi_n) + Q'_p(x,\xi_n)\mathrm{d}\theta\, \dot{\phi}_p(\xi_n) \right\} \tag{41}$$

where u^{p}_α and t^{p}_α are, for convenience, defined in axisymmetry case

$$\{u^{\mathrm{p}}_\alpha\} = \{u^{\mathrm{p}}_r\ u^{\mathrm{p}}_z\}^{\mathrm{T}} \tag{42}$$

$$\{t^{\mathrm{p}}_\alpha\} = \{t^{\mathrm{p}}_r\ t^{\mathrm{p}}_z\}^{\mathrm{T}} \tag{43}$$

Note that when transforming a line integral to an integral with respect to angle, one usually uses

$$\mathrm{d}l = r_{\mathrm{o}}\,\mathrm{d}\theta \tag{44}$$

In Eqs. (35)–(41), the radius r_{o} has been absorbed in the fictitious functions $\phi(\xi_n)$, $\dot{\phi}_k(\xi_n)$ and $\dot{\phi}_p(\xi_n)$ terms, so that it does not need to appear explicitly.

Considering purely axisymmetry body in cylindrical coordinates (Fig. 1), we have

$$\begin{aligned} &\theta = \theta_x - \theta_{\xi_n} \\ &y_1 = r_x\cos\theta - r_{\xi_n}, \quad y_2 = r_x\sin\theta, \quad y_3 = Z = z_x - z_{\xi_n} \\ &n_1 = n_r\cos\theta, \quad n_2 = n_r\sin\theta, \quad n_3 = n_z \end{aligned} \tag{45}$$

and then

$$y_i n_i = \left(r_x - r_{\xi_n}\cos\theta\right)n_r + Zn_z \tag{46}$$

where n_r, n_z = components of normal vector at point x in r and z-directions, respectively.

Substituting Eqs. (45), (46) into Eqs. (22)–(31) the integration of kernels $C'(x,\xi_n)$, $U'_\alpha(x,\xi_n)$, $T'_\alpha(x,\xi_n)$, $D'_{\alpha k}(x,\xi_n)$, $K'_p(x,\xi_n)$, $P'_k(x,\xi_n)$, $P'_p(x,\xi_n)$, $Q'_k(x,\xi_n)$ and $Q'_p(x,\xi_n)$ can be achieved in terms of elliptic integrals (see Appendix A).

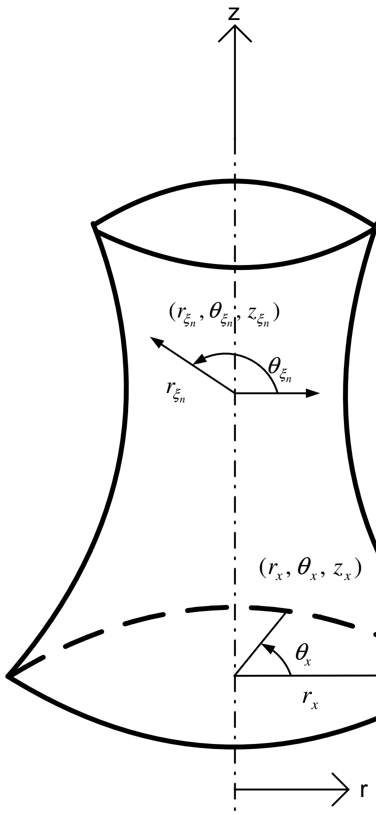

Fig. 1. Axisymmetry body in cylindrical coordinates.

4. Numerical implementation

The boundary integral equation related to the complementary functions u_α^c, t_α^c, p^c and q^c of Eqs. (9) and (10) can be written as (Banerjee, 1994)

$$\begin{Bmatrix} C_{\alpha\beta}(\xi)u_\alpha^c(\xi) \\ C_{pp}(\xi)p^c(\xi) \end{Bmatrix} = \int_C \left(\begin{bmatrix} G_{\alpha\beta}(x,\xi) & 0 \\ 0 & G_{pp}(x,\xi) \end{bmatrix} \begin{Bmatrix} t_\alpha^c(x) \\ q^c(x) \end{Bmatrix} - \begin{bmatrix} F_{\alpha\beta}(x,\xi) & 0 \\ 0 & F_{pp}(x,\xi) \end{bmatrix} \begin{Bmatrix} u_\alpha^c(x) \\ p^c(x) \end{Bmatrix} \right) \mathrm{d}C(x) \tag{47}$$

where $G_{\alpha\beta}$, $F_{\alpha\beta}$, G_{pp} and F_{pp} are the fundamental solutions for axisymmetric elastostatic and steady-state potential flow equations and $C_{\alpha\beta}(\xi)$, $C_{pp}(\xi)$ represent the jump terms resulting from the singular nature of $F_{\alpha\beta}$ and F_{pp}, respectively.

After a usual discretization of boundary C, Eq. (47) can be written in matrix form as

$$\begin{bmatrix} G_{\alpha\beta} & 0 \\ 0 & G_{pp} \end{bmatrix} \begin{Bmatrix} t_\alpha^c \\ q^c \end{Bmatrix} - \begin{bmatrix} F_{\alpha\beta} & 0 \\ 0 & F_{pp} \end{bmatrix} \begin{Bmatrix} u_\alpha^c \\ p^c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{48}$$

Considering the total solutions of Eq. (13) the complementary functions in Eq. (48) can be eliminated as

$$\begin{bmatrix} G_{\alpha\beta} & 0 \\ 0 & G_{pp} \end{bmatrix} \begin{Bmatrix} t_\alpha \\ q \end{Bmatrix} - \begin{bmatrix} F_{\alpha\beta} & 0 \\ 0 & F_{pp} \end{bmatrix} \begin{Bmatrix} u_\alpha \\ p \end{Bmatrix} = \begin{bmatrix} G_{\alpha\beta} & 0 \\ 0 & G_{pp} \end{bmatrix} \begin{Bmatrix} t_\alpha^p \\ q^p \end{Bmatrix} - \begin{bmatrix} F_{\alpha\beta} & 0 \\ 0 & F_{pp} \end{bmatrix} \begin{Bmatrix} u_\alpha^p \\ p^p \end{Bmatrix} \tag{49}$$

If a finite number of ξ_n, N, are chosen, the particular integrals for displacement, traction, pore pressure and flux can be written as

$$\{u_\alpha^{\mathrm{p}}\} = [U_\alpha]\{\phi\} \tag{50}$$

$$\{t_\alpha^{\mathrm{p}}\} = [T_\alpha]\{\phi\} \tag{51}$$

$$\{p^{\mathrm{p}}\} = [P_k \quad P_p]\begin{Bmatrix} \dot{\phi}_k \\ \dot{\phi}_p \end{Bmatrix} \tag{52}$$

$$\{q^{\mathrm{p}}\} = [Q_k \quad Q_p]\begin{Bmatrix} \dot{\phi}_k \\ \dot{\phi}_p \end{Bmatrix} \tag{53}$$

Substituting Eqs. (50)–(53) into (49) and considering the fictitious nodal values as

$$\{\phi\} = [C]^{-1}\{p\} \tag{54}$$

$$\begin{Bmatrix} \dot{\phi}_k \\ \dot{\phi}_p \end{Bmatrix} = \begin{bmatrix} D_{k\alpha}^{-1} & 0 \\ 0 & K_p^{-1} \end{bmatrix}\begin{Bmatrix} \dot{u}_\alpha \\ \dot{p} \end{Bmatrix} \tag{55}$$

one can obtain the following equation

$$\begin{bmatrix} G_{\alpha\beta} & 0 \\ 0 & G_{pp} \end{bmatrix}\begin{Bmatrix} t_\alpha \\ q \end{Bmatrix} - \begin{bmatrix} F_{\alpha\beta} & M_{\alpha p} \\ 0 & F_{pp} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ p \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ M_{p\alpha} & M_{pp} \end{bmatrix}\begin{Bmatrix} \dot{u}_\alpha \\ \dot{p} \end{Bmatrix} \tag{56}$$

where

$$[M_{\alpha p}] = \big([G_{\alpha\beta}][T_\alpha] - [F_{\alpha\beta}][U_\alpha]\big)[C]^{-1} \tag{57}$$

$$[M_{p\alpha} \quad M_{pp}] = \big([G_{pp}][Q_k \quad Q_p] - [F_{pp}][P_k P_p]\big)\begin{bmatrix} D_{k\alpha}^{-1} & 0 \\ 0 & K_p^{-1} \end{bmatrix} \tag{58}$$

Using an explicit time integration scheme, Eq. (56) can be expressed as

$$\begin{bmatrix} G_{\alpha\beta} & 0 \\ 0 & G_{pp} \end{bmatrix}\begin{Bmatrix} t_\alpha \\ q \end{Bmatrix}^t - \begin{bmatrix} F_{\alpha\beta} & M_{\alpha p} \\ \frac{1}{\Delta t}M_{p\alpha} & F_{pp} + \frac{1}{\Delta t}M_{pp} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ p \end{Bmatrix}^t = -\frac{1}{\Delta t}\begin{bmatrix} 0 & 0 \\ M_{p\alpha} & M_{pp} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ p \end{Bmatrix}^{t-\Delta t} \tag{59}$$

Since the right side of Eq. (59) involves known values of displacement and pore pressure specified either as initial conditions or calculated previously, the final system equation can be written as

$$[B]\{X\} = \{b\} \tag{60}$$

where X is unknown vector of displacement, traction, pore pressure and flux, b is a known vector and B is the coefficient matrix. Therefore, the unknown displacement or traction can be obtained together with the unknown pore pressure or flux.

As mentioned in the previous works (Park and Banerjee, 2006) the interior points can be used for a better representation of the particular integrals. It can be also noted that the present computer program for axisymmetric coupled poroelastic analysis is developed from the axisymmetric elastostatic and steady-state potential flow programs available in Banerjee (1994).

5. Numerical examples

In order to test the validity and accuracy of the present formulations, three example problems are solved. The example problems are described for consolidation problems of a saturated sphere of soil subjected to a uniform surface load and a single poroelastic layer of a finite thickness subjected to axisymmetric loading as well as unidirectional consolidation.

The material properties used in all example problems are : $\kappa = \frac{k}{\gamma_w} = 1.0$, $E = 1.0$, $v = 0$, $v_u = 0.5$ and $B = 1$. Notice, for this set of properties, that the diffusivity is unity.

5.1. Example 1: unidirectional consolidation

The first example is the unidirectional consolidation of a fully saturated soil. The uniform compression traction of unity is applied instantaneously at time $t = 0$ and thereafter held constant with drainage occurring only at the top surface. The soil sample is assumed to be axisymmetric with the remaining two faces which are impermeable and restrained from normal displacement. The modeling mesh with 12 quadratic boundary elements and six interior points is shown in Fig. 2.

The analytical solutions of pore pressure p and displacement u for this example problem can be obtained as (Biot, 1941)

$$p(z = 0, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin \frac{(2n-1)\pi}{2} e^{-\frac{(2n-1)^2 \pi^2 \kappa t}{4}}$$

$$u(z = 1, t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left(1 - e^{-\frac{(2n-1)^2 \pi^2 \kappa t}{4}}\right)$$

Some computed values of pore pressure at the point $(r, z) = (0, 0)$ and displacement at the point $(r, z) = (0, 1)$, for a time step of 0.0025, are shown in Figs. 3 and 4, respectively. For all figures shown hereafter, the number in the parenthesis represents the number of elements used for the analysis. Plus (+) sign indicates the additional number of the interior points involved in the analysis. For example in Fig. 3, (12+6) means 12 quadratic boundary elements and 6 interior points. Good agreement between analytical and numerical solutions can be seen, with the addition of interior points.

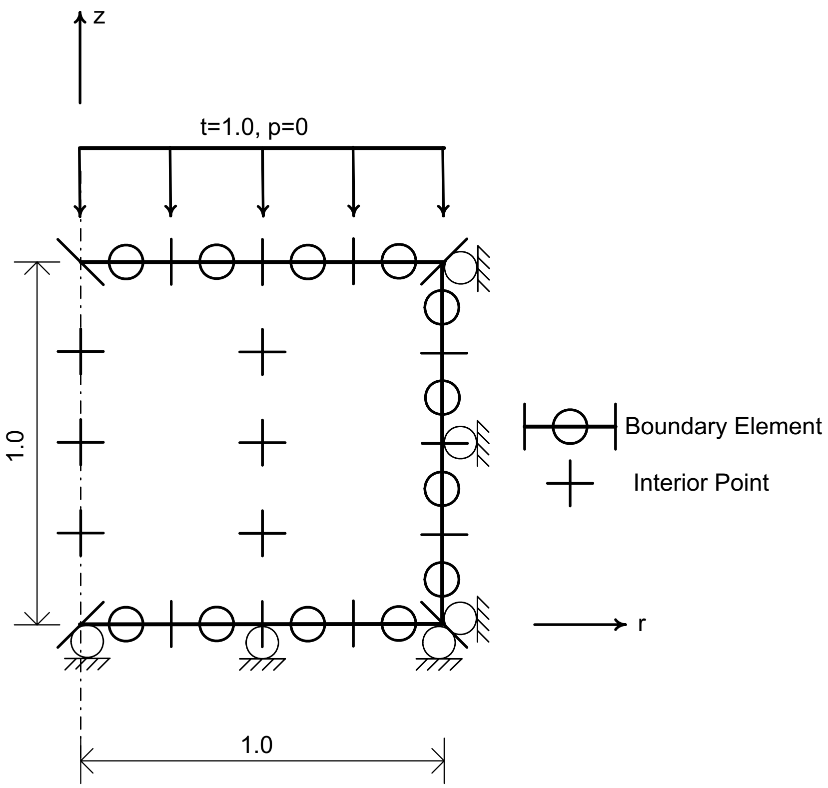

Fig. 2. Modeling mesh for unidirectional consolidation.

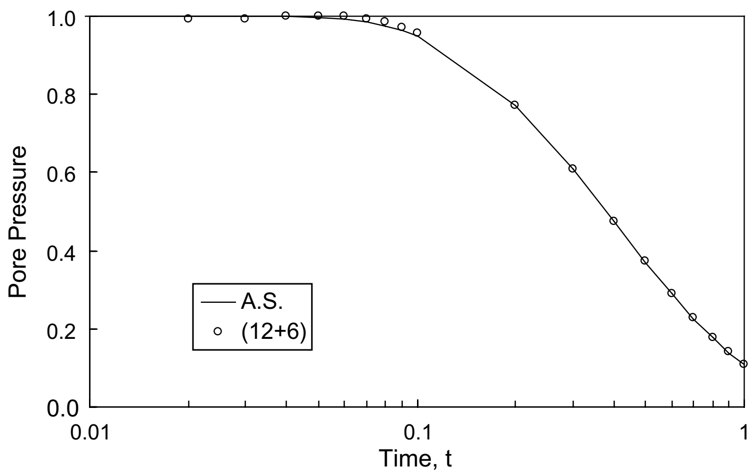

Fig. 3. Example 1: pore pressure at $r = 0$, $z = 0$.

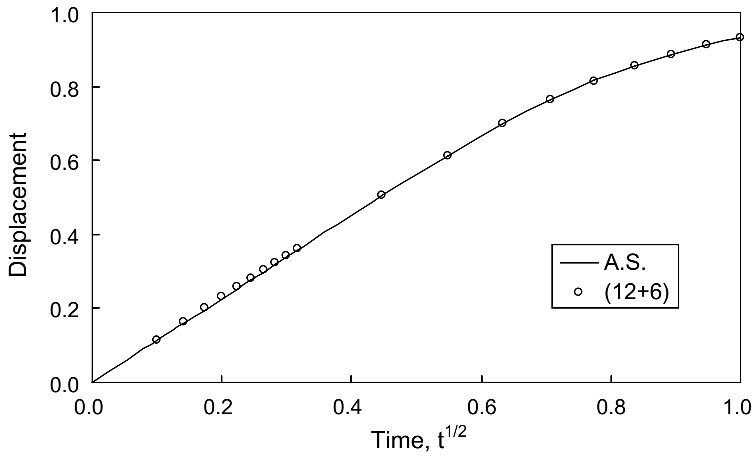

Fig. 4. Example 1: displacement at $r = 0$, $z = 1$.

5.2. Example 2: Consolidation of a solid sphere

The second problem is the consolidation of a saturated solid sphere of soil subjected to a uniform load. After Mandel (1953) discovered the difference between Biot's and Terzaghi's theories in the prediction of pore pressure, the closed-form solutions for the surface displacement and the pore pressure change at the center of the sphere were first presented by Cryer (1963). It is notable that Cryer's formulations are constructed in terms of non-dimensional quantities.

By using the non-dimensional quantities, by definition,

$$T = \frac{c_v t}{a^2}, \quad U_p = \frac{u_p}{P}, \quad R = \frac{r}{a}, \quad \mu_c = \frac{\mu}{(\lambda + 2\mu)} \text{ and } \lambda_c = \frac{\lambda}{(\lambda + 2\mu)}$$

where a is the radius of the sphere, c_v the coefficient of consolidation, U_p the pore pressure, P the applied load intensity, t the time and r the radial distance from the center, the non-dimensional pore pressure at the center and displacement of the surface at time T are given by (Cryer, 1963)

$$U_{\mathrm{P}}(0,T)=\sum_{n=1}^{\infty}\frac{-8\mu+2(4\mu-s_n)/\cos\sqrt{s_n}}{s_n-12\mu_{\mathrm{c}}+16\mu_{\mathrm{c}}^2}\mathrm{e}^{-s_nT}$$

$$U_R(1,T)=1-2\sum_{n=1}^{\infty}\frac{3-4\mu}{s_n-12\mu_{\mathrm{c}}+16\mu_{\mathrm{c}}^2}\mathrm{e}^{-s_nT}$$

where $-s_n$ are the roots of

$$(s+4\mu)\sinh\sqrt{s}-4\mu\sqrt{s}\cosh\sqrt{s}=0.$$

A quarter of sphere is analyzed here and the axisymmetric modeling mesh with 8 quadratic boundary elements and 6 (+ mark only) or 12 (+ and circle marks) interior points is shown in Fig. 5.

Some numerical results of pore pressure at the center and displacement of the surface, for a time step of 0.0025, are shown in Figs. 6 and 7, together with the analytical solutions.

Again, good agreement can be seen, except the discrepancy of pore pressure at the early time $t = 0.01$. From Fig. 6, the well-known Mandel–Cryer effect, of increasing pore pressure during the early stages of the process, is evident.

5.3. Example 3: Consolidation of a flexible circular footing

The final example problem deals with the consolidation of a poroelastic layer of a finite thickness, resting on a smooth impervious base and subjected to axisymmetric loading. This problem was solved by Gibson et al. (1970).

The modeling mesh with 20 quadratic boundary elements and 21 interior points is shown in Fig. 8. Axisymmetric load of radius a with a uniform intensity is applied instantaneously at time $t = 0$ and thereafter held constant with drainage occurring only at the top surface.

The numerical result of displacement at the point $(r,z) = (0,1)$ with respect to time, for a time step of 0.001, is shown in Fig. 9. A good agreement is observed between the numerical and analytical solutions.

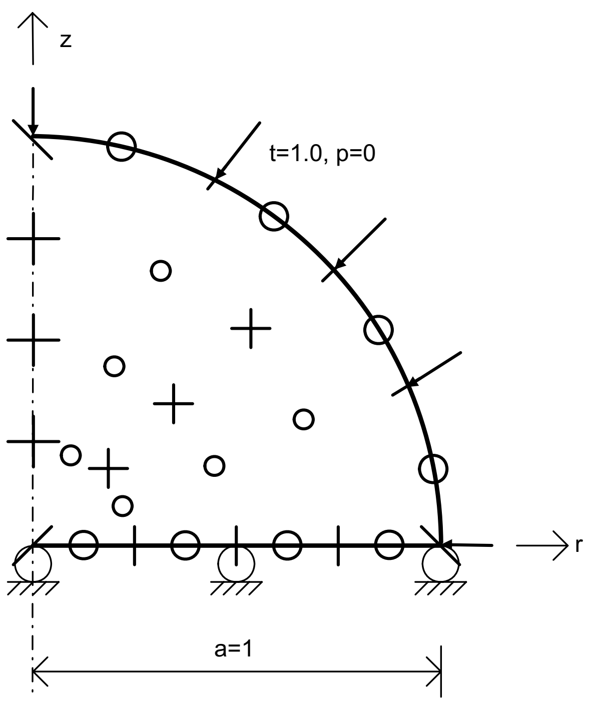

Fig. 5. Modeling mesh for consolidation of a solid sphere.

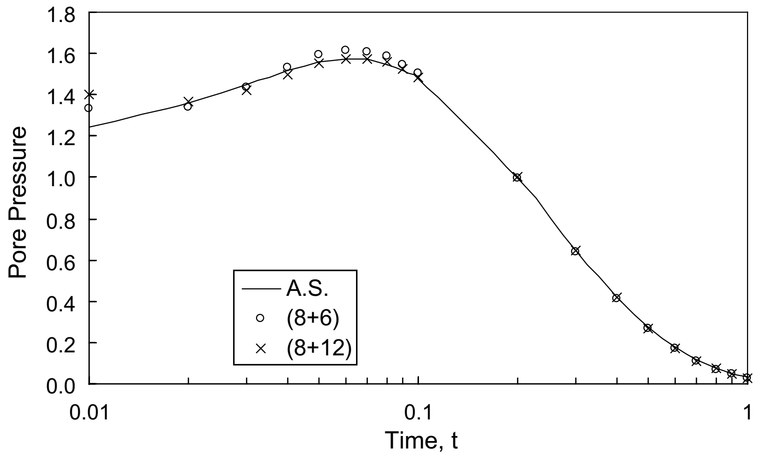

Fig. 6. Example 2: pore pressure at $r = 0$, $z = 0$.

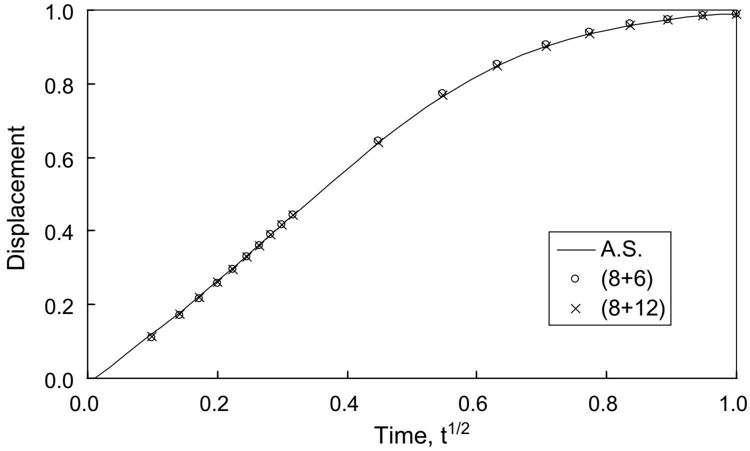

Fig. 7. Example 2: displacement at $r = 1$, $z = 0$.

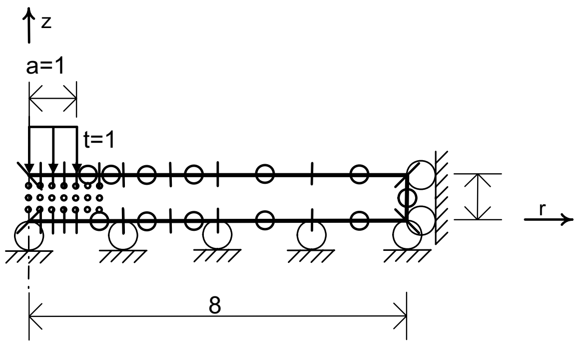

Fig. 8. Modeling mesh for consolidation of a flexible circular footing.

6. Discussions and conclusions

The simple particular integral formulation has been developed for axisymmetric coupled poroelastic analysis. The equations of axisymmetric elastostatic and steady-state potential flow have been used as the

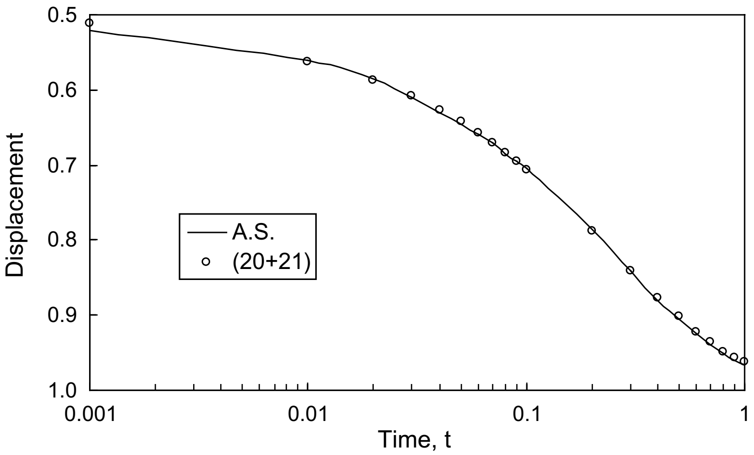

Fig. 9. Example 3: displacement at $r = 0$, $z = 1$.

complementary functions. The particular integrals of displacement, traction, pore pressure and flux are obtained by integrating three-dimensional BEM formulation along the circumferential direction and converting them into elliptic integrals.

The present formulation is first verified by comparing the results of three axisymmetric problems of soil consolidation with their analytical solutions. Good agreement among all of those results has been obtained by including some interior points. It has been demonstrated that axisymmetric coupled poroelastic problems can be solved using the present simple particular integral formulation.

The present formulation needs to be extended to a multi-region form so that in a large scale practical application, the inversion of matrices embodied in Eqs. (54), (57) and (58) does not present a major impediment in the analyses.

Appendix A. Integrations of kernels $C'(x,\xi_n)$, $U'_\alpha(x,\xi_n)$, $T'_\alpha(x,\xi_n)$, $D'_{\alpha k}(x,\xi_n)$, $K'_p(x,\xi_n)$, $P'_k(x,\xi_n)$, $P'_p(x,\xi_n)$, $Q'_k(x,\xi_n)$ and $Q'_p(x,\xi_n)$

This appendix provides the details of integrations of kernels $C'(x,\xi_n)$, $U'_\alpha(x,\xi_n)$, $T'_\alpha(x,\xi_n)$, $D'_{\alpha k}(x,\xi_n)$, $K'_p(x,\xi_n)$, $P'_k(x,\xi_n)$, $P'_p(x,\xi_n)$, $Q'_k(x,\xi_n)$ and $Q'_p(x,\xi_n)$ in terms of elliptic integrals.

For the general form of $r_x \neq r_{\xi n}$ and $z_x \neq z_{\xi n}$,

$$\int_0^{2\pi} C' \, d\theta = 2\pi C_1 A - C_2 G_1 \tag{A.1}$$

$$\int_0^{2\pi} U'_r \, d\theta = 2\pi U_1 A r_x - U_2 r_x G_1 + U_2 r_{\xi_n} G_2 \tag{A.2}$$

$$\int_0^{2\pi} U'_z \, d\theta = (2\pi U_1 A - U_2 G_1) Z \tag{A.3}$$

$$\int_0^{2\pi} T'_r \, d\theta = \left[2\pi S_1 A - S_2 G_1 - S_3\left(r_x^2 F_1 - 2 r_x r_{\xi_n} F_2 + r_{\xi_n}^2 F_3\right)\right] n_r - \left[S_3 Z\left(r_x F_1 - r_{\xi_n} F_2\right)\right] n_z \tag{A.4}$$

$$\int_0^{2\pi} T'_z \, d\theta = \left[-S_3 Z\left(r_x F_1 - r_{\xi_n} F_2\right)\right] n_r + \left[2\pi S_1 A - S_2 G_1 - S_3 Z^2 F_1\right] n_z \tag{A.5}$$

$$\int_0^{2\pi} D'_{rr} \, d\theta = -2 D_1 D_2 A G_2 + D_2^2 L_2 \tag{A.6}$$

$$\int_0^{2\pi} D'_{zz} \, d\theta = 2\pi D_1^2 A^2 - 2 D_1 D_2 A G_1 + D_2^2 L_1 \tag{A.7}$$

$$\int_0^{2\pi} D'_{rz}\,\mathrm{d}\theta = \int_0^{2\pi} D'_{zr}\,\mathrm{d}\theta = 0 \tag{A.8}$$

$$\int_0^{2\pi} K'_p\,\mathrm{d}\theta = 2\pi A K_1 - K_2 G_1 \tag{A.9}$$

$$\int_0^{2\pi} P'_r\,\mathrm{d}\theta = -P_1 A\left(r_x G_2 - r_{\xi_n} G_1\right) + P_2\left(r_x L_2 - r_{\xi_n} L_1\right) \tag{A.10}$$

$$\int_0^{2\pi} P'_z\,\mathrm{d}\theta = -Z(P_1 A G_1 - P_2 L_1) \tag{A.11}$$

$$\int_0^{2\pi} P'_p\,\mathrm{d}\theta = P_3 A L_1 - P_4 M_1 \tag{A.12}$$

$$\int_0^{2\pi} Q'_r\,\mathrm{d}\theta = k\left[P_1 A G_2 - P_2 L_2 - P_1 A\left\{r_x r_{\xi_n} F_1 - \left(r_x^2 + r_{\xi_n}^2\right)F_2 + r_x r_{\xi_n} F_3\right\} + 6\pi P_2 r_x r_{\xi_n}\right] n_r + kZ\left[P_1 A\left(r_x F_2 - r_{\xi_n} F_1\right) + 4\pi P_2 r_{\xi_n}\right] n_z \tag{A.13}$$

$$\int_0^{2\pi} Q'_z\,\mathrm{d}\theta = kZ\left[P_1 A\left(r_x F_1 - r_{\xi_n} F_2\right) - 4\pi P_2 r_x\right] n_r + k\left[P_1 A G_1 - P_2 L_1 + Z^2(P_1 A F_1 - 4\pi P_2)\right] n_z \tag{A.14}$$

$$\int_0^{2\pi} Q'_p\,\mathrm{d}\theta = -k\left[4\pi P_3 A r_x - 3P_4\left(r_x G_1 - r_{\xi_n} G_2\right)\right] n_r - kZ[4\pi P_3 A - 3P_4 G_1] n_z \tag{A.15}$$

where

$$G_1 = \int_0^{2\pi} r\,\mathrm{d}\theta = 4RE \tag{A.16}$$

$$G_2 = \int_0^{2\pi} r\cos\theta\,\mathrm{d}\theta = 4mR(B_3 - B_2) \tag{A.17}$$

$$L_1 = \int_0^{2\pi} r^2\,\mathrm{d}\theta = (2-m)\pi R^2 \tag{A.18}$$

$$L_2 = \int_0^{2\pi} r^2\cos\theta\,\mathrm{d}\theta = -\frac{1}{2} m\pi R^2 \tag{A.19}$$

$$M_1 = \int_0^{2\pi} r^3\,\mathrm{d}\theta = 2R^3\left[(2-m)E - m^2(B_3 - B_2)\right] \tag{A.20}$$

$$F_1 = \int_0^{2\pi} \frac{1}{r}\,\mathrm{d}\theta = \frac{4}{R} B_1 \tag{A.21}$$

$$F_2 = \int_0^{2\pi} \frac{\cos\theta}{r}\,\mathrm{d}\theta = -\frac{4}{R}(2B_2 - B_1) \tag{A.22}$$

$$F_3 = \int_0^{2\pi} \frac{\cos^2\theta}{r}\,\mathrm{d}\theta = \frac{4}{R}(4B_3 - 4B_2 + B_1) \tag{A.23}$$

$$B_1 = K(m) \tag{A.24}$$

$$B_2 = \int_0^{2\pi} \frac{\cos^2\theta}{\sqrt{1 - m\sin^2\theta}}\,\mathrm{d}\theta = \frac{E - (1-m)K}{m} \tag{A.25}$$

$$B_3 = \int_0^{2\pi} \frac{\cos^4\theta}{\sqrt{1 - m\sin^2\theta}}\,\mathrm{d}\theta = \frac{2E(2m-1) + K(1-m)(2-3m)}{3m^2} \tag{A.26}$$

$$K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2\theta}}\,\mathrm{d}\theta: \quad \text{the elliptic integral of first kind}$$

$$E(m)=\int_0^{\pi/2}\sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta: \quad \text{the elliptic integral of second kind.}$$

$$r=R\left[1-\frac{m(1+\cos\theta)}{2}\right]^{1/2}, \quad R=\left[(r_x+r_{\xi_n})^2+Z^2\right]^{1/2}, \quad m=\frac{4r_xr_{\xi_n}}{R^2}$$

If $r_x=0$ and/or $r_{\xi n}=0$ and $z_x\neq z_{\xi n}$ (in this case, $m=0$ and $E(m)=\pi/2$), then the following substitutions should be made as

$$G_1=2\pi R, \quad G_2=0 \tag{A.27}$$

$$L_1=2\pi R^2, \quad L_2=0 \tag{A.28}$$

$$M_1=2\pi R^3 \tag{A.29}$$

$$F_1=\frac{2\pi}{R}, \quad F_2=0, \quad F_3=\frac{\pi}{R} \tag{A.30}$$

When $r_x=r_{\xi n}$ and $z_x=z_{\xi n}$ (in this case, $R=2r_x$ and $m=E(m)=1$), the limiting form should be used as

$$\int_0^{2\pi}C'\,\mathrm{d}\theta=2\pi C_1A-8r_xC_2 \tag{A.31}$$

$$\int_0^{2\pi}U_r'\,\mathrm{d}\theta=2\pi U_1Ar_x-32/3U_2r_x^2 \tag{A.32}$$

$$\int_0^{2\pi}U_z'\,\mathrm{d}\theta=0 \tag{A.33}$$

$$\int_0^{2\pi}T_r'\,\mathrm{d}\theta=[2\pi S_1A-8r_xS_2-16/3S_3r_x]n_r \tag{A.34}$$

$$\int_0^{2\pi}T_z'\,\mathrm{d}\theta=[2\pi S_1A-8r_xS_2]n_z \tag{A.35}$$

$$\int_0^{2\pi}D_{rr}'\,\mathrm{d}\theta=16/3D_1D_2Ar_x-2\pi D_2^2r_x^2 \tag{A.36}$$

$$\int_0^{2\pi}D_{zz}'\,\mathrm{d}\theta=2\pi D_1^2A^2-16D_1D_2Ar_x+4\pi D_2^2r_x^2 \tag{A.37}$$

$$\int_0^{2\pi}K_p'\,\mathrm{d}\theta=2\pi AK_1-8K_2r_x \tag{A.38}$$

$$\int_0^{2\pi}P_r'\,\mathrm{d}\theta=32/3P_1Ar_x^2-6\pi P_2r_x^3 \tag{A.39}$$

$$\int_0^{2\pi}P_z'\,\mathrm{d}\theta=0 \tag{A.40}$$

$$\int_0^{2\pi}P_p'\,\mathrm{d}\theta=4\pi P_3Ar_x^2-64/3P_4r_x^3 \tag{A.41}$$

$$\int_0^{2\pi}Q_r'\,\mathrm{d}\theta=k\left[-8P_1Ar_x+8\pi P_2r_x^2\right]n_r \tag{A.42}$$

$$\int_0^{2\pi}Q_z'\,\mathrm{d}\theta=k\left[8P_1Ar_x-4\pi P_2r_x^2\right]n_z \tag{A.43}$$

$$\int_0^{2\pi}Q_p'\,\mathrm{d}\theta=-k\left[4\pi P_3Ar_x-32P_4r_x^2\right]n_r \tag{A.44}$$

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