ON GREEN'S FUNCTIONS FOR THE REDUCED WAVE EQUATION IN A CIRCULAR ANNULAR DOMAIN WITH DIRICHLET, NEUMANN AND RADIATION TYPE BOUNDARY CONDITIONS

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Summary

In this paper Green's functions for the reduced wave equation (Helmholtz equation) in a circular annular domain with the Dirichlet, the radiation, and Neumann boundary conditions are derived. The convergence of the series representing Green's functions is then established. Finally it is shown that these functions reduce to Green's function for the exterior of a circle as given by Franz and Etiènne when the outer radius is moved towards infinity.

§ 1. *Introduction.* Green's functions for a circular ring with Dirichlet's boundary conditions have been known (Courant and Hilbert[1]) for the Laplace equation for some time. Recently (Heyda[2]) obtained Green's function for Laplace's equation in a circular ring with radiation type boundary conditions. The latter case includes the Neumann boundary value problem. In other modern works (Thyssen[3]) and Etiènne[4]) Green's functions in the interior and exterior of a circle for the reduced wave equation with Dirichlet and Neumann boundary conditions have been studied.

§ 2. *The Dirichlet boundary value problem.* Let $G(r, \theta)$ be Green's function for the reduced wave equation

$$(\Delta + k^2)\, G(r, \theta) = -\delta(r\, \mathrm{e}^{i\theta} - r_0\, \mathrm{e}^{i\theta_0}), \tag{1}$$

in the annular domain between the two circles $r = r_1$ and $r = r_2$

$(r_1 < r_2)$, where the fundamental singularity of $G(r, \theta)$ is located on $r = r_0$ $(r_1 < r_0 < r_2)$ with the Dirichlet boundary condition

$$G(r, \theta)|_{r=r_i} = 0 \qquad (i = 1, 2). \tag{2}$$

The $G(r, \theta)$ in (1) can be written in the form

$$G(r, \theta) = G^*(r, \theta) + S(r, \theta), \tag{3}$$

where the singular component of G is given by Sommerfeld[5]) in terms of Bessel and Hankel functions in the form

$$S(R) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} J_n(kr_0)\, e^{in(\theta-\theta_0)}\, H_n^{(1)}(kr) & \text{for } r > r_0 \\ \sum\limits_{n=-\infty}^{\infty} H_n^{(1)}(kr_0)\, e^{in(\theta-\theta_0)}\, J_n(kr) & \text{for } r < r_0 \end{cases} \tag{4}$$

with

$$R = |r\, e^{i\theta} - r_0\, e^{i\theta_0}|,$$

where $S(R)$ is singular at $R = 0$.

The non-singular component can be written in the form (Reichardt[6])):

$$G^*(r, \theta) = \sum_{n=-\infty}^{n=\infty} c_n H_n^{(1)}(kr)\, e^{in\theta} + \sum_{n=-\infty}^{n=\infty} d_n J_n(kr)\, e^{in\theta}, \tag{5}$$

where non-singular refers to the domain excluding the origin and infinity. It may be verified that each term of the series (5) satisfies the homogeneous Helmholtz equation

$$\Delta G^* + k^2 G^* = 0 \tag{6}$$

in the specified domain. The coefficients c_n and d_n can be uniquely determined with the aid of our boundary conditions (2) and the series (4) and (5). The resulting solution does therefore not introduce any singularity other than the one of $S(R)$ into the annular region. We note that the series expansion (5) is similar to the Laurent series in complex function theory.

§ 3. *Derivation of Green's function for the Dirichlet boundary conditions.* In order to determine the coefficients c_n and d_n, we substitute the expressions (4) and (5) into (2) and obtain for the n-th terms the equations

$$c_n H_n^{(1)}(kr_1) + d_n J_n(kr_1) = -H_n^{(1)}(kr_0)\, J_n(kr_1)\, \mathrm{e}^{-in\theta_0} \tag{7}$$

and

$$c_n H_n^{(1)}(kr_2) + d_n J_n(kr_2) = -J_n(kr_0)\, H_n^{(1)}(kr_2)\, \mathrm{e}^{-in\theta_0}.$$

From (7) we find c_n and d_n and hence Green's function for $r < r_0$ in the form

$$\begin{aligned} G(r, \theta) = \sum_{n=-\infty}^{n=\infty} & \Bigg[\mathrm{e}^{in(\theta-\theta_0)}\, J_n(kr_1) \\ & \times \frac{H_n^{(1)}(kr_0)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_0)}{H_n^{(1)}(kr_1)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_1)} \Bigg] H_n^{(1)}(kr) \\ & + \sum_{n=-\infty}^{n=\infty} \Bigg\{ \mathrm{e}^{in(\theta-\theta_0)} \Bigg[H_n^{(1)}(kr_0) - H_n^{(1)}(kr_2) \\ & \times \frac{H_n^{(1)}(kr_1)\, J_n(kr_2) + H_n^{(1)}(kr_2)\, J_n(kr_0)}{H_n^{(1)}(kr_1)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_1)} \Bigg] \Bigg\} I_n(kr), \end{aligned} \tag{8}$$

and Green's function for $r > r_0$ becomes

$$\begin{aligned} G(r, \theta) = \sum_{n=-\infty}^{n=\infty} & \Bigg\{ \mathrm{e}^{in(\theta-\theta_0)} \Bigg[J_n(kr_0) + J_n(kr_1) \\ & \times \frac{H_n^{(1)}(kr_0)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_0)}{H_n^{(1)}(kr_1)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_1)} \Bigg] \Bigg\} H_n^{(1)}(kr) \\ & - \sum_{n=-\infty}^{n=\infty} \Bigg[\mathrm{e}^{in(\theta-\theta_0)}\, H_n^{(1)} kr_2) \\ & \times \frac{H_n^{(1)}(kr_1)\, J_n(kr_0) + H_n^{(1)}(kr_0)\, J_n(kr_1)}{H_n^{(1)}(kr_1)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_1)} \Bigg] I_n(kr). \end{aligned} \tag{9}$$

The denominators of (8) and (9) are different from zero except possibly for a very special choice of k, r_1, and r_2. It should be pointed out that, given r_1 and r_2 (together with h_1, h_2 in the radiation problem) there exist countably many k-values (the eigenvalues of the boundary value problem) for which no proper Green's function exists. For each of these values the denominators in (8) and (9) vanish for at least one n.

The convergence of the series (8) and (9) has still to be established. For this purpose we make use of the asymptotic formula as n goes to plus infinity (Gray, Mathews and Macrobert [7])

$$|H_n^{(1)}(kr)\, J_n(kd)| \sim \frac{1}{\pi n} \exp\{-n \log (r/a)\} \qquad (n > 0). \tag{10}$$

In order to apply this formula to (9) we first write that equation in the form

$$\begin{aligned} G(r,\theta) = &\Big\{ I_0(kr_0) + J_0(kr_1) \\ &\times \frac{H_0^{(1)}(kr_0)\, J_0(kr_2) - H_0^{(1)}(kr_2)\, J_0(kr_0)}{H_0^{(1)}(kr_1)\, J_0(kr_2) - H_0^{(1)}(kr_2)\, J_0(kr_1)} \Big\} H_0^{(1)}(kr) \\ &+ 2 \sum_{n=1}^{\infty} [\cos n(\theta - \theta_0)] \Big\{ I_n(kr_0) + J_n(kr_1) \\ &\times \frac{H_n^{(1)}(kr_0)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_0)}{H_n^{(1)}(kr_1)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_1)} \Big\} H_n^{(1)}(kr) \\ &+ H_0^{(1)}(kr_2)\, \frac{H_0^{(1)}(kr_1)\, J_0(kr_0) + H_0^{(1)}(kr_0)\, J_0(kr_1)}{H_0^{(1)}(kr_2)\, J_0(kr_1) - H_0^{(1)}(kr_1)\, J_0(kr_2)}\, I_0(kr) \\ &- 2 \sum_{n=1}^{\infty} [\cos n(\theta - \theta_0)]\, H_n^{(1)}(kr_2) \\ &\times \frac{H_n^{(1)}(kr_1)\, J_n(kr_0) + H_n^{(1)}(kr_0)\, J_n(kr_1)}{H_n^{(1)}(kr_1)\, J_n(kr_2) - H_n^{(1)}(kr_2)\, J_n(kr_1)}\, I_n(kr), \end{aligned} \tag{11}$$

where we have made use of the relations

$$J_{-n}(z) = (-1)^n J_n(z) \quad \text{and} \quad H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z). \tag{12}$$

Since the summation index n runs through positive values only in (11), we may use the asymptotic formula (10) for the appraisal of the absolute values of the terms of the series in (11). Consider for example, the general term of the first series in (11). We can write it in the form

$$\cos n(\theta-\theta_0)\left\{J_n(kr_0)\,H_n^{(1)}(kr)\right.$$

$$\left.+\frac{H_n^{(1)}(kr_0)\,J_n(kr_2)\left[1-\dfrac{H_n^{(1)}(kr_2)\,J_n(kr_0)}{H_n^{(1)}(kr_0)\,J_n(kr_2)}\right]}{H_n^{(1)}(kr_1)\,J_n(kr_2)\left[1-\dfrac{H_n^{(1)}(kr_2)\,J_n(kr_1)}{H_n^{(1)}(kr_1)\,J_n(kr_2)}\right]}\right\}J_n(kr_1)\,H_n^{(1)}(kr). \quad (13)$$

The first factor is less than or equal to 1 in absolute value. The first term in the braces satisfies in view of (10), the relation

$$|J_n(kr_0)\,H_n^{(1)}(kr)|\sim\frac{1}{\pi n}\exp\{-n\log(r/r_0)\}=\frac{1}{\pi n}\left(\frac{r_0}{r}\right)^n. \quad (14)$$

Since $r>r_0$, this term is less than the n-th term of a geometric series with ratio less than 1.

Next we examine each of the ratios which occur in the second term within the braces.

$$\left|\frac{H_n^{(1)}(kr_2)\,J_n(kr_0)}{H_n^{(1)}(kr_0)\,J_n(kr_2)}\right|\sim\frac{\exp\{-n\log(r_2/r_0)\}}{\exp\{-n\log(r_0/r_2)\}}=\left(\frac{r_0}{r_2}\right)^{2n}\to 0, \quad (15)$$

since $r_2>r_0$;

$$\left|\frac{H_n^{(1)}(kr_2)\,J_n(kr_1)}{H_n^{(1)}(kr_1)\,J_n(kr_2)}\right|\sim$$

$$\sim\frac{\exp\{-n\log(r_2/r_1)\}}{\exp\{-n\log(r_1/r_2)\}}=\frac{(r_1/r_2)^n}{(r_2/r_1)^n}=\left(\frac{r_1}{r_2}\right)^{2n}\to 0, \quad (16)$$

since $r_1<r_2$;

$$\left|\frac{H_n^{(1)}(kr_0)\,J_n(kr_2)}{H_n^{(1)}(kr_1)\,J_n(kr_2)}\right|\sim\frac{(r_2/r_0)^n}{(r_2/r_1)^n}=\left(\frac{r_1}{r_0}\right)^n\to 0, \quad (17)$$

since $r_1<r_0$; and

$$|J_n(kr_1)\,H_n^{(1)}(kr)|=\frac{1}{\pi n}\left(\frac{r_1}{r}\right)^n\to 0, \quad (18)$$

since $r_1<r$. This shows that the second term in the braces of (14) is less than the n^{th} term of a geometric series whose common ratio is less than 1. Hence the first series in (11) converges for all r

such that $r_1 < r_0 < r \leqslant r_2$. In a similar way one can show that the second series in (11) converges in the same layer $r_1 < r_0 < < r \leqslant r_2$. (They obviously do not converge at $r = r_0$ since the series $\sum$ represents $S(R)$ which is singular on $r = r_0$).

The convergence of the series in (8) can be established in an analogous manner.

Therefore (8) and (9) (or (11) and a similar equation corresponding to (8)) are indeed Green's function for the circular annular domain with Dirichlet's boundary conditions for the reduced wave equation.

In the limiting case when r_2 goes to infinity in (8) for instance, we can recover the known results by Etiènne[4]) for Green's function related to the reduced wave equation for the Dirichlet boundary condition exterior to a circle. We have then

$$\frac{1}{i\pi} G(r, \theta) = \sum_{n=-\infty}^{n=\infty} e^{in(\theta-\theta_0)} \frac{J_n(kr_1)\, H_n^{(1)}(kr_0)}{H_n^{(1)}(kr_1)} H_n^{(1)}(kr) + \sum_{n=-\infty}^{n=\infty} e^{in(\theta-\theta_0)} H_n^{(1)}(kr_0)\, J_n(kr). \quad (19)$$

This equation is essentially the equation (13) of Etiènne, where

$$K_n(z) = \frac{\pi}{2} e^{in\pi/2} H_n^{(1)}(iz) \quad (20)$$

and

$$J_m(z) = e^{-im\pi/2} J_m(iz).$$

§ 4. *Green's function for the radiation boundary conditions.* Let $G(r, \theta)$ be Green's function for the reduced wave equation (1) in the annular domain between the circles $r = r_1$ and $r = r_2$ $(r_1 < r_2)$, where the fundamental singularity of $G(r, \theta)$ is located on $r = r_0$ $(r_1 < r_0 < r_2)$ with the radiation type boundary condition

$$\left. \frac{\partial G(r, \theta)}{\partial r} + (-1)^i h_i\, G(r, \theta) \right|_{r=r_i} = 0 \qquad (i = 1, 2). \quad (21)$$

When $h_i = 0$ this Green's function reduces immediately to Green's function for the Neumann boundary value problem.

The $G(r, \theta)$ in (21) can again be written in the form

$$G = G^* + S, \quad (22)$$

where the singular component S is given by (4) and the non-singular component is given by (5). The constants c_n and d_n are now to be determined so that

$$\left.\frac{\partial G^*}{\partial r} + (-1)^i\, h_i\, G^*\right|_{r=r_i} = -\left.\frac{\partial S}{\partial r} + (-1)^{i-1}\, h_i\, S\right|_{r=r_i}, \qquad (i = 1, 2). \tag{23}$$

Substituting G^* from (5) and S from (4) into (23) we can determine the coefficients c_n and d_n. Deriving the expressions in (21) and inserting these expressions and (4), (5) into (23) we find the coefficients c_n and d_n as

$$c_n = \frac{|AB|}{D}\, \mathrm{e}^{-in\theta_0}, \tag{24}$$

$$d_n = \frac{|CA|}{D}\, \mathrm{e}^{-in\theta_0}, \tag{25}$$

where A and B are the columns

$$A = \begin{Bmatrix} -H_n^{(1)}(kr_0)\left[\dfrac{\partial}{\partial r} J_n(kr)\right]_{r=r_1} + h_1 H_n^{(1)}(kr_0)\, J_n(kr_1) \\ -J_n(kr_1)\left[\dfrac{\partial}{\partial r} H_n^{(1)}(kr)\right]_{r=r_2} - h_2 J_n(kr_0)\, H_n^{(1)}(kr_2) \end{Bmatrix},$$

$$B = \begin{Bmatrix} \dfrac{\partial}{\partial r} J_n(kr) - h_1 J_n(kr_1) \\ \dfrac{\partial}{\partial r} J_n(kr) + h_2 J_n(kr_2) \end{Bmatrix},$$

and C is the same as B except that $J_n(kr)$ is replaced by $H_n^{(1)}(kr)$ and the denominator common to c_n and d_n is

$$D = |CB|.$$

Thus Green's function for the mixed boundary value problem is given by (5) with the derived c_n and d_n, (24) and (25).

For the Neumann boundary value problem we set $h_i = 0$ $(i = 1, 2)$. Then the c_n and d_n of (24) and (25), after evaluating the determinants, reduce considerably and the Green's function for the Neumann boundary value problem becomes

$$G(r,\theta) = -\sum_{n=-\infty}^{n=\infty} H_n^{(1)}(kr) \,.$$
$$. \, e^{in(\theta-\theta_0)} J_n'(kr_1) \frac{H_n^{(1)}(kr_0) J_n'(kr_2) - H_n^{(1)'}(kr_2) J_n(kr_1)}{H_n^{(1)'}(kr_1) J_n'(kr_2) - H_n^{(1)'}(kr_2) J_n'(kr_1)}$$
$$+ \sum_{n=-\infty}^{n=\infty} J_n(kr) e^{in(\theta-\theta_0)} \Big[H_n^{(1)}(kr_0) - H_n^{(1)'}(kr_2)$$
$$\times \frac{H_n^{(1)'}(kr_1) J_n(kr_1) - J_n'(kr_1) H_n^{(1)}(kr_0)}{H_n^{(1)'}(kr_1) J_n'(kr_2) - H_n^{(1)'}(kr_2) J_n'(kr_1)} \Big] \text{ if } r < r_0, \qquad (26)$$

and

$$G(r,\theta) = \sum_{n=-\infty}^{n=\infty} H_n^{(1)}(kr) e^{in(\theta-\theta_0)} \Big[J_n(kr_0) - J_n'(kr)$$
$$\times \frac{H_n^{(1)}(kr_0) J_n'(kr_2) - H_n^{(1)'}(kr_2) J_n(kr_1)}{H_n^{(1)'}(kr_1) J_n'(kr_2) - H_n^{(1)'}(kr_2) J_n'(kr_1)} \Big]$$
$$- \sum_{n=-\infty}^{n=\infty} J_n(kr) e^{in(\theta-\theta_0)} H_n^{(1)'}(kr_2)$$
$$\times \frac{H_n^{(1)'}(kr_1) J_n(kr_1) - J_n'(kr_1) H_n^{(1)}(kr_0)}{H_n^{(1)'}(kr_1) J_n'(kr_2) - H_n^{(1)'}(kr_2) J_n'(kr_1)}, \text{ if } r > r_0. \qquad (27)$$

The convergence of the series in (26) and (27) can now be proven in a manner similar to that used to prove the convergence of (8) and (9).

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