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Degenerate scale problem for the Laplace equation in the multiply connected region with outer elliptic boundary

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Abstract This paper investigates the degenerate scale problem for the Laplace equation in a multiply connected region with an outer elliptic boundary. Inside the elliptic boundary, there are many voids with arbitrary configurations. The problem is studied on the relevant homogenous boundary integral equation. The suggested solution is derived from a solution of a relevant problem. It is found that the degenerate scale and the eigenfunction along the elliptic boundary in the problem is the same as in the case of a single elliptic contour without voids, or the involved voids have no influence on the degenerate scale. The present study mainly depends on the integrations of two integrals, which can be integrated in closed form.

1 Introduction

Comparing with the finite element method (abbreviated as FEM), the boundary integral equation (abbreviated as BIE) has some advantages. For example, if choosing N divisions along the edges of a square plate, there are $N * N$ unknowns in the formulation after discretization in FEM. However, under the same condition for divisions, there are $4N$ unknowns in the formulation after discretization in BIE. Therefore, the boundary integral equation (abbreviated as BIE) was widely used in elasticity [1–4]. Recently, the historical background and formulation of the boundary element method was summarized [5].

The degenerate scale problem in BIE is a particular boundary value problem in plane elasticity as well as in the Laplace equation. The problem typically arises from the exterior Dirichlet problem in plane elasticity and the Laplace equation. The problem can be easily explained by the following example. We know that the Laplace equation $\nabla^2 F(r,\theta) = 0$ has a particular solution $F(r,\theta) = \ln r$. It is assumed that the solution is defined on the outside region to a circle with radius "a". If $a = 1$, the function $F(r,\theta) = \ln r$ takes zero value at the circular boundary from $\ln r\ |_{r=a=1} = 0$. Alternatively speaking, there is a non-trivial solution $F(r,\theta) = \ln r$ for the Laplace equation, even though it takes a vanishing boundary value at the circle with radius $a = 1$. In fact, once the degenerate scale is reached, the relevant homogenous equation for the boundary tractions has a non-trivial solution. Alternatively speaking, the non-homogeneous equation has a non-unique solution, or multiple solutions. Clearly, the degenerate scale represents an illness condition. Therefore, one must avoid meeting an illogical solution caused by the occurrence of the degenerate scale.

The degenerate scale problems were studied by many researchers using a variety of methods. Among the suggested techniques for evaluating the degenerate scale, the coordinate transform technique for evaluating the degenerate scale is significant [6,7]. In the technique, one only needs to solve a BIE in the normal scale, and the equation for evaluating the degenerate scale follows immediately.

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Some researchers studied the analytical solution of degenerate scales in the BEM for the two-dimensional Laplace equation by using degenerate kernels and Fourier series expansion [8]. A numerical procedure was developed to evaluate the degenerate scale directly from the zero value of a determinant, or det $U(a) = 0$ [9]. In the procedure, the influence matrix is denoted by $U(a)$, where "a" is a size. In the computation, the size "a" is changed gradually. Once the condition det $U(a) = 0$ is satisfied, the relevant scale "a" is the degenerate scale. The conformal mapping technique is used to find the degenerate scale for a variety of the boundary configurations [10].

The degenerate scale for multiply connected Laplace problems was studied in [11]. In the paper, the authors utilized the null-field integral equation to analytically study the degenerate scale problem. The problem was studied in the discrete system. When the radius of the outer circular boundary is equal to unity, all elements in one column of the influence matrix become zero. Alternatively speaking, the influence matrix is singular when the used size reaches the degenerate scale. The degenerate scale problem for an annular region in plane elasticity was studied in [12,13].

The perspective on the degenerate problems in plane elasticity was provided in [14]. Those problems include: (1) the degenerate boundary, (2) the degenerate scale, (3) the spurious eigensolution and (4) the fictitious frequency in the boundary integral formulation. All the degenerate problems originate from the rank deficiency in the influence matrix.

The question of unique solvability of the boundary integral equation of the first kind given by the single-layer potential operator was studied in the case of plane isotropic elasticity [15]. The critical scales of a bounded multiply connected domain were studied. The critical scale in the problem is the same as in the case of a single outer contour.

This paper investigates the degenerate scale problem for the Laplace equation in the multiply connected region with an outer elliptic boundary. Inside the elliptic boundary, there are many voids with arbitrary configurations. The problem is studied on the relevant homogenous BIE. The merit of this study is to evaluate a non-trivial solution for the homogenous BIE. It is assumed that the tractions on all the inner void boundaries are equal to zero and tractions on the outer elliptic boundary are equal to some functions. Therefore, all the integrations are performed on the outer elliptic boundary only. The degenerate scale for the problem is found. It is found that the degenerate scale and the eigenfunction along the elliptic boundary in the problem is the same as in the case of a single elliptic contour without voids, or the involved voids have no influence on the degenerate scale. This conclusion in the case of plane elasticity was proposed previously [15]. The present study mainly depends on integrations of two integrals, which can be integrated in closed form.

2 Degenerate scale problem for the Laplace equation in the multiply connected region with outer elliptic boundary

In the first part of this section, the formulation and solution of the degenerate scale problem for a single elliptic contour is studied, which in turn becomes a foundation for studying the relevant degenerate scale problem for the multiply connected region with outer elliptic boundary.

2.1 Formulation and solution of the degenerate scale problem for a single elliptic contour

Similar to the case of plane elasticity, the degenerate scale in the BIE for antiplane elasticity for the Laplace equation can also be studied by using the complex variable. After using the complex potential $\phi(z)$ in antiplane elasticity, all the physical quantities can be expressed through $\phi(z)$ [16],

$$Gw(x, y) + \mathrm{i} f(x, y) = \phi(z), \tag{1}$$

$$f(x, y) = \int_{z_o}^{z} \sigma_{xz}\mathrm{d}y - \sigma_{yz}\mathrm{d}x, \tag{2}$$

$$G\frac{\partial w}{\partial x} + \mathrm{i}\frac{\partial f}{\partial x} = \sigma_{xz} - \mathrm{i}\sigma_{yz} = \Phi(z) = \phi'(z), \tag{3}$$

where G is the shear modulus of elasticity, w is the out of plane displacement, f is the longitudinal resultant force, and σ_{xz} and σ_{yz} are the stress components, and $z = x + \mathrm{i}y$. Without losing generality, we always assume

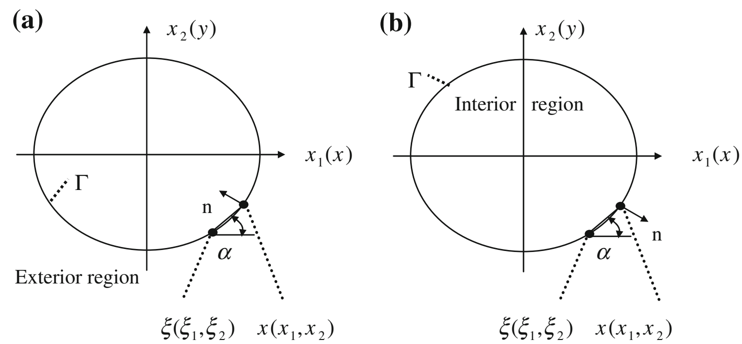

Fig. 1 **a** An exterior region to the elliptic boundary. **b** An interior region to the elliptic boundary

$G = 1$ in the following analysis. Clearly, the displacement component $w(x, y)$ satisfies the following Laplace equation:

$$\nabla^2 w(x, y) = 0, \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{4}$$

Without losing generality, we introduce the BIE for the Laplace equation for the exterior problem with an elliptic boundary Γ (Fig. 1a). The source point is denoted by $\xi(\xi_1, \xi_2)$, and the field point is denoted by $x(x_1, x_2)$. After some manipulations, the following BIE for the exterior boundary value problem is obtained [11]:

$$\frac{1}{2}w(\xi) + \int_\Gamma P^*(\xi, x)w(x)\mathrm{d}s(x) = \int_\Gamma U^*(\xi, x)p(x)\mathrm{d}s(x), \quad (\text{for } \xi \in \Gamma), \tag{5}$$

where the kernels $U^*(\xi, x)$ and $P^*(\xi, x)$ are defined by

$$U^*(\xi, x) = -\frac{1}{2\pi}\ln r, \tag{6}$$

$$P^*(\xi, x) = \frac{\partial U^*(\xi, x)}{\partial n} = -\frac{1}{2\pi}\frac{1}{r}(r_{,1}n_1 + r_{,2}n_2), \tag{7}$$

$$r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}, \quad r_{,1} = \frac{x_1 - \xi_1}{r} = \cos\ \alpha, \quad r_{,2} = \frac{x_2 - \xi_2}{r} = \sin\alpha, \tag{8}$$

where the angle "α" is indicated in Fig. 1a. In Eq. (7), the normal $n(n_1, n_2)$ at the boundary point always directs at the outward side.

Similarly, we can introduce the BIE for the Laplace equation for the interior problem with an elliptic boundary Γ (Fig. 1b). After some manipulations, the following BIE is obtained:

$$\frac{1}{2}w(\xi) + \int_\Gamma P^*_{\text{in}}(\xi, x)w(x)\mathrm{d}s(x) = \int_\Gamma U^*(\xi, x)p(x)\mathrm{d}s(x), \quad (\text{for } \xi \in \Gamma), \tag{9}$$

where the integral kernel $U^*(\xi, x)$ has been defined by Eq. (6). In Eq. (9), the suffix "in" means that the integral kernel is formulated in the interior boundary value problem. Since the normal at a point in the exterior problem is just opposite to the normal in the interior problem, we have $P^*(\xi, x) = -P^*_{\text{in}}(\xi, x)$. Therefore, from Eq. (9), the BIE for the interior boundary value problem can be rewritten as

$$\frac{1}{2}w(\xi) - \int_\Gamma P^*(\xi, x)w(x)\mathrm{d}s(x) = \int_\Gamma U^*(\xi, x)p(x)\mathrm{d}s(x), \quad (\text{for } \xi \in \Gamma). \tag{10}$$

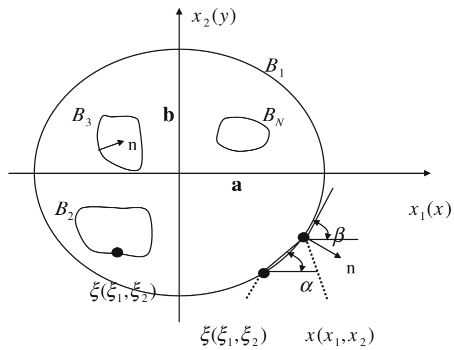

Fig. 2 A multiply connected region with outer elliptic boundary

Substituting $w(\xi) = 0$ and $w(x) = 0$ into the left-hand terms of Eq. (5) or (10), in both cases, we can obtain the following homogeneous equation:

$$\int_{\Gamma} U^*(\xi, x)p(x)\mathrm{d}s(x) = 0, \quad (\text{for } \xi \in \Gamma). \tag{11}$$

Equation (11) reveals that the same homogeneous equation can be formulated without regarding the exterior and the interior problem.

In the degenerate scale problem, one needs to find the same particular size such that Eq. (11) has a non-trivial solution for $p(x)$, or $p(x) \neq 0$.

Previously, for the case of a single exterior elliptic contour (Fig. 1a), the following mapping function is suggested [17]:

$$z = \omega(\varsigma) = R\left(\varsigma + \frac{m}{\varsigma}\right), \quad \text{with } a = R(1+m), b = R(1-m), \ \ 0 \le m < 1. \tag{12}$$

After using some manipulations [10], the following results were found:

1. The degenerate scale is $R = 1$. Alternatively speaking, the degenerate scale for ellipse is $a = 1 + m$ and $b = 1 - m$, where $0 \le m < 1$ (Fig. 1).
2. The non-trivial solution for the case of a single elliptic contour is as follows:

$$\phi_1(\varsigma) = \phi(z)\mid_{z=\omega(\varsigma)} = \ln(R\varsigma) = \ln R + \ln\varsigma, \quad (\text{or} = c\ln(R\varsigma), \quad c\text{–constant}) \tag{13}$$

Alternatively speaking, the eigenfunction $p(x)$ in Eq. (11) is obtained from the complex potential shown by Eq. (13).

2.2 Formulation and solution of the degenerate scale problem for a multiply connected region with outer elliptic boundary

Without losing generality, we introduce the BIE for the Laplace equation for the multiply connected region with outer elliptic boundary B_1 (Fig. 2). Inside the elliptic boundary B_1, there are many voids $B_k (k = 2, 3, \ldots, N)$ with arbitrary configurations (Fig. 2). The source point is denoted by $\xi(\xi_1, \xi_2)$, and the field point is denoted by $x(x_1, x_2)$. Similarly, the following BIE is obtained:

$$\frac{1}{2}w(\xi) + \int_{B} P^*(\xi, x)w(x)\mathrm{d}s(x) = \int_{B} U^*(\xi, x)p(x)\mathrm{d}s(x),$$
$$(\text{for } \xi \in B, B = B_1 + B_2 + \cdots + B_N), \tag{14}$$

where the kernels $U^*(\xi, x)$ and $P^*(\xi, x)$ have been defined by Eqs. (6) and (7) previously.

Suppose that the assumed displacements $w(x, y)$ on all contours B_k $(k = 1, 2, 3, \ldots, N)$ are vanishing. Therefore, substituting this result in the left-hand side of Eq. (14) yields the following homogeneous equation:

$$\int_B U^*(\xi, x)p(x)\mathrm{d}s(x) = 0, \quad (\text{for } \xi \in B,\ B = B_1 + B_2 + \cdots + B_N). \tag{15}$$

By using Eqs. (6) and (15) it can be rewritten in an alternative form,

$$\sum_{k=1}^{N} \int_{B_k} \ln r(x, \xi)p(x)\mathrm{d}s(x) = 0, \quad (\text{for } \xi \in B,\ B = B_1 + B_2 + \cdots + B_N), \tag{16}$$

where B_1 is the outer elliptic contour, and B_k $(k = 2, 3, \ldots, N)$ are voids with arbitrary configurations.

In the degenerate scale problem, one needs to find the same particular size such that Eq. (16) has a nontrivial solution for $p(x)$, or $p(x) \neq 0$ (for $x \in B$, $B = B_1 + B_2 + \cdots + B_N$). This is the normal demand for the formulation. However, we can propose an alternative demand for the formulation. For example, the alternative demand is as follows: $p(x) \neq 0$ for $x \in B_1$, and $p(x) = 0$ for $x \in B_j$, $j = 2, 3, \ldots, N$.

We can prove that the homogeneous Eq. (16) can be satisfied if the following conditions are satisfied:

1. $p(x) = 0$ for $x \in B_k$, $k = 2, 3, \cdots, N$. Note that this is the alternative demand mentioned above.
2. $p(x)$obtained from Eq. (13), for $x \in B_1$, or from $\phi_1(\varsigma) = \phi(z)\mid_{z=\omega(\varsigma)} = \ln(R\varsigma) = \ln R + \ln \varsigma$ for ς on the unit circle. This condition means that $p(x) \neq 0$, for $x \in B_1$.
3. $R = 1$ in Eq. (12), or $a = 1 + m$ and $b = 1 - m$ for the elliptic boundary with $0 \leq m < 1$.

This solution is not easy to obtain from Eq. (16) directly. However, it can be obtained from a guess of the previously obtained result for a single elliptic contour case [10].

Clearly, from the first condition, Eq. (16) can be reduced to

$$\int_{B_1} \ln r(x, \xi)p(x)\mathrm{d}s(x) = 0, \quad (\text{for } \xi \in B, \quad B = B_1 + B_2 + \cdots + B_N). \tag{17}$$

Further, Eq. (17) can be rewritten alternatively

$$\int_{B_1} \ln r(x, \xi)p(x)\mathrm{d}s(x) = 0, \quad (\text{for } \xi \in B_1), \tag{18}$$

$$\int_{B_1} \ln r(x, \xi)p(x)\mathrm{d}s(x) = 0, \quad (\text{for } \xi \in B_k, \quad k = 2, 3, \ldots, N). \tag{19}$$

From Eqs. (2) and (13), a factor $p(x)\mathrm{d}s(x)$ in Eq. (16) can be rewritten as

$$p(x)\mathrm{d}s(x) = \mathrm{d}f = \mathrm{Im}\ \phi'(z)\mathrm{d}z = \mathrm{Im}\ \phi_1'(\varsigma)\mathrm{d}\varsigma = \mathrm{Im}\frac{\mathrm{d}\varsigma}{\varsigma} = \mathrm{d}\theta,$$
$$(\text{using the relation } \varsigma = \mathrm{e}^{\mathrm{i}\theta} \quad \text{and} \quad \frac{\mathrm{d}\varsigma}{\mathrm{i}\varsigma} = \mathrm{d}\theta). \tag{20}$$

By using Eqs. (18) and (20), we need to prove the following equation:

$$I_1 = 0, \quad \text{where} \quad I_1 = \int_{B_1} \ln|z - t|\,\mathrm{d}\theta \tag{21}$$

where z is a point on the ellipse which is relating to $x(x_1, x_2)$, and t is also a point on the ellipse which is relating to $\xi(\xi_1, \xi_2)$ (Fig. 3).

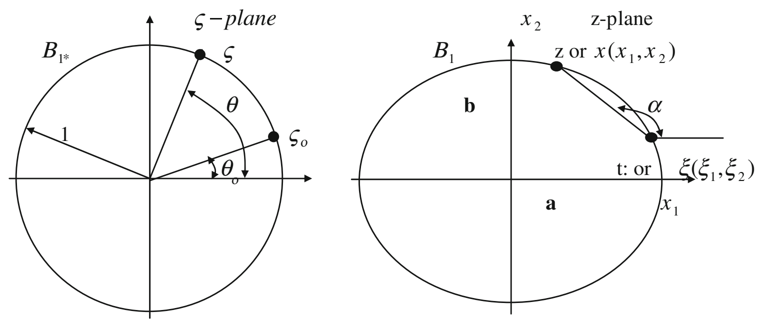

Fig. 3 Mapping relation for "z" to "ς" and "t" to "ς_o", with both "z" and "t" on the elliptic contour

Clearly, the integral I_1 can be rewritten as

$$I_1 = \mathrm{Re}\int_{B_1} \ln(z-t)\mathrm{d}\theta = \mathrm{Re}\int_{B_1} \ln(z-t)\frac{\mathrm{d}\varsigma}{i\varsigma} = \mathrm{Im}\int_{B_1} \ln(z-t)\frac{\mathrm{d}\varsigma}{\varsigma}. \tag{22}$$

If both z (for integration) and t (observation point) are located on the elliptic boundary (Fig. 3), we can let

$$z = R\left(\varsigma + \frac{m}{\varsigma}\right), \quad t = R\left(\varsigma_o + \frac{m}{\varsigma_o}\right), \quad z - t = R(\varsigma - \varsigma_o)\left(1 - \frac{m}{\varsigma_o\varsigma}\right) \tag{23}$$

where both ς and ς_o are located on the unit circle.

Therefore, we have

$$I_1 = I_{11} + I_{12} + I_{13}, \tag{24}$$

where

$$I_{11} = \mathrm{Im}\int_{B_1} \ln R\frac{\mathrm{d}\varsigma}{\varsigma} = 2\pi \ln R, \tag{25}$$

$$I_{12} = \mathrm{Im}\int_{B_1} \ln(\varsigma - \varsigma_o)\frac{\mathrm{d}\varsigma}{\varsigma}, \tag{26}$$

$$I_{13} = \mathrm{Im}\int_{B_1} \ln\left(1 - \frac{m}{\varsigma_o\varsigma}\right)\frac{\mathrm{d}\varsigma}{\varsigma} = 0. \tag{27}$$

Clearly, in Eq. (27), the function $\ln(1 - \frac{m}{\varsigma_o\varsigma})$ can be considered as a boundary value of the analytic function $\ln(1 - \frac{m}{\varsigma_o\varsigma})$ defined outside of the unit circle. Therefore, the result shown by Eq. (27) is proved [17].

For the integral I_{12}, from Fig. 3, we have

$$\begin{aligned} I_{12} &= \mathrm{Im}\int_{B_1} \ln(\varsigma - \varsigma_o)\frac{\mathrm{d}\varsigma}{\varsigma} = \mathrm{Re}\int_{B_1} \ln(\varsigma - \varsigma_o)\frac{\mathrm{d}\varsigma}{i\varsigma} = \mathrm{Re}\int_{B_1} \ln(\varsigma - \varsigma_o)\mathrm{d}\theta = \int_0^{2\pi} \ln\left|2\sin\frac{\theta-\theta_o}{2}\right|\mathrm{d}\theta \\ &= \int_{\theta_o}^{2\pi+\theta_o} \ln\left|2\sin\frac{\theta-\theta_o}{2}\right|\mathrm{d}\theta = \int_0^{2\pi} \ln\left(2\sin\frac{u}{2}\right)\mathrm{d}u = 2\int_0^{\pi} \ln(2\sin v)\,\mathrm{d}v = 4\int_0^{\pi/2} \ln(2\sin v)\,\mathrm{d}v = 0. \end{aligned} \tag{28}$$

Finally, we have

$$I_1 = \int_{B_1} \ln|z-t|\,\mathrm{d}\theta = \mathrm{Im}\int_{B_1} \ln(z-t)\frac{\mathrm{d}\varsigma}{\varsigma} = 2\pi \ln R. \tag{29}$$

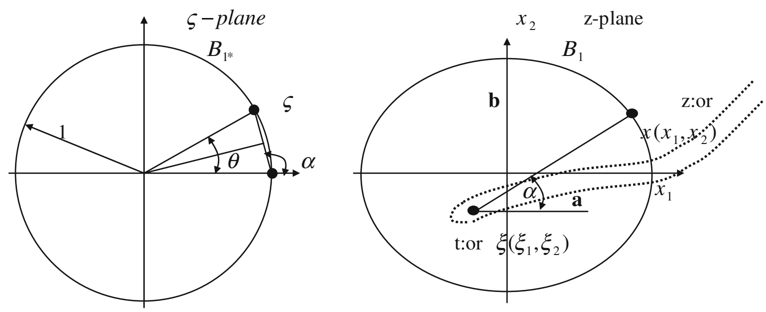

Fig. 4 Mapping relation for "z" to "ς" with "z" on the elliptic contour and "t" in the elliptic contour

If substituting $R = 1$ into Eq. (29), we have

$$I_1 = \int_{B_1} \ln|z-t|\,\mathrm{d}\theta = \mathrm{Im}\int_{B_1} \ln(z-t)\frac{\mathrm{d}\varsigma}{\varsigma} = 0, \quad \text{(for } z \text{ and } t \text{ on the elliptic boundary).} \tag{30}$$

Finally, we reach the following conclusion. Under the aforementioned three conditions, $I_1 = 0$, or Eq. (18) is satisfied.

Similarly, to prove Eq. (19) is equivalent to prove the following equation:

$$I_2 = 0, \quad \text{where} \quad I_2 = \int_{B_1} \ln|z-t|\,\mathrm{d}\theta \tag{31}$$

where z is a point on the ellipse which is relating to $x(x_1, x_2)$, Different to the aforementioned case, now t is a point in the interior region to the ellipse contour which is relating to $\xi(\xi_1, \xi_2)$ (Fig. 4).

Similarly, we have

$$I_2 = \int_{B_1} \ln|z-t|\,\mathrm{d}\theta = \mathrm{Re}\int_{B_1} \ln(z-t)\mathrm{d}\theta = \mathrm{Re}\int_{B_1} \ln(z-t)\frac{\mathrm{d}\varsigma}{\mathrm{i}\varsigma} = \mathrm{Im}\int_{B_1} \ln(z-t)\frac{\mathrm{d}\varsigma}{\varsigma}. \tag{32}$$

Note that $\ln|z-t|$ is a single-valued function. However, the imaginary part of $\ln(z-t)$ is multiply-valued. In this case, one may define a branch cut line from point "t" to $z = a$ to infinity (Fig. 4). However, from the definition shown by $\mathrm{Re}\int_{B_1} \ln(z-t)\mathrm{d}\theta$, the assumed branch will not affect the final result of I_2.

In Eq. (32), the term $\ln(z-t)$ can be written in the form

$$\ln(z-t) = \ln z + \ln\left(1 - \frac{t}{z}\right) = \ln R + \ln\varsigma + \ln\left(1 + \frac{m}{\varsigma^2}\right) + \ln\left(1 - \frac{t}{R(\varsigma + m/\varsigma)}\right). \tag{33}$$

Note that the function $\ln(1-\frac{t}{z}) = \ln(z-t) - \ln z$ (where t is located in the inside region to the elliptic contour) is an analytic single-valued function for z defined in the region outside the elliptic contour. Because both functions $\ln(z-t)$ and $\ln z$ have an increment $2\pi\mathrm{i}$ when z goes around the elliptic contour, this property is easily seen. This property does not change after the mapping relation is used. Thus, $\ln(1 - \frac{t}{R(\varsigma+m/\varsigma)}) = \ln(1-\frac{t}{z})$ is also an analytic single-valued function defined in the region outside the unit circle. Clearly, $\ln(1+\frac{m}{\varsigma^2})$ is also a single-valued function in the region outside the unit circle. From Eqs. (32) and (33), we have

$$I_2 = I_{21} + I_{22} + I_{23} + I_{24}, \tag{34}$$

where

$$I_{21} = \mathrm{Im}\int_{B_1} \ln R\frac{\mathrm{d}\varsigma}{\varsigma} = 2\pi\ln R, \tag{35}$$

$$I_{22} = \mathrm{Im} \int\limits_{B_1} \ln \varsigma \frac{\mathrm{d}\varsigma}{\varsigma} = \mathrm{Im} \int\limits_{B_1} \ln \varsigma \mathrm{d}(\ln \varsigma) = \mathrm{Im} \frac{(\ln \varsigma)^2}{2} \Bigg|_{\ln \varsigma = 0}^{\ln \varsigma = 2\pi i} = \mathrm{Im}(-4\pi^2) = 0, \quad (36)$$

$$I_{23} = \mathrm{Im} \int\limits_{B_1} \ln \left(1 + \frac{m}{\varsigma^2}\right) \frac{\mathrm{d}\varsigma}{\varsigma} = 0, \quad (37)$$

$$I_{24} = \mathrm{Im} \int\limits_{B_1} \ln \left(1 - \frac{t\varsigma}{R(\varsigma^2 + m)}\right) \frac{\mathrm{d}\varsigma}{\varsigma} = 0. \quad (38)$$

In Eqs. (37) and (38), the two functions $\ln(1 + \frac{m}{\varsigma^2})$ and $\ln(1 - \frac{t\varsigma}{R(\varsigma^2+m)})$ can be considered as the boundary values of the single-valued functions defined in the region outside of the unit circle. Finally, we have

$$I_2 = \int\limits_{B_1} \ln |z - t| \, \mathrm{d}\theta = 2\pi \ln R. \quad (39)$$

If substituting $R = 1$ into Eq. (39), we obtain

$$I_2 = \int\limits_{B_1} \ln |z - t| \, \mathrm{d}\theta = \mathrm{Im} \int\limits_{B_1} \ln(z - t) \frac{\mathrm{d}\varsigma}{\varsigma} = 0,$$

(for z on the boundary, and t an interior point to the elliptic boundary). (40)

Finally, we can get the following conclusion: under the aforementioned three conditions, $I_2 = 0$, or Eq. (19) is satisfied.

3 Conclusions

The merit of the present study is as follows. This paper not only provides the degenerate scale but also the non-trivial solution [see the statement after Eq. (12)], which is derived from the case of a single elliptic contour. However, the arising problem is whether the mentioned degenerate scale and the non-trivial solution satisfy the conditions for the case of a multiply connected region with outer elliptic boundary. After evaluating some integrals in closed form, the assertion is proved.

From the aforementioned analysis, we can obtain the following conclusion. Under the aforementioned three conditions, Eqs. (18) and (19) will be satisfied. Alternatively speaking, the aforementioned three conditions provide the degenerate scale (or eigenvalue) and boundary traction on the boundaries (or eigenfunction) for the problem addressed.

As claimed previously, the degenerate scale will cause an illness solution for the BIE. It is assumed that the ellipse has a ratio $b/a = 1/3$ in the multiply connected region with outer elliptic boundary. Therefore, from Eq. (12), we have $b/a = 1/3 = (1 - m)/(1 + m)$, or $m = 0.5$. In this case, one should avoid using the size $a = 1 + m = 1.5$ and $b = 1 - m = 0.5$ in computation of the BIE, particularly, in the Dirichlet boundary value problem.

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