A Green's function for the annulus

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gutta cavat lapidum, consumitor anulus usu
OVIDIUS, Epistulae ex Ponto

Contents

0. Introduction. Let $\Omega=\{1<|z|<R\}$ be a (circular) annulus in the complex plane $\mathbb{C}$ with generic point $z=x+iy=re^{i\theta}$. In this paper we shall consider the problem of determining Green's function for the operator Δ^2 (the square of the Laplace operator $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$) in Ω subject to Dirichlet boundary conditions $u=\dfrac{\partial u}{\partial N}=0$ on the boundary $\partial\Omega=\{|z|=1\}\cup\{|z|=R\}$ of Ω, N being the outer normal on $\partial\Omega$.

A solution of the analogous question for the operator Δ itself can be found in the book [8], p. 335-337 (cf. Appendix V of the present paper). There the problem is reduced to the study of a functional equation and in this way the appropriate Green's function gets expressed in terms of Jacobi theta functions. We record that, physically speaking, it is question of a clamped elastic plane respectively a membrane.

Remark 1. We note also that the homogeneous Dirichlet problem, on the other hand, was considered by H. Villat [20] in 1912. From the formulae in his paper one can read off expressions for the corresponding Poisson kernels in terms of the Weierstrass zeta function. (Villat's result is not quoted in [8].) □

Our approach is rather simple-minded and depends on separation of variables (the method of Bernoulli and Fourier). In principle it works also for any power Δ^p ($p=1,2,\dots$) of Δ and for many other operators as well (cf. the appendices) and also in higher dimensions ($\mathbb{R}^d$ in place of $\mathbb{C}$); in this paper we consider the two dimensional case exclusively. Let us give an outline of our method.

For simplicity we assume first that the pole of our Green's function U sits at a point t on the positive real axis, $1<t<R$. In accordance with Almansi's theorem [1] (see Appendix I) we have the Fourier expansion

$$U=\sum{}''(A_n^*r^n+B_n^*r^{2+n}+C_n^*r^{-n}+D_n^*r^{2-n})e^{in\theta}\qquad\text{in }\{1<|z|<t\};$$
$$U=\sum{}''(A_n^{**}r^n+B_n^{**}r^{2+n}+C_n^{**}r^{-n}+D_n^{**}r^{2-n})e^{in\theta}\qquad\text{in }\{t<|z|<R\}.$$

The double stroke $''$ after the sum sign means that the expression has to be modified if $n=0,\pm1$ due to the presence of logarithmic terms (we turn to this case only in Section 3). The corresponding basis of

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biharmonic functions consists of the functions z^n and $z^n|z|^2$ and their conjugates (plus the logarithmic terms when $n = 0, \pm 1$; cf. again Section 3). The boundary conditions give

$$\begin{cases} A_n^* & + \quad B_n^* & + \quad C_n^* & + \quad D_n^* & = 0; \\ nA_n^* & + (2+n)B_n^* & + (-n)C_n^* & + (2-n)D_n^* & = 0; \\ A_n^{**}R^n & + \quad B_n^{**}R^{2+n} & + \quad C_n^{**}R^{-n} & + \quad D_n^{**}R^{2-n} & = 0; \\ nA_n^{**}R^n & + (2+n)B_n^{**}R^{2+n} & + (-n)C_n^{**}R^{-n} & + (2-n)D_n^{**}R^{2-n} & = 0. \end{cases}$$

On the other hand, let u^* and u^{**} denote the one-sided distributional boundary values of u on the circle $\{|z| = t\}$. Exploiting the partial differential equation $\Delta^2 u = \delta_t(z)$ (Dirac function) then gives

$$\begin{cases} U^{**} - U^* = 0; \\ \dfrac{\partial U^{**}}{\partial N} - \dfrac{\partial U^*}{\partial N} = 0; \\ \dfrac{\partial^2 U^{**}}{\partial N^2} - \dfrac{\partial^2 U^*}{\partial N^2} = 0; \\ \dfrac{\partial^3 U^{**}}{\partial N^3} - \dfrac{\partial^3 U^*}{\partial N^3} = t^{-1}\delta_0(\theta), \end{cases}$$

where $\delta_=$ is the one dimensional delta function at the point 0.

Remark 2. The presence of the factor t^{-1} in the last equation is due to the relation

$$\iint \delta_t(r,\theta)\, r dr d\theta = 1,$$

where as above δ_t stands for the delta function placed at the point t. □

It follows that we must have

$$\begin{cases} \Delta A_n t^n + & \Delta B_n t^{2+n} + & \Delta C_n t^{-n} + & \Delta D_n t^{2-n} = 0; \\ n\ \Delta A_n t^n + (2+n) & \Delta B_n t^{2+n} + (-n) & \Delta C_n t^{-n} + & (2-n)\Delta D_n t^{2-n} = 0; \\ n^2 \Delta A_n t^n + (2+n)^2 & \Delta B_n t^{2+n} + (-n)^2 & \Delta C_n t^{-n} + & (2-n)^2 \Delta D_n t^{2-n} = 0; \\ n^3 \Delta A_n t^n + (2+n)^3 & \Delta B_n t^{2+n} + (-n)^3 & \Delta C_n t^{-n} + & (2-n)^3 \Delta D_n t^{2-n} = \frac{1}{2\pi} t^2, \end{cases}$$

where we have put $\Delta A_n = A_n^{**} - A_n^*$ etc. Thus for each index n we have *in toto* a system of 8 linear equations in the 8 unknowns A_n^*, B_n^*, C_n^*, D_n^*, A_n^{**}, B_n^{**}, C_n^{**}, D_n^{**}.

Why have we undertaken this research? One reason is that we thought that this might shed some new light on the time honored problem of the positivity of Green's function for the clamped plate (see e. g. the discussion in [13], where the author evokes the names Boggio and Hadamard, speaking of the Boggio-Hadamard conjecture). Indeed, using the explicit formulae obtained, we are able to show that, regardless of the size of R, the Green function U is always negative at some point. On the other hand, as the aforementioned formula in the case of the operator Δ involves theta functions, we thought that the generalization to the case of Δ^2 might also involve interesting special functions. From this point of view this paper belongs to a series of papers which the senior author has been engaged in over a long period, the first of these being perhaps [15]. That the resulting formulae become so very complicated is of course a source of some disappointment.

The disposition of matters is as follows. The solution of the above mentioned linear system is given in Section 1. Indeed, it is convenient to treat a more general case with general multipliers x_1, x_2, x_3, x_4 in place of the particular set of numbers $n, 2+n, -n, 2-n$. In this special case the full expression of the Fourier coefficients is written out in Section 2, where we likewise verify the convergence of our series. The exceptional case $n = 0, \pm 1$ is considered in Section 3. Finally, in Section 4 we collect the information obtained so far writing out the series expansion of the function U in full. The result is expressed in terms of certain "transcendental" functions, apparently, of a new type. The simplest of these is a certain meromorphic function $\mathbf{X}(\lambda)$, in the punctured plane $\mathbb{C}\backslash\{0\}$, which for $R^{-2} < |\lambda| < R^2$ is given by the series development

$$\mathbf{X}(\lambda) = \sum_{|n|>1} \frac{\lambda^n}{(R^n - R^{-n})^2 - n^2(R - R^{-1})^2}.$$

Remark 3. Actually, the expression occurring in the above denominator occurs already in Almansi's great paper [1], p. 24. Perhaps we should call it the *Almansi determinant.* (A related expression occurs also in [12], formula (36), p. 512 in the case of a domain bounded by an ellipse.) Note that we assume basically that $R > 1$, that is, that $|z| = R$ is the outer circle. However, it is very easy to modify our formulae to the case $R < 1$ when $|z| = R$ is the inner circle (see Remark 4 in Section 2) or even, in order to obtain more symmetric formulations, to adapt the result to the case when we have two circles $|z| = R$ and $|z| = R'$ with $R \neq R'$, as in [1]. It is the ratio $\frac{R}{R'}$ that matters. □

We note that in the case of the operator Δ (cf. Appendix V) one encounters instead the series

$$\sum_{|n|>0} \frac{\lambda^n}{R^n - R^{-n}},$$

which series is closely connected with the Weierstrass zeta function, or rather with its multiplicative analogue. Thus our function $\mathbf{X}$, say, must be viewed as a natural generalization of the latter.[2] In the case of Δ^2 the same function $\mathbf{X}$ enters also in the expression for the corresponding Poisson kernels, which calculations are set forth in Section 5. Section 6 is devoted to the limiting cases $R \to 0$ and $R \to \infty$ (the punctured disc and the exterior of the disc, respectively) and also contains a partly new counterexample to the aforementioned Boggio-Hadamard conjecture. The latter is completely solved (for the case of the annuli) in the next Section 7: we show that U can never be positive in the whole annulus. In the proof a decisive rôle is played by the famous Schur algorithm [18]. In Section 8 the function $\mathbf{X}$ and the other transcendental functions entering in our expression for U are studied in some detail.

There are also several appendices where we discuss auxiliary topics. In Appendix I we have assembled some salient facts connected with biharmonic functions in general, including a proof of Almansi's theorem [1]. In Appendix II we extend our results to the more general case of Hedenmalm's famous weighted bi-Laplacean $\Delta |z|^{-2\alpha} \Delta$ [13]. In Appendix III we consider briefly the related case of the strip, which may be viewed as a limiting case of the annulus. Appendix IV deals with the singularities of the biharmonic continuation of the Green function. In Appendix V we give, mainly for the benefit of the reader, the corresponding computations of Green's function for the operator Δ (not Δ^2) in the annulus. Note that this gives also, in principle, an alternative derivation of the formula in [8]. Finally, in Appendix VI we put the basic computation in Section 1 in a broader perspective by connecting it with a certain interpolation problem.

The sign □ is used liberally to design not only end of proofs, but also end of remarks, examples etc.

1. Solution of a system of linear equations. It will be convenient to consider a somewhat more general system of equations, viz.

$$(1) \qquad \begin{cases} A^* + B^* + C^* + D^* = 0; \\ x_1 A^* + x_2 B^* + x_3 C^* + x_4 D^* = 0; \\ A^{**} R^{x_1} + B^{**} R^{x_2} + C^{**} R^{x_3} + D^{**} R^{x_4} = 0; \\ x_1 A^{**} R^{x_1} + x_2 B^{**} R^{x_2} + x_3 C^{**} R^{x_3} + x_4 D^{**} R^{x_4} = 0; \\ \Delta A t^{x_1} + \Delta B t^{x_2} + \Delta C t^{x_3} + \Delta D t^{x_4} = 0; \\ x_1 \Delta A t^{x_1} + x_2 \Delta B t^{x_2} + x_3 \Delta C t^{x_3} + x_4 \Delta D t^{x_4} = 0; \\ x_1^2 \Delta A t^{x_1} + x_2^2 \Delta B t^{x_2} + x_3^2 \Delta C t^{x_3} + x_4^2 \Delta D t^{x_4} = 0; \\ x_1^3 \Delta A t^{x_1} + x_2^3 \Delta B t^{x_2} + x_3^3 \Delta C t^{x_3} + x_4^3 \Delta D t^{x_4} = c. \end{cases}$$

with, similarly as before, $\Delta A = A^{**} - A^*$ etc. and arbitrary exponents x_1, x_2, x_3, x_4, c being an arbitrary constant. It is clear that when

$$(2) \qquad x_1 = n, \quad x_2 = 2 + n, \quad x_3 = -n, \quad x_4 = 2 - n, \quad c = \tfrac{t^2}{2\pi}$$

then (1) reduces to the system in the Introduction. Note that, in the general case, the exponents x_1, x_2, x_3, x_4 enter in a symmetric fashion. In the four last equations (1) it is essentially question of inverting a 4 dimensional "Vandermonde matrix". Indeed, it is readily seen that one has

$$(3) \qquad \Delta A = \frac{c t^{-x_1}}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \text{ etc.}$$

[2]In a way, what we are dealing with in this paper may be viewed as a generalization of quantum- or q-function theory; cf. Remark 2 in Section 8. Often, for instance in [13], one puts $q = R^2$ and considers this quantity as the modulus of the annulus.

This allows one to eliminate, say, the variables A^{**} etc., writing $A^{**} = A^* + \Delta A$, in the first four equations (1). Therefore what remains is a system of four equations for the determination of the quantities A^* etc., namely

$$\begin{array}{lllll} A^* & + \quad B^* & + \quad C^* & + \quad D^* & = 0; \\ x_1 A^* & + x_2 B^* & + x_3 C^* & + x_4 D^* & = 0; \\ A^* R^{x_1} & + \quad B^* R^{x_2} & + \quad C^* R^{x_3} & + \quad D^* R^{x_4} & = \heartsuit; \\ x_1 A^* R^{x_1} & + x_2 B^* R^{x_2} & + x_3 C^* R^{x_3} & + x_4 D^* R^{x_4} & = \heartsuit\heartsuit, \end{array} \tag{4}$$

where

$$\heartsuit = -c\left[\frac{(R/t)^{x_1}}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{(R/t)^{x_2}}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + \right. \\ \left. + \frac{(R/t)^{x_3}}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{(R/t)^{x_4}}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}\right]$$

and

$$\heartsuit\heartsuit = -c\left[\frac{x_1(R/t)^{x_1}}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{x_2(R/t)^{x_2}}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + \right. \\ \left. + \frac{x_3(R/t)^{x_3}}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{x_4(R/t)^{x_4}}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}\right].$$

In treating the system (4) we begin by expanding the corresponding determinant, viz.

$$\Lambda = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ R^{x_1} & R^{x_2} & R^{x_3} & R^{x_4} \\ x_1 R^{x_1} & x_2 R^{x_2} & x_3 R^{x_3} & x_4 R^{x_4} \end{vmatrix},$$

using Laplace's theorem [16], p. 38. We find

$$\Lambda = (x_2-x_1)(x_4-x_3)R^{x_3+x_4} - (x_3-x_1)(x_4-x_2)R^{x_2+x_4} + \\ + (x_4-x_1)(x_3-x_2)R^{x_2+x_3} + (x_3-x_2)(x_4-x_1)R^{x_1+x_4} - \\ - (x_4-x_2)(x_3-x_1)R^{x_1+x_3} + (x_4-x_3)(x_2-x_1)R^{x_1+x_2},$$

which formula can also be written as

$$\boxed{\begin{aligned} \Lambda = & (x_1-x_2)(x_3-x_4)(R^{x_1+x_2} + R^{x_3+x_4}) + \\ & + (x_1-x_3)(x_4-x_2)(R^{x_1+x_3} + R^{x_2+x_4}) + \\ & + (x_1-x_4)(x_2-x_3)(R^{x_1+x_4} + R^{x_2+x_3}) \end{aligned}}. \tag{5}$$

Example 1. Let us look at the special case when the exponents are given by (2). In this case the differences and the sums of the exponents x_1 etc. are given by

$$(x_i - x_k) = \begin{pmatrix} 0 & -2 & 2n & -2+2n \\ 2 & 0 & 2+2n & 2n \\ -2n & -2-2n & 0 & -2 \\ 2-2n & -2n & 2 & 0 \end{pmatrix}$$

and

$$(x_i + x_k) = \begin{pmatrix} \bullet & 2+2n & 0 & 2 \\ 2+2n & \bullet & 2 & 4 \\ 0 & 2 & \bullet & 2-2n \\ 2 & 4 & 2-2n & \bullet \end{pmatrix}$$

respectively.[3] Using this information one finds

$$\begin{aligned} \Lambda = \Lambda_n = \Lambda_n(R) = & \\ & = (-2)\cdot(-2)(R^{2+2n} + R^{2-2n}) - 2n\cdot 2n(1+R^4) + \\ & + (-2+2n)(2+2n)\cdot 2R^2 = \\ & = 4R^2\big[R^{2n} + R^{-2n} - n^2(R^2+R^{-2}) + 2(n^2-1)\big], \end{aligned} \tag{6}$$

[3]Note that the latter is a Hankel matrix; since the diagonal terms are of no interest for us we have indicated them by the sign • ("bullet"). This is of course just a restatement of the fact that $x_1 + x_4 = x_2 + x_3 = 2$ in this case.

which also can be written

$$\boxed{\Lambda = 4R^2\big[(R^n - R^{-n})^2 - n^2(R - R^{-1})^2\big]}\,. \tag{7}$$

For later use we also set

$$\boxed{M_n = M_n(R) = (R^n - R^{-n})^2 - n^2(R - R^{-1})^2}\,, \tag{8}$$

so that $\Lambda = 4R^2 M_n$ then. $\square$

Remark 1. The quantity M_n will play a major rôle in what follows. Let us note that $M_n > 0$ if $|n| > 1$. As recorded already in Remark 3 in the Introduction, M_n occurs essentially already in [1], p. 24. We suggested there that it be called the *Almansi determinant.* See also Example 1 in Appendix VI. $\square$

Returning to the general case we can now determine the coefficients with the help of Cramer's rule. Let us begin with writing down an expression for the unknown A^*, say. We have

$$\Lambda \cdot A^* = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & x_2 & x_3 & x_4 \\ \heartsuit & R^{x_2} & R^{x_3} & R^{x_4} \\ \heartsuit\heartsuit & x_2R^{x_2} & x_3R^{x_3} & x_4R^{x_4} \end{vmatrix}$$

or, upon subtracting a suitable multiple of the third row from the fourth,

$$\Lambda \cdot A^* = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ (\heartsuit x_2 - \heartsuit\heartsuit)R^{x_2} & (\heartsuit x_3 - \heartsuit\heartsuit)R^{x_3} & (\heartsuit x_4 - \heartsuit\heartsuit)R^{x_4} \end{vmatrix}$$

or expanding

$$\begin{aligned}\Lambda \cdot A^* = {} & (x_4 - x_3)(\heartsuit x_2 - \heartsuit\heartsuit)R^{x_2} + (x_2 - x_4)(\heartsuit x_3 - \heartsuit\heartsuit)R^{x_3} + \\ & + (x_3 - x_2)(\heartsuit x_4 - \heartsuit\heartsuit)R^{x_4}.\end{aligned}$$

On the other hand, we find

$$\begin{aligned}\heartsuit x_2 - \heartsuit\heartsuit &= c \cdot \left[\frac{(R/t)^{x_1}}{(x_1 - x_3)(x_1 - x_4)} + 0 + \frac{(R/t)^{x_3}}{(x_3 - x_1)(x_3 - x_4)} + \right. \\ &\qquad \left. + \frac{(R/t)^{x_4}}{(x_4 - x_1)(x_4 - x_3)}\right]; \\ \heartsuit x_3 - \heartsuit\heartsuit &= c \cdot \left[\frac{(R/t)^{x_1}}{(x_1 - x_2)(x_1 - x_4)} + \frac{(R/t)^{x_2}}{(x_2 - x_1)(x_2 - x_4)} + 0 + \right. \\ &\qquad \left. + \frac{(R/t)^{x_4}}{(x_4 - x_1)(x_4 - x_2)}\right]; \\ \heartsuit x_4 - \heartsuit\heartsuit &= c \cdot \left[\frac{(R/t)^{x_1}}{(x_1 - x_2)(x_1 - x_3)} + \frac{(R/t)^{x_2}}{(x_2 - x_1)(x_2 - x_3)} + \right. \\ &\qquad \left. + \frac{(R/t)^{x_3}}{(x_3 - x_1)(x_3 - x_2)} + 0\right].\end{aligned}$$

Using this we obtain

$$\begin{aligned}A^* = \frac{c}{\Lambda}\Bigg[& \left(\frac{(x_4 - x_3)R^{x_1+x_2}t^{-x_1}}{(x_1 - x_3)(x_1 - x_4)} - \frac{R^{x_3+x_2}t^{-x_3}}{x_3 - x_1} + \frac{R^{x_4+x_2}t^{-x_4}}{x_4 - x_1}\right) + \\ & + \left(\frac{(x_2 - x_4)R^{x_1+x_3}t^{-x_1}}{(x_1 - x_2)(x_1 - x_4)} + \frac{R^{x_2+x_3}t^{-x_2}}{x_2 - x_1} - \frac{R^{x_4+x_3}t^{-x_4}}{x_4 - x_1}\right) + \\ & + \left(\frac{(x_3 - x_2)R^{x_1+x_4}t^{-x_1}}{(x_1 - x_2)(x_1 - x_3)} - \frac{R^{x_2+x_4}t^{-x_2}}{x_2 - x_1} + \frac{R^{x_3+x_4}t^{-x_3}}{x_3 - x_1}\right)\Bigg].\end{aligned}$$

After some simplifications we can write this as

$$\boxed{\begin{aligned}A^* = \frac{c}{\Lambda}\Bigg[& -\left(\frac{(x_3 - x_4)R^{x_1+x_2}}{(x_1 - x_3)(x_1 - x_4)} + \frac{(x_4 - x_2)R^{x_1+x_3}}{(x_1 - x_2)(x_1 - x_4)} + \right. \\ & \qquad \left. + \frac{(x_2 - x_3)R^{x_1+x_4}}{(x_1 - x_2)(x_1 - x_3)}\right)t^{-x_1} + \\ & + \frac{(R^{x_3} - R^{x_4})R^{x_2}}{x_2 - x_1}t^{-x_2} + \frac{(R^{x_4} - R^{x_2})R^{x_3}}{x_3 - x_1}t^{-x_3} + \frac{(R^{x_2} - R^{x_3})R^{x_4}}{x_4 - x_1}t^{-x_4}\Bigg]\end{aligned}}\,, \tag{9}$$

which formula can again be written in condensed form as

$$A^* = \frac{c}{\Lambda}\left[-\sum_{jkl}\frac{(x_k - x_l)R^{x_1+x_j}}{(x_1-x_k)(x_1-x_l)}t^{-x_1} + \sum_{jkl}\frac{(R^{x_k}-R^{x_l})R^{x_j}}{x_j - x_1}t^{-x_j}\right],$$

where the summation is carried over all cyclic permutations jkl of the indices 234. Exploiting the symmetry we can now likewise write down the corresponding expressions for the remaining coefficients B^* etc. We find thus corresponding to (9):

$$\boxed{\begin{aligned} B^* = \frac{-c}{\Lambda}\Bigg[-\bigg(&\frac{(x_3-x_4)R^{x_2+x_1}}{(x_2-x_3)(x_2-x_4)} + \frac{(x_4-x_1)R^{x_2+x_3}}{(x_2-x_1)(x_2-x_4)} + \\ &+\frac{(x_1-x_3)R^{x_2+x_4}}{(x_2-x_1)(x_2-x_3)}\bigg)t^{-x_2} + \\ +\frac{(R^{x_3}-R^{x_4})R^{x_1}}{x_1-x_2}t^{-x_1} &+ \frac{(R^{x_4}-R^{x_1})R^{x_3}}{x_3-x_2}t^{-x_3} + \frac{(R^{x_1}-R^{x_3})R^{x_4}}{x_4-x_2}t^{-x_4}\Bigg]\end{aligned}}, \tag{10}$$

$$\boxed{\begin{aligned} C^* = \frac{c}{\Lambda}\Bigg[-\bigg(&\frac{(x_4-x_1)R^{x_3+x_2}}{(x_3-x_1)(x_3-x_4)} + \frac{(x_2-x_4)R^{x_3+x_1}}{(x_3-x_2)(x_3-x_4)} + \\ &+\frac{(x_1-x_2)R^{x_3+x_4}}{(x_3-x_2)(x_3-x_1)}\bigg)t^{-x_3} + \\ +\frac{(R^{x_4}-R^{x_1})R^{x_2}}{x_2-x_3}t^{-x_2} &+ \frac{(R^{x_2}-R^{x_4})R^{x_1}}{x_1-x_3}t^{-x_1} + \frac{(R^{x_1}-R^{x_2})R^{x_4}}{x_4-x_3}t^{-x_4}\Bigg]\end{aligned}} \tag{11}$$

and

$$\boxed{\begin{aligned} D^* = \frac{-c}{\Lambda}\Bigg[-\bigg(&\frac{(x_3-x_1)R^{x_4+x_2}}{(x_4-x_3)(x_4-x_1)} + \frac{(x_1-x_2)R^{x_4+x_3}}{(x_4-x_2)(x_4-x_1)} + \\ &+\frac{(x_2-x_3)R^{x_4+x_1}}{(x_4-x_2)(x_4-x_3)}\bigg)t^{-x_4} + \\ +\frac{(R^{x_3}-R^{x_1})R^{x_2}}{x_2-x_4}t^{-x_2} &+ \frac{(R^{x_1}-R^{x_2})R^{x_3}}{x_3-x_4}t^{-x_3} + \frac{(R^{x_2}-R^{x_3})R^{x_1}}{x_1-x_4}t^{-x_1}\Bigg]\end{aligned}}. \tag{12}$$

Note that if we move the ith column to position 1 then the sign of the determinant Λ changes; this explains the presence of a minus in front of c in the above expressions (10) and (12).

As $A^{**} = A^* + \Delta A$, we can using (4) also easily get formulae for the quantities A^{**} etc. For instance, we have

$$\begin{aligned} A^{**} = \frac{c}{\Lambda}\Bigg[\bigg(&\frac{(x_3-x_4)R^{x_3+x_4}}{(x_1-x_3)(x_1-x_4)} + \frac{(x_4-x_2)R^{x_2+x_4}}{(x_1-x_2)(x_1-x_4)} + \\ &+\frac{(x_2-x_3)R^{x_2+x_3}}{(x_1-x_3)(x_1-x_4)}\bigg)t^{-x_1} + \\ +\frac{(R^{x_3}-R^{x_4})R^{x_2}}{x_2-x_1}t^{-x_2} &+ \frac{(R^{x_4}-R^{x_2})R^{x_3}}{x_3-x_1}t^{-x_3} + \frac{(R^{x_2}-R^{x_3})R^{x_4}}{x_4-x_1}t^{-x_4}\Bigg],\end{aligned} \tag{13}$$

and similar formulae for the remaining coefficients B^{**}, C^{**} and D^{**}.

Remark 2. The apparent similarity between the expressions for A^* etc., on the one hand, and A^{**} etc., on the other hand, is connected with the fact that if we replace A^* etc. by $R^{x_1}A^{**}$ etc., at the same time writing $-c$ in place of c, we get as solution of the same system (1) with $\frac{1}{R}$ in place of R and $\frac{t}{R}$ in place of t. □

2. The Fourier coefficients for $|n| > 1$. Specializing to the case $x_1 = n$, $x_2 = 2{+}n$, $x_3 = -n$, $x_4 = 2{-}n$ (with $n \neq 0, \pm 1$) the formulae (9)-(12) in Section 1 give us at once the corresponding coefficients in the Fourier expansion of our Green's function U (see Introduction). Indeed, we find after some rearrangement:

$$\boxed{\begin{aligned} A_n^* = \frac{1}{16\pi M_n}&\Big[\tfrac{1}{n(n-1)}\big\{(R^{2n}-1) - n^2(R^{-2}-1)\big\}t^{2-n} - \\ &-(R^2-1)t^{-n} - \tfrac{1}{n}(R^{-2n}-1)t^{2+n} + \tfrac{1}{n-1}(R^{-2n}-R^2)t^n\Big]\end{aligned}}; \tag{1}$$

$$B_n^* = \frac{1}{16\pi M_n}\Big[-\tfrac{1}{n(n+1)}\big\{(R^{2n}-1)-n^2(R^2-1)\big\}t^{-n}+ +(R^{-2}-1)t^{2-n}+\tfrac{1}{n+1}(R^{-2n}-R^{-2})t^{2+n}-\tfrac{1}{n}(R^{-2n}-1)t^n\Big]; \tag{2}$$

$$C_n^* = \frac{1}{16\pi M_n}\Big[\tfrac{1}{n(n+1)}\big\{(R^{-2n}-1)-n^2(R^{-2}-1)\big\}t^{2+n}- -\tfrac{1}{n+1}(R^{2n}-R^2)t^{-n}+\tfrac{1}{n}(R^{2n}-1)t^{2-n}-(R^2-1)t^n\Big]; \tag{3}$$

$$D_n^* = \frac{1}{16\pi M_n}\Big[-\tfrac{1}{n(n-1)}\big\{(R^{-2n}-1)-n^2(R^2-1)\big\}t^{n}+ +\tfrac{1}{n}(R^{2n}-1)t^{-n}+(R^{-2}-1)t^{2+n}-\tfrac{1}{n-1}(R^{2n}-R^{-2})t^{2-n}\Big]; \tag{4}$$

Here M_n is as in formula (8) in Section 1.

Remark 1. Note that we have $C_n^* = A_{-n}^*$, $D_n^* = B_{-n}^*$, which is a reflection of the fact that the function U is real valued. This remark will be exploited in connection with our final result in Section 4. □

Similar formulae prevail for A_n^{**} etc. For instance, we have

$$A_n^{**} = \frac{1}{16\pi M_n}\Big[-\tfrac{1}{n(n-1)}\big\{(R^{-2n}-1)-n^2(R^2-1)\big\}t^{2-n}- -(R^2-1)t^{-n}-\tfrac{1}{n}(R^{-2n}-1)t^{2+n}+\tfrac{1}{n-1}(R^{-2n}-R^2)t^n\Big]. \tag{5}$$

Remark 2. The rule by which a formula like (5) is obtained is, apparently, the following: *We keep the last three terms but modify the first one by first subtracting the quantity M_n from the expression within curly brackets $\{\}$ and then changing the sign of the whole so modified term.* □

For the sake of completeness we write also down the expressions for the quantities ΔA_n etc. (cf. formula (3) in Section 1):

$$\Delta A_n = -\frac{1}{16\pi}\frac{t^{2-n}}{n(n-1)};\quad \Delta B_n = \frac{1}{16\pi}\frac{t^{-n}}{n(n+1)};$$
$$\Delta C_n = -\frac{1}{16\pi}\frac{t^{2+n}}{n(n+1)};\quad \Delta D_n = \frac{1}{16\pi}\frac{t^{n}}{n(n-1)}.$$

From formulae (1)-(4) we can obtain the following asymptotic expressions for our coefficients for $n\to\infty$:

$$A_n^* \sim \frac{1}{16\pi}\cdot\frac{t^2}{n^2}\cdot\frac{1}{t^n}; \tag{6}$$

$$B_n^* \sim -\frac{1}{16\pi}\cdot\frac{1}{n^2}\cdot\frac{1}{t^n}; \tag{7}$$

$$C_n^* \sim \frac{1}{16\pi}\cdot\frac{t^2-1}{n}\cdot\frac{1}{t^n}; \tag{8}$$

$$D_n^* \sim -\frac{1}{16\pi}\cdot\frac{t^2-1}{n}\cdot\frac{1}{t^n}. \tag{9}$$

Here we have used the fact that by our hypothesis $1<t<R$.

Proof of (6): (The proof of (7)-(9) is entirely parallel). It follows from (1) that

$$t^nA_n^* = \frac{1}{16\pi}\frac{R^{2n}}{M_n}\Big[\frac{t^2}{n(n-1)}[(1-R^{-2n})-n^2(R^{-2-2n}-R^{-2n})]- -(R^{2-2n}-R^{-2n})-\tfrac{1}{n}(R^{-2n}-1)t^2(\frac{t}{R})^{2n}+\tfrac{1}{n-1}(R^{-2n}-R^2)(\frac{t}{R})^{2n}\Big].$$

It follows from (8) in Section 1 that

$$\lim_{n\to\infty} \frac{R^{2n}}{M_n} = 1.$$

Moreover, by hypothesis we have $1 < t < R$. Therefore we find

$$\lim_{n\to\infty} n^2 t^n A_n^* = \frac{t^2}{16\pi},$$

which is precisely the meaning of the symbol $\sim$ in (6). □

Now we observe that $C_n^* = A_{-n}^*$ and $D_n^* = B_{-n}^*$. This follows by inspection from (1)-(4), but is also a reflection of the fact that the function U must be real valued. Therefore (6)-(9) can likewise be used to get asymptotic estimates for the same coefficients for $n \to -\infty$. In particular, we can draw from here the following important conclusion.

Lemma 1. *Assume that $1 < t < R$. Let $E_n^*(r)$ denote the n-th coefficient in the Fourier expansion of our Green's function u in the region $1 < |z| < t$, i.e.* (see the Introduction)

$$E_n^*(r) = A_n^* r^n + B_n^* r^{2+n} + C_n^* r^{-n} + D_n^* r^{2-n} \quad (n \neq 0, \pm 1).$$

Then we have the estimate

$$|E_n^*(r)| \leq \text{const.} \Big(\max(\frac{t}{r}, \frac{1}{rt}) \Big)^{|n|},$$

with a constant independent of r. □

Again it follows from here that our series converges not only for $1 < |z| < t$, as it is expected to do, but also for $\frac{1}{t} < |z| \leq 1$. *Thus, as a corollary, we have obtained an explicit biharmonic continuation of the function u to the whole region $\frac{1}{t} < |z| < t$.*

Remark 3. We have treated only the case $1 < |z| < t$. The case $t < |z| < R$ can be treated in an analogous way. Alternatively one could have relied on the fact that the whole set-up is invariant under inversion, $z \mapsto \frac{R}{z}$. □

Remark 4. (The case $R < 1$.) Finally we remark that up to now we have assumed that $R > 1$. On the other hand, if $R < 1$, *i.e.* we have the annulus $\Omega = \{R < |z| < 1\}$, we can use exactly the same expressions for the Fourier coefficient: *The only thing that we have to do is to change the sign in the above formulae* $(1) - (4)$. Namely, there appears a minus sign in the right hand side of the last 4 of the 8 linear equations for the Fourier coefficients, because change of "orientation": the coefficients with a single star $*$ now correspond to the "exterior" portion $\{t < |z| < 1\}$ of the annulus, while the ones with a double star $**$ correspond to the "interior" portion $\{R < |z| < t\}$. □

3. The case $|n| \leq 1$. In this section we quickly go through the computations of the Fourier coefficients in the exceptional case $n = 0, \pm 1$. If, as before, the nth Fourier coefficient is denoted by $E_n(r)$ (with an additional superscript $*$ if $1 < |z| < t$, and a superscript $**$ if $t < |z| < R$), then we have (cf. Appendix I)

$$\begin{aligned} E_0(r) &= A_0 + B_0 r^2 + C_0 \log r^2 + D_0 r^2 \log r^2; \\ E_1(r) &= A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 r \log r^2; \\ E_{-1}(r) &= A_{-1} r^{-1} + B_{-1} r \log r^2 + C_{-1} r + D_{-1} r^3 \end{aligned}$$

corresponding to the follow bases of biharmonic functions:

$$\{1, r^2, \log r^2, r^2 \log r^2\}; \quad \{z, zr^2, \frac{1}{\bar{z}}, z \log r^2\}; \quad \{z^{-1}, \bar{z} \log r^2, \bar{z}, \bar{z} r^2\}.$$

We treat each of these three cases separately.

$\boxed{n = 0}$. In this case we are lead to the system of equations

$$\begin{cases} A_0^* + B_0^* + 0 + 0 = 0; \\ 0 + 2B_0^* + 2C_0^* + 2D_0^* = 0; \\ A_0^{**} + B_0^{**} R^2 + C_0^{**} \log R^2 + D_0^{**} R^2 \log R^2 = 0; \\ 0 + 2B_0^{**} R + 2C_0^{**} R^{-1} + 2D_0^{**} (R \log R^2 + R) = 0; \\ \Delta A_0 + \Delta B_0 t^2 + \Delta C_0 \log t^2 + \Delta D_0 t^2 \log t^2 = 0; \\ 0 + 2\Delta B_0 t^2 + 2\Delta C_0 + 2\Delta D_0 (t^2 \log t^2 + t^2) = 0; \\ 0 + 2\Delta B_0 t^2 + (-2)\Delta C_0 + 2\Delta D_0 (t^2 \log t^2 + 3t^2) = 0; \\ 0 + 0 + 4\Delta C_0 + 4\Delta D_0 t^2 = \frac{1}{2\pi} t^2, \end{cases}$$

where $\Delta A_0 = A_0^{**} - A_0^*$ etc. Following the same policy as in Section 1 we begin by first solving the last four equations. One finds

(1) $$\Delta A_0 = \tfrac{1}{16\pi}(2t^2 - t^2\log t^2); \quad \Delta B_0 = \tfrac{1}{16\pi}(-2-\log t^2); \quad \Delta C_0 = \tfrac{1}{16\pi}t^2; \quad \Delta D_0 = \tfrac{1}{16\pi}.$$

Using the relations $A_0^{**} = A_0^* + \Delta A_0$ etc., we can eliminate the quantities A_0^{**} etc. in the first four equations. This gives the system

$$\begin{cases} A_0^* + B_0^* + \quad + 0 = 0; \\ 0 + 2B_0^* + 2C_0^* + 2D_0^* = 0; \\ 0 + B_0^{**}(R^2-1) + C_0^{**}\log R^2 + D_0^{**}R^2\log R^2 = \heartsuit; \\ 0 + 2B_0^{**}R^2 + 2\, C_0^{**} + 2D_0^{**}(R^2\log R^2 + R^2) = \heartsuit\heartsuit. \end{cases}$$

where

$$\heartsuit = \tfrac{1}{16\pi}\left[(2R^2 - R^2\log R^2) + t^2(-2-\log R^2)t^2 + R^2\log t^2 + t^2\log t^2\right]$$

and

$$\heartsuit\heartsuit = \tfrac{1}{16\pi}\left[(R^2 - R^2\log R^2) - t^2 + R^2\log t^2\right].$$

The determinant of this system is of the form $4R^2M_0$ where

(2) $$\boxed{M_0 = (\log R^2)^2 - (R - R^{-1})^2}.$$

We notice that

$$M_0 = \lim_{n\to 0}\frac{M_n}{n^2}.$$

Solving out gives then

(3) $$\boxed{\begin{aligned} A_0^* = -B_0^* = \tfrac{1}{16\pi M_0}\big[&2(R^2-1) + 2\log R^2 - (\log R^2)^2 + \\ &+ \big(2(R^{-2}-1) - 2\log R^2 - (\log R^2)^2\big)t^2 + \\ &+ \big(R^2 - 1 + \log R^2\big)\log t^2 + \\ &+ \big(-(R^{-2}-1) + \log R^2\big)t^2\log t^2\big] \end{aligned}};$$

(4) $$\boxed{\begin{aligned} C_0^* = \tfrac{1}{16\pi M_0}\big[&(R^2-1) + \log R^2 + \\ &+ \big(R^{-2} - 1 - \log R^2 - (\log R^2)^2\big)t^2 + \\ &+ (R^2-1)\log t^2 + \\ &+ \log R^2 t^2\log t^2\big] \end{aligned}};$$

(5) $$\boxed{\begin{aligned} D_0^* = \tfrac{1}{16\pi M_0}\big[&(R^2-1) + \log R^2 - (\log R^2)^2 + \\ &+ \big(R^{-2} - 1 - \log R^2\big)t^2 + \\ &+ \log R^2\log t^2 - \\ &- (R^{-2}-1)t^2\log t^2\big] \end{aligned}}.$$

The coefficients A_0^{**} etc. are found using the relations $A_0^{**} = A_0^* + \Delta A_0$ etc. along with the expressions for ΔA_0 etc. obtained in (1).

$\boxed{n=1}$. In this case the system takes the form

$$\begin{cases} A_1^* + B_1^* + C_1^* + 0 = 0; \\ A_1^* + 3B_1^* + (-1)C_1^* + 2D_1^* = 0; \\ A_1^{**}R + B_1^{**}R^3 + C_1^{**}R^{-1} + D_1^{**}R\log R^2 = 0; \\ A_1^{**}R + 3B_1^{**}R^3 + (-1)C_1^{**}R^{-1} + D_1^{**}R(2+\log R^2) = 0; \\ \Delta A_1 t + \Delta B_1 t^3 + \Delta C_1 t^{-1} + \Delta D_1 t\log t^2 = 0; \\ \Delta A_1 t + 3\Delta B_1 t^3 + (-1)\Delta C_1 t^{-1} + \Delta D_1(2t + t\log t^2) = 0; \\ 0 + 6\Delta B_1 t^3 + 2\Delta C_1 t^{-1} + 2\Delta D_1 t = 0; \\ 0 + 6\Delta B_1 t^3 + (-6)\Delta C_1 t^{-1} + (-2)\Delta D_1 t = \tfrac{1}{2\pi}t^2. \end{cases}$$

It can be treated in the same way as in the case $n = 0$. In particular, we find that the first four equations give the determinant $-8R^2(R - R^{-1})M_1$ with

$$\boxed{M_1 = (R^2 - R^{-2}) \log R^2 - 2(R - R^{-1})^2}. \tag{6}$$

This time we have the limes relation

$$M_1 = \lim_{n\to 1} \frac{M_n}{n-1}.$$

The last four equations are solved by

$$\Delta A_1 = \frac{1}{16\pi} \cdot t \log t^2; \quad \Delta B_1 = \frac{1}{16\pi} \cdot \frac{t^{-1}}{2};$$
$$\Delta C_1 = \frac{1}{16\pi} \cdot \frac{-t^3}{2}; \quad \Delta D_1 = \frac{1}{16\pi} \cdot (-t).$$

Finally, the sought solution is found to be given by the expressions

$$\boxed{\begin{aligned} A_1^* = \tfrac{1}{16\pi M_1} \big[&(R^2 - R^{-2} - (R^2 + R^{-2}) \log R^2)t + \\ &+ (R^{-2} - 1 + R^{-2} \log R^2)t^3 + \\ &+ (1 - R^2 + R^2 \log R^2)t^{-1} + \\ &+ (R - R^{-1})^2 t \log t^2\big] \end{aligned}}; \tag{7}$$

$$\boxed{\begin{aligned} B_1^* = \tfrac{1}{16\pi M_1} \big[&(R^{-2} - 1 + R^{-2} \log R^2)t - \\ &- \tfrac{1}{2} R^{-2} \log R^2 t^3 + \\ &+ (R^2 - 1 - \tfrac{1}{2} R^2 \log R^2)t^{-1} + \\ &+ (1 - R^{-2})t \log t^2\big] \end{aligned}}; \tag{8}$$

$$\boxed{\begin{aligned} C_1^* = \tfrac{1}{16\pi M_1} \big[&(1 - R^2 + R^2 \log R^2)t + \\ &+ (1 - R^{-2} - \tfrac{1}{2} R^{-2} \log R^2)t^3 - \\ &- \tfrac{1}{2} R^2 \log R^2 t^{-1} + \\ &+ (1 - R^2)t \log t^2\big] \end{aligned}}. \tag{9}$$

$$\boxed{\begin{aligned} D_1^* = \tfrac{1}{16\pi M_1} \big[&((R^2 - R^{-2}) \log R^2 - (R - R^{-1})^2)t + \\ &+ (1 - R^{-2})t^3 + \\ &+ (1 - R^2)t^{-1} + \\ &+ (R^{-2} - R^2)t \log t^2\big] \end{aligned}}. \tag{10}$$

$\boxed{n = -1}$. We have the obvious relations $A_{-1} = C_1$, $B_{-1} = D_1$, $C_{-1} = A_1$ and $D_{-1} = B_1$. Therefore we need not even write down the result in this case. Note also that the determinant is the same, $M_{-1} = M_1$.

Remark 1. Alternatively, we could have obtained the coefficient formulae in this section by a passage to the limit ($n \to 0, 1$ or -1). Let us indicate how this goes in the case $n = 0$. We note that if the basis r^n, r^{2+n}, r^{-n}, r^{2-n} is replaced by r^n, r^{2+n}, $\frac{1}{n}(r^n - r^{-n})$, $\frac{1}{n}(r^{2+n} - r^{2-n})$ then the coefficients A_n, B_n, C_n, D_n are replaced by $A_n + C_n$, $B_n + D_n$, $-nC_n$, $-nD_n$; note that the expressions for the coefficients make sense even if we treat n as a continuous variable, rather than a discrete one, as up to now, of course, as long as we avoid the values $n = 0, \pm 1$. Thus we obtain

$$A_0 = \lim_{n\to 0}(A_n + C_n); \qquad B_0 = \lim_{n\to 0}(B_n + D_n);$$
$$C_0 = -\lim_{n\to 0} nC_n; \quad D_0 = -\lim_{n\to 0} nD_n.$$

Analogous formulae hold for $n = \pm 1$. However, the calculations become hardly any simpler this way. □

4. Main result. Using the formulae for the Fourier coefficients A_n etc. derived in Section 2 and 3 we are, finally, in a position to write down an expression for our Green's function U. First we recall that up to this moment we have assumed that U has its pole at a point on the positive real axis, this mainly for notational simplicity. On the other hand, if the pole is situated on a ray forming the angle ψ with the x-axis at the distance $t > 0$ from the origin, we have simply to replace θ by $\theta - \psi$. It will be convenient to put $\boxed{w = te^{i\psi}}$.

In order to obtain a compact formulation it will further be convenient to introduce certain "transcendental" functions denoted $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Y}_+$, $\mathbf{Y}_-$, $\mathbf{Z}_+$, $\mathbf{Z}_-$. They are defined by the following series developments:

$$
(1)\qquad
\begin{aligned}
\mathbf{X}(\lambda) &= \sum_{|n|>1} \frac{1}{M_n}\lambda^n & &(R^{-2} < |\lambda| < R^2);\\
\mathbf{Y}(\lambda) &= \sum_{|n|>1} \frac{1}{n}\frac{R^{-2n}-1}{M_n}\lambda^n & &(1 < |\lambda| < R^2);\\
\mathbf{Y}_+(\lambda) &= \sum_{|n|>1} \frac{1}{n-1}\frac{R^{-2n}-R^2}{M_n}\lambda^n & &(1 < |\lambda| < R^2);\\
\mathbf{Y}_-(\lambda) &= \sum_{|n|>1} \frac{1}{n+1}\frac{R^{-2n}-R^{-2}}{M_n}\lambda^n & &(1 < |\lambda| < R^2);\\
\mathbf{Z}_+(\lambda) &= \sum_{|n|>1} \frac{1}{n(n+1)}\frac{(R^{2n}-1)-n^2(R^2-1)}{M_n}\lambda^n & &(R^{-2} < |\lambda| < 1);\\
\mathbf{Z}_-(\lambda) &= \sum_{|n|>1} \frac{1}{n(n-1)}\frac{(R^{2n}-1)-n^2(R^{-2}-1)}{M_n}\lambda^n & &(R^{-2} < |\lambda| < 1),
\end{aligned}
$$

where as before (see formula (8) in Section 1)

$$
(2)\qquad
\begin{aligned}
M_n = M_n(R) &= (R^n - R^{-n})^2 - n^2(R - R^{-1})^2 =\\
&= R^{2n} + R^{-2n} - n^2(R^2 + R^{-2}) + 2(n^2 - 1),
\end{aligned}
$$

and where we have indicated, to the right, their respective ranges of convergence.

Remark 1. These functions will be investigated in some detail in Section 6. Let us note here right away only that the simplest and most basic of them is undoubtedly $\mathbf{X}(\lambda)$. This function admits a meromorphic continuation to the whole punctured plane $\mathbb{C}\backslash\{0\}$ (= doubly punctured Riemann sphere $S^2\backslash\{0,\infty\}$) with poles at the points $R^{\pm 2}, R^{\pm 4}, R^{\pm 6}, \ldots$, while the remaining ones are multivalued and display logarithmic singularities. We note also that the function $\mathbf{X}$ has the obvious symmetry $\mathbf{X}(\lambda) = \mathbf{X}(\frac{1}{\lambda})$, which follows from the fact that M_n is an even function of n, $M_{-n} = M_n$. Furthermore the three functions $\mathbf{Y}_+$, $\mathbf{Y}_-$ and $\mathbf{Y}$ can be unified by introducing the function $\mathbf{Y}_\kappa(\lambda)$, depending on an auxiliary parameter κ ($\neq \pm 2, \pm 3, \ldots$) with the expansion:

$$
\mathbf{Y}_\kappa(\lambda) = \sum_{|n|>1} \frac{1}{n-\kappa}\frac{R^{-2n}-R^{2\kappa}}{M_n}\lambda^n \qquad (1 < |\lambda| < R^2).
$$

Clearly we obtain the previous functions by taking $\kappa = 0, \pm 1$. It is likewise tempting to set

$$
\mathbf{Z}_\kappa(\lambda) = \sum_{|n|>1} \frac{1}{n(n+\kappa)}\frac{(R^{2n}-1)-n^2(R^{2\kappa}-1)}{M_n}\lambda^n.
$$

Then one covers in one stroke not only $\mathbf{Z}_+$ and $\mathbf{Z}_-$ (the case $\kappa = \pm 1$), but also the function $\mathbf{Y} = \mathbf{Y}_0$. Indeed, one has $\mathbf{Y}(\lambda) = -\mathfrak{E}\mathbf{Z}_0(\frac{1}{\lambda})$ where $\mathfrak{E} = \lambda\frac{d}{d\lambda}$ (Euler operator). □

Now we can state the following theorem.

Theorem 1. *Let $U = U(z) = U(z, w)$ be Green's function of the bi-Laplacean Δ^2 in the annulus $\Omega = \{1 < |z| < R\}$ with Dirichlet boundary conditions on the boundary $\partial\Omega$ and pole at the interior point $w \in \Omega$, i.e. if $\delta = \delta(z) = \delta(z, w)$ is Dirac's function at w, we have*

$$
\Delta^2 U = \delta \text{ in } \Omega;\ U = \frac{\partial U}{\partial N} = 0 \text{ on } \partial\Omega.
$$

Then U comes as a sum $U = U_{\text{transc}} + U_{\text{elem}}$ of a "transcendental" part U_{transc} and an "elementary" part U_{elem}. Again the elementary part comes as a sum $U_{\text{elem}} = U^0_{\text{elem}} + U^1_{\text{elem}}$ of a "zeroth order" part U^0_{elem} and a "first order" part U^1_{elem}. If $1 < |z| < |w|$ the transcendental part is given by

$$\boxed{\begin{aligned} U_{\text{transc}} = \frac{1}{8\pi}\,\mathrm{Re}\Big(&-(R^2-1)\mathbf{X}(\frac{z}{w}) + \mathbf{Y}_+(z\bar{w}) + \\ &+ |z|^2\Big(-\mathbf{Y}(z\bar{w}) - \mathbf{Z}_+(\frac{z}{w})\Big) + |w|^2\Big(-\mathbf{Y}(z\bar{w}) + \mathbf{Z}_-(\frac{z}{w})\Big) + \\ &+ |z|^2|w|^2\Big((R^{-2}-1)\mathbf{X}(\frac{z}{w}) + \mathbf{Y}_-(z\bar{w})\Big)\Big)\end{aligned}}, \tag{3}$$

while the elementary one is given by

$$\boxed{\begin{aligned} U^0_{\text{elem}} = \frac{1}{16\pi M_0}\Big\{&2(R^2-1) + 2\log R^2 - (\log R^2)^2 - \\ &-\big(2(R^2-1) + 2\log R^2 - (\log R^2)^2\big)|z|^2 + \\ &+\big(2(R^{-2}-1) - 2\log R^2 - (\log R^2)^2\big)|w|^2 - \\ &-\big(2(R^{-2}-1) - 2\log R^2 - (\log R^2)^2\big)|z|^2|w|^2 + \\ &+\big(R^2 - 1 + \log R^2\big)\log|z|^2 + \big(R^2 - 1 + \log R^2\big)\log|w|^2 + \\ &+\big(R^2 - 1 + \log R^2 - (\log R^2)^2\big)|z|^2\log|z|^2 + \big(1 - R^{-2} + \log R^2\big)|w|^2\log|w|^2 - \\ &-\big(R^2 - 1 + \log R^2\big)|z|^2\log|w|^2 + \big(R^{-2} - 1 - \log R^2 - (\log R^2)^2\big)\log|z|^2|w|^2 - \\ &-(1 - R^{-2} + \log R^2)|z|^2|w|^2\log|w|^2 + (R^{-2} - 1 - \log R^2)|z|^2\log|z|^2|w|^2 + \\ &+(R^2-1)\log|z|^2\log|w|^2 - (R^{-2}-1)|z|^2\log|z|^2|w|^2\log|w|^2 + \\ &+\log R^2\log|z|^2|w|^2\log|w|^2 + \log R^2|z|^2\log|z|^2\log|w|^2\Big\}\end{aligned}} \tag{4}$$

and

$$\boxed{\begin{aligned} U^1_{\text{elem}} = \frac{1}{8\pi M_1}\,\mathrm{Re}\Big\{&\big(1 - R^2 + R^2\log R^2\big)\Big(\frac{z}{w} + \frac{w}{z}\Big) + \\ &+\big(R^2 - R^{-2} - (R^2 + R^{-2})\log R^2\big)z\bar{w} - \frac{1}{2}R^2\log R^2\frac{1}{z\bar{w}} + \\ &+\big((R^{-2} - 1 + R^{-2}\log R^2)z\bar{w} + (R^2 - 1 - \tfrac{1}{2}R^2\log R^2)\frac{z}{w}\big)|z|^2 + \\ &+\big((R^{-2} - 1 + R^{-2}\log R^2)z\bar{w} + (1 - R^{-2} - \tfrac{1}{2}R^{-2}\log R^2)\frac{w}{z}\big)|w|^2 - \\ &-\tfrac{1}{2}R^{-2}\log R^2 z\bar{w}|z|^2|w|^2 + \\ &+\big(((R^2 - R^{-2})\log R^2 - (R - R^{-1})^2)z\bar{w} + (1 - R^2)\frac{z}{w}\big)\log|z|^2 + \\ &+\big((R - R^{-1})^2 z\bar{w} + (1 - R^2)\frac{w}{z}\big)\log|w|^2 + \\ &+(1 - R^{-2})z\bar{w}|z|^2\log|w|^2 + (1 - R^{-2})z\bar{w}\log|z|^2|w|^2 + \\ &+(R^{-2} - R^2)z\bar{w}\log|z|^2\log|w|^2\Big\}\end{aligned}}, \tag{5}$$

where M_0 and M_1 are given by (2) *and* (3) *in Section 3.* □

Remark 2. Our policy in presenting the elementary part (see (4) and (5)), as well as the transcendental part (see (3)) has been to write out everything as a linear combination of non-analytic functions with analytic ones as coefficients. □

As we have the symmetry $U(z,w) = U(w,z)$ – a standard consequence of the fact that Δ^2 with Dirichlet boundary conditions determines a self-adjoint operator – we obtain as a corollary the following result.

Corollary. *If $|w| < |z| < R$ we can, interchanging the rôle of z and w, use for U the same expressions as in the theorem. In particular, the transcendental part comes as*

$$\boxed{\begin{aligned}U_{\text{transc}} = \frac{1}{8\pi}\,\mathrm{Re}\Big(&-(R^2-1)\mathbf{X}(\frac{z}{w}) + \mathbf{Y}_+(z\bar{w}) + \\ &+ |z|^2(-\mathbf{Y}(z\bar{w}) + \mathbf{Z}_-(\frac{w}{z})) + |w|^2(-\mathbf{Y}(z\bar{w}) - \mathbf{Z}_+(\frac{w}{z})) + \\ &+ |z|^2|w|^2((R^{-2}-1)\mathbf{X}(\frac{z}{w}) + \mathbf{Y}_-(z\bar{w}))\Big)\end{aligned}}.$$

Thus the only significant change occurs in the two **Z***-terms.* □

Remark 3. The above formulae thus look rather symmetric in z and w. To make this symmetry perfect let us put into play the following well-known fundamental solution of the operator Δ^2:

$$E = E(z) = E(z,w) = \frac{1}{16\pi}|z-w|^2 \log|z-w|^2.$$

This function has (for w fixed) the Fourier expansion

$$\begin{aligned}E = \frac{1}{16\pi}\Bigg\{ &|w|^2\log|w|^2 + 2|z|^2 + |z|^2\log|w|^2 + \\ &+ 2\,\mathrm{Re}\big(-z\bar{w}\log|w|^2 - \frac{z}{w}|w|^2 - \tfrac{1}{2}\frac{z}{w}|z|^2\big) + \\ &+ 2\,\mathrm{Re}\left(\sum_{n=2}^{\infty} -\tfrac{1}{n(n+1)}\left(\frac{z}{w}\right)^n |z|^2 + \sum_{n=2}^{\infty}\tfrac{1}{n(n-1)}\left(\frac{z}{w}\right)^n |w|^2\right).\Bigg\}\end{aligned}$$

The functions U and E have the same singularities in Ω. It follows that their difference $U^\dagger = U - E$, which obviously is symmetric too, is biharmonic in the whole of Ω and thus must be represented by the <u>same</u> analytic expression there. That this is so can be easily reflected at the hand of the formulae (3)-(4). For instance, if we compare the coefficients of the terms $z\bar{w}\log|z|^2$ and $z\bar{w}\log|w|^2$ in (4), we see that their difference amounts precisely to M_1! If we turn instead our attention to the transcendental part of U, we express this in terms of the functions $\mathbf{Z}_\pm$. Let us introduce two more functions $\mathbf{Z}_+^\dagger$ and $\mathbf{Z}_-^\dagger$ defined as follows:

$$\begin{aligned}\mathbf{Z}_+^\dagger(\lambda) &= \mathbf{Z}_+(\lambda) + (1-\frac{1}{\lambda})\log(1-\lambda) - 1 + \frac{\lambda}{2};\\ \mathbf{Z}_-^\dagger(\lambda) &= \mathbf{Z}_-(\lambda) - (1-\lambda)\log(1-\lambda) - \lambda.\end{aligned}$$

It is clear that these functions are analytic for $R^{-2} < |\lambda| < R^2$ and are in this range represented by the series:

$$\begin{aligned}\mathbf{Z}_+^\dagger(\lambda) = &-\sum_{n=2}^{\infty}\tfrac{1}{n(n+1)}\frac{(R^{-2n}-1) - n^2(R^{-2}-1)}{M_n}\lambda^n + \\ &+\sum_{-\infty}^{n=-2}\tfrac{1}{n(n+1)}\frac{(R^{2n}-1) - n^2(R^2-1)}{M_n}\lambda^n;\end{aligned}$$

$$\begin{aligned}\mathbf{Z}_-^\dagger(\lambda) = &-\sum_{n=2}^{\infty}\tfrac{1}{n(n-1)}\frac{(R^{-2n}-1) - n^2(R^2-1)}{M_n}\lambda^n + \\ &+\sum_{-\infty}^{n=-2}\tfrac{1}{n(n-1)}\frac{(R^{2n}-1) - n^2(R^{-2}-1)}{M_n}\lambda^n.\end{aligned}$$

In particular, these series are clearly more advantageous from the <u>numerical</u> point of view than those for $\mathbf{Z}_+^\dagger$ and $\mathbf{Z}_-^\dagger$. □

5. The Poisson kernels. We begin by recalling *Green's formula* which in the case of the operator Δ^2 takes the form

(1) $$\int_\Omega \Delta^2 f\cdot g = \int_\Omega f\Delta^2 g + \int_{\partial\Omega}\left(\frac{\partial\Delta f}{\partial N}g - \Delta f\frac{\partial g}{\partial N} + \frac{\partial f}{\partial N}\Delta g - f\frac{\partial\Delta g}{\partial N}\right)$$

where the integration is with respect to area measure on Ω and arc length on $\partial\Omega$. In this connection Ω may be any bounded planar domain with smooth boundary $\partial\Omega$.

Let us apply (1) in the case when Ω is our annulus $\Omega = \{1 < |z| < R\}$, with the boundary consisting of the two circles $\Gamma^* = \{|z| = 1\}$ and $\Gamma^{**} = \{|z| = r\}$, taking $f = U =$ our Green's function with pole at the interior point $w \in \Omega$ and, furthermore, letting g be biharmonic in Ω, $\Delta^2 g = 0$. Writing $\varphi = g|\partial\Omega$, $\psi = \dfrac{\partial g}{\partial N}|\partial\Omega$ (restriction) we obtain the following representation formula for the solution of the homogeneous Dirichlet problem with data φ, ψ:

$$g(w) = \int_{\partial\Omega} (P\varphi + Q\psi), \tag{2}$$

where we have put $P = \dfrac{\partial \Delta U}{\partial N}|\partial\Omega$, $Q = -\Delta U|\partial\Omega$. The functions P and Q are known as the Poisson kernels at the point w corresponding to this problem. As the boundary $\partial\Omega$ has connectivity two, the integral in (1) comes as the sum of two, one extended over Γ^* and the other over Γ^{**}, so there are *in toto* four kernels denoted P^*, P^{**}, Q^* and Q^{**}. We remind that, as functions of w, they are biharmonic functions.

We wish to find explicit expressions for these kernels. To fix the ideas we shall concentrate our discussion on P^* and Q^*. Assuming that $1 < |z| < |w| < R$ let us write the function $U = U(z) = U(z, w)$ in the form

$$\begin{aligned} U = \frac{1}{8\pi} \operatorname{Re} \Big\{ & \sum{}'' \frac{1}{M_n} \Big[a^*_{n1} (z\bar{w})^n + a^*_{n2} (z\bar{w})^n |w|^2 + a^*_{n3} (\frac{z}{w})^n + a^*_{n4} (\frac{z}{w})^n |w|^2 \Big] + \\ & + \sum{}'' \frac{1}{M_n} \Big[b^*_{n1} (z\bar{w})^n + b^*_{n2} (z\bar{w})^n |w|^2 + b^*_{n3} (\frac{z}{w})^n + b^*_{n4} (\frac{z}{w})^n |w|^2 \Big] |z|^2 \Big\}, \end{aligned} \tag{3}$$

where a^*_{nj} and b^*_{nj} ($n \in \mathbb{Z}$, $j = 1, 2, 3, 4$) is a somewhat *ad hoc* notation for coefficients that were in principle determined in Section 2 (see the formula (1)-(4) there); the double stroke $''$ is, as in the Introduction, a reminder that the sum has to be conveniently modified if $n = 0, \pm 1$. In fact, we shall concentrate on the "transcendental" parts of our Poisson kernels, leaving it to the reader to work out the corresponding computations in the "elementary" case, $|n| \leq 1$.

We begin by writing down the corresponding expression for ΔU. We first note the formulae

$$\Delta(z^n |z|^2) = 4(n+1) z^n; \quad \Delta(z^n) = 0.$$

Using (3) we find

$$\begin{aligned} \Delta U = \frac{1}{2\pi} \operatorname{Re} \Big\{ & \sum{}'' \frac{1}{M_n} \Big[(n+1) b^*_{n1} (z\bar{w})^n + (n+1) b^*_{n2} (z\bar{w})^n |w|^2 + \\ & + (n+1) b^*_{n3} (\frac{z}{w})^n + (n+1) b^*_{n4} (\frac{z}{w})^n |w|^2 \Big] \Big\}; \end{aligned} \tag{4}$$

the quantities a^* have thus disappeared, as they should! Let now $z \in \Gamma^*$, that is, $|z| = 1$. From (2) in Section 2 we infer

$$b^*_{n1} = -\tfrac{1}{n}(R^{-2n} - 1); \quad b^*_{n2} = \tfrac{1}{n+1}(R^{-2n} - R^{-2});$$
$$(n+1) b^*_{n3} = -\tfrac{1}{n}\{(R^{2n} - 1) - n^2(R^2 - 1)\}; \quad b^*_{n4} = R^{-2} - 1.$$

Thus we find

$$\begin{aligned} Q^* = \frac{1}{2\pi} \operatorname{Re} \Big\{ & \sum{}'' \tfrac{n+1}{n} \frac{R^{-2n} - 1}{M_n} (z\bar{w})^n - \sum{}'' \frac{R^{-2n} - R^{-2}}{M_n} (z\bar{w})^n |w|^2 + \\ & + \sum{}'' \tfrac{1}{n} \frac{(R^{2n} - 1) - n^2(R^2 - 1)}{M_n} (\frac{z}{w})^n - \sum{}'' (n+1) \frac{R^{-2} - 1}{M_n} (\frac{z}{w})^n |w|^2. \end{aligned} \tag{5}$$

But this formula (5) is susceptible of further simplifications! Let us have a look at the first and the third term in (5). In the first term we can, taking complex conjugates, replace the factor $(z\bar{w})^n$ by $(\bar{z}w)^n$, while in the third term the factor $(\frac{z}{w})^n$ can, due to the relation $z\bar{z} = 1$, be written as $(\bar{z}w)^{-n}$. Changing the summation index to $-n$ in the last referred to sum, we can merge these two terms into one:

$$\frac{(R^{-2n} - 1) + n(R^2 - 1)}{M_n} (\bar{z}w)^n.$$

We can treat the second and the fourth term in a similar way. One finds that they also can be combined into one and the same expression:

$$-\frac{(R^{-2n}-1)-n(R^{-2}-1)}{M_n}(\bar{z}w)^n|w|^2.$$

It follows that (5) can be rewritten as

$$Q^* = \frac{1}{2\pi}\,\mathrm{Re}\left\{\sum{}''\frac{(R^{-2n}-1)+n(R^2-1)}{M_n}(\bar{z}w)^n - \right. \\ \left. -\sum{}''\frac{(R^{-2n}-1)-n(R^{-2}-1)}{M_n}(\bar{z}w)^n|w|^2.\right\} \tag{6}$$

In particular, we see Q^* can be expressed in terms of the function $\mathbf{X}(\lambda)$ (see Section 4), actually with the aid of this function and its first derivative.

In a similar way we can determine P^*. Instead of working with the normal derivative $\frac{\partial}{\partial N}$ we shall use the Euler derivative (cf. Appendix I):

$$\mathfrak{E} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}.$$

We note that $\mathfrak{E}(z^n|z|^2) = (n+2)z^n|z|^2$. Using this fact we find with the aid of (4) as a generalization:

$$\mathfrak{E}\Delta U = \frac{1}{2\pi}\,\mathrm{Re}\left\{\sum{}''\frac{1}{M_n}\left[(n+1)(n+2)b^*_{n1}(z\bar{w})^n + (n+1)(n+2)b^*_{n2}(z\bar{w})^n|w|^2 + \right.\right. \\ \left.\left. +(n+1)(n+2)b^*_{n3}(\frac{z}{w})^n + (n+1)(n+2)b^*_{n4}(\frac{z}{w})^n|w|^2\right]|z|^2\right\}. \tag{7}$$

Taking $|z|=1$ this gives after some simplifications

$$P^* = \frac{1}{2\pi}\,\mathrm{Re}\left\{\sum{}''\frac{(n+4)(R^{-2n}-1)-n(n-2)(R^2-1)}{M_n}(\bar{z}w)^n - \right. \\ \left. -\sum{}''\frac{(n+2)(R^{-2n}-1)+n(n-4)(R^{-2}-1)}{M_n}(\bar{z}w)^n|w|^2.\right\} \tag{8}$$

Remark 1. By formally setting $\varphi = 0$, $\psi = \delta = \delta_z =$ delta function with the mass placed at the point $z \in \Gamma^*$, we see using (2) that Q^*, as a function of w, must satisfy the boundary conditions $Q^* = 0$, $\frac{\partial Q^*}{\partial N} = \delta$ on $\partial\Omega$. Similarly, one finds $P^* = \delta$, $\frac{\partial P^*}{\partial N} = 0$. It is indeed an amusing exercise to verify that this is indeed the case. In doing this one has to take account of the following elementary fact: *Consider any series* $\sum \alpha_n z^n$ *where* $|z|=1$ *and the* α_n*'s are arbitrary complex numbers. Then the value of the real part* $\mathrm{Re}\sum\alpha_n z^n$ *depends only on the numbers* $\frac{\alpha_n+\alpha_{-n}}{2}$. In our case the coefficients are real valued so we can count modulo odd terms in the index n. □

Remark 2. We conclude this section by emphasizing that the above formulae (6) and (8) for P^* and Q^* respectively, as well as their counterparts for P^{**} and Q^{**} – which we have not bothered to write down –, are in complete harmony with the results of Villat [20] recalled in Remark 1 in the Introduction. In particular, we see that our function $\mathbf{X}$ must be viewed as a direct generalization of Weierstrass's function ζ. □

With but a little more work, we can also use the computations above to identify the harmonic Bergman kernel for the annulus. Indeed, consider a function g which vanishes on $\partial\Omega$ and is biharmonic in Ω, $\Delta^2 g = 0$. Applying a Laplacian in the w-variable in the formula (2), we find

$$\Delta g(w) = -\int_{\partial\Omega}\frac{\partial g}{\partial N}\cdot \Delta_w\Delta_z U. \tag{9}$$

Since the fundamental solution for Δ^2 is $E = \frac{1}{16\pi}|z|^2\log|z|^2$ (see Remark 3 in Section 4), we have

$$U(z,w) = \tfrac{1}{16\pi}|z-w|^2\log|z-w|^2 + \text{ a } C^\infty \text{ function on } \Omega\times\Omega.$$

Thus

$$\Delta_w \Delta_z U(z,w) = \delta(z-w) + h(z,w)$$

where $h(z,w)$ is C^∞ on $\Omega \times \Omega$. Moreover, the function h is also harmonic in each variable, a fact that we will need shortly. Since $h(z,w)$ coincides with $\Delta_z \Delta_w U$ for $z \neq w$, we can write (9) also as

$$\Delta g(w) = -\int_{\partial\Omega} \frac{\partial g}{\partial N} \cdot h(z,w).$$

As h has no singularities in Ω, we are at liberty to apply Green's formula once again. In view of the above mentioned harmonicity of h and the hypothesis that g vanish on $\partial\Omega$, this gives

$$\Delta g(w) = -\int_{\Omega} \Delta g(z) \cdot h(z,w),$$

or, in terms of the function $G = \Delta g$,

$$G(w) = -\int_{\Omega} G(z) h(z,w). \tag{10}$$

Since $\Delta^2 g = 0$, the function $G = \Delta g$ is harmonic; conversely, for any harmonic function G on Ω which is, say, in $L^1(\Omega)$, there exists a biharmonic function g such that $\Delta g = G$ and $g = 0$ on the boundary (just take $g(z) = \int_\Omega \Gamma(z,\zeta) G(\zeta)$, where $\Gamma(z,\zeta)$ is the ordinary Green function for the Dirichlet problem $\Delta G = g$ on Ω). It follows that (10) holds for all integrable harmonic functions G on Ω. In particular, denoting by $L^2_h(\Omega)$, the <u>harmonic Bergman space</u>, the subspace of all harmonic functions in $L^2(\Omega)$, we see that

$$k(z,w) = -h(z,w) = -\Delta_w \Delta_z U(z,w) \qquad \text{for } z \neq w \tag{11}$$

is the reproducing kernel for $L^2_h(\Omega)$. (See Garabedian [12], where a variant of this relation for Bergman spaces with weights is also established.)

Example 1. For Ω the unit disc, (11) reduces to the identity

$$\pi \Delta_w \Delta_z V(z,w) = 1 - 2\,\mathrm{Re}(1 - \bar{w}z)^{-2},$$

easily verified directly by a short computation. Here $V(z,w)$ is the biharmonic Green function for the unit disc, to be described in Section 6 below. □

Returning to the particular case of the annulus, we first let $1 < |z| < |w|$. Using the formula (4) for $\Delta_z U$, we get

$$\Delta_w \Delta_z U(z,w) = \frac{2}{\pi} \mathrm{Re} \left\{ \sum{}'' \frac{1}{M_n} \left[(n+1)^2 b^*_{n2} (z\bar{w})^n - (n^2-1) b^*_{n4} (\frac{z}{w})^n \right] \right\}; \tag{12}$$

the quantities b^*_{n1} and b^*_{n3} disappear, as they should, since the corresponding terms are harmonic in w. Similarly, we get an analogous expression for the double-star case $|w| < |z| < R$. However, since

$$b^*_{n1} = b^{**}_{n1}, \quad b^*_{n2} = b^{**}_{n2}, \quad b^*_{n4} = b^{**}_{n4},$$

we see that the formula (12) *is actually valid in both cases*, i.e. for all $w, z \in \Omega_R$ — a reflection of the fact that $h(z,w)$ is regular in Ω_R. Using the expressions for b^*_{n2} and b^*_{n4} mentioned after (4) and supplying the terms corresponding to the special values $n = 0, \pm 1$ (which is done in a completely analogous fashion, so we omit the details here), we finally arrive at the formula

$$\begin{aligned} k(z,w) = &- \sum_{|n|>1} \frac{1}{\pi M_n} \Big[(n^2-1)(1-R^{-2})(\frac{z^n}{w^n} + \frac{\bar{z}^n}{\bar{w}^n}) + \\ &\qquad + (n+1)(R^{-2n} - R^{-2})(z^n \bar{w}^n + \bar{z}^n w^n) \Big] - \\ &- \frac{1}{\pi M_0} \Big[\big(2(1-R^{-2}) + 2\log R^2 + (\log R^2)^2\big) + \\ &\qquad + (R^{-2} - 1 - \log R^2)(\log|z|^2 + \log|w|^2 + 4) + \\ &\qquad + (1-R^{-2})(\log|z|^2 + 2)(\log|w|^2 + 2) \Big] - \\ &- \frac{1}{\pi M_1} \Big[-2R^{-2}\log R^2 (z\bar{w} + \bar{z}w) + 2(1-R^{-2})(\frac{z}{w} + \frac{\bar{z}}{\bar{w}} + \frac{w}{z} + \frac{\bar{w}}{\bar{z}}) + \\ &\qquad + (R^{-2} - R^2)(\frac{1}{\bar{z}w} + \frac{1}{z\bar{w}}) \Big]. \end{aligned}$$

It is a really amusing exercise to verify the reproducing property for $k(z,w)$ directly from this.

Remark 3. Garabedian used the explicit formula for $k(z,w)$ to disprove the Boggio-Hadamard conjecture for a sufficiently eccentric ellipse: it suffices to show that $k(z,w) > 0$ for some points z and w on the boundary [12], p. 511. However, this does not seem to be easily seen in our case, even for $z = -w = 1$. □

6. A limiting case: the (punctured) disc. The methods of the main body of the this paper work of course also for the punctured disc. Consider for instance the case of the exterior of the unit circle, $\Omega = \Omega_\infty = \{1 < |z| < \infty\}$. Then the corresponding Fourier coefficients $A_n^* = A_{n\infty}^*$ and $A_n^{**} = A_{n\infty}^{**}$ etc. are determined by a certain system of linear equations. If $n \neq 0, \pm 1$ we have the six equations

$$(1)\qquad \begin{cases} A_n^* + B_n^* + C_n^* + D_n^* = 0; \\ n\, A_n^* + (2+n)B_n^* + (-n)\, C_n^* + (2-n)D_n^* = 0; \\ \Delta A_n t^n + \Delta B_n t^{2+n} + \Delta C_n t^{-n} + \Delta D_n t^{2-n} = 0; \\ n\, \Delta A_n t^n + (2+n)\Delta B_n t^{2+n} + (-n)\, \Delta C_n t^{-n} + (2-n)\Delta D_n t^{2-n} = 0. \\ n^2 \Delta A_n t^n + (2+n)^2 \Delta B_n t^{2+n} + (-n)^2 \Delta C_n t^{-n} + (2-n)^2 \Delta D_n t^{2-n} = 0. \\ n^3 \Delta A_n t^n + (2+n)^3 \Delta B_n t^{2+n} + (-n)^3 \Delta C_n t^{-n} + (2-n)^3 \Delta D_n t^{2-n} = \frac{1}{2\pi} t^2. \end{cases}$$

where, as before, $\Delta A_n = A_n^{**} - A_n^*$. But there are still eight unknowns. In order to have a unique solution, which is tantamount to the Green's function $U = U_\infty$ to be of order $o(|z| \log |z|^2)$ as z tends to infinity, we impose the additional conditions

$$A_{n\infty}^{**} = B_{n\infty}^{**} = 0 \text{ if } n > 1, C_{n\infty}^{**} = D_{n\infty}^{**} = 0 \text{ if } n < 1.$$

Alternatively, we could directly have passed to the limit $R = \infty$ in the formulae already available to us (see (1)-(4) in Section 2). In any case, we find

$$A_{n\infty}^* = \lim_{R\to\infty} A_n^* = \begin{cases} \frac{1}{16\pi}\left(\frac{1}{n(n-1)} t^{2-n}\right) & \text{if } n > 1 \\ \frac{1}{16\pi}\left(\frac{1}{n-1} t^n - \frac{1}{n} t^{2+n}\right) & \text{if } n < -1 \end{cases};$$

$$B_{n\infty}^* = \lim_{R\to\infty} B_n^* = \begin{cases} \frac{1}{16\pi}\left(-\frac{1}{n(n+1)} t^{-n}\right) & \text{if } n > 1 \\ \frac{1}{16\pi}\left(-\frac{1}{n} t^n + \frac{1}{n+1} t^{2+n}\right) & \text{if } n < -1 \end{cases};$$

and analogous expressions with $C_{n\infty}^*$ and $D_{n\infty}^*$. (As $C_{n\infty}^* = A_{-n\infty}^*$ and $B_{n\infty}^* = D_{-n\infty}^*$ we need not write down these expressions.) In the same way we find e.g.

$$A_{n\infty}^{**} = \lim_{R\to\infty} A_n^{**} = \begin{cases} 0 & \text{if } n > 1 \\ \frac{1}{16\pi}\left(-\frac{1}{n(n-1)} t^{2-n} - \frac{1}{n} t^{2+n} + \frac{1}{n-1} t^n\right) & \text{if } n < -1 \end{cases}.$$

Likewise we can determine the coefficients for $n = 0, \pm 1$. In this case it is possible to sum the series (it is essentially question of the formula

$$(2)\qquad \log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

due to Leibnitz.) We defer the details to Theorem 1 below. The resulting formula can be compared with the following known formula for Green's function of the exterior disc $\{1 < |z| \leq \infty\}$:

$$(3)\qquad V(z,w) = \frac{1}{16\pi}\left(|z-w|^2 \log\left|\frac{z-w}{1-z\bar{w}}\right|^2 + (1-|z|^2)(1-|w|^2)\right).$$

See e.g. [13], p. 52, where the normalization is a different one, so that the constant $\frac{1}{16\pi}$ can be suppressed; it is also stated there for the unit disc $\{|z| < 1\}$ itself, not the exterior disc $\{1 < |z| \leq \infty\}$ as here, but is easy to convince oneself that the same expression (1) can be used in either case.

Remark 1. The simplest way of proving (2) is otherwise via conformal invariance using Bojarski's theorem [5], which reduces everything to the case $w = 0$ (in the case of the disc). The corresponding general formula for the iterates Δ^p ($p = 1, 2, \dots$) is due to Hayman and Korenblum [14]; it can be established in an analogous fashion. (For details see [11].) $\square$

Remark 2. We can also treat the case Hedenmalm's operator (see Appendix II). This is the limiting case $R = \infty$ of the formulae given in that appendix. If the parameter β is an integer we get then a new proof of Hedenmalm's generalization of (1), [13], Theorem 4.6:

$$V_\beta(z,w) = \tfrac{1}{16\pi}\beta^{-2}\left(|z-w|^{2\beta}\log\left|\frac{z-w}{1-z\bar{w}}\right|^2 + \text{finitely many lower order terms}\right). \quad \square$$

Finally, let us clarify the point that was skipped over in the above discussion. Namely, one might easily be led to believe that the "exterior" Green's function V coincides with the limit, say, U_∞ of the Green's function $U = U_R$ for the annulus $\Omega = \Omega_R = \{1 < |z| < R\}$ as R tends to infinity. But this by all means not the case: Although, as we have seen, the Fourier coefficients agree for $|n| > 1$, they do not agree for $n = 0, \pm 1$. We shall set forth this in a moment but first we must make a slight detour.

As it is somewhat cumbersome to deal with biharmonic functions at the point at infinity, we prefer to change our set up, taking instead $R < 1$ and eventually letting R tend to zero. By a previous remark (see Remark 3 in Section 2) we know that in this new situation the only thing we have to do is to change the sign of the Fourier coefficients A_n etc. Let the unit disc – the interior of the unit circle – be denoted by $\Omega_0 = \{|z| < 1\}$ and its corresponding Green's function by V; we know that for V we can use the same analytic expression as given by (3) in the exterior case. Similarly, we retain the notation U_0 for the limit of the Green's function $U = U_R$ for the annulus:

$$U_0 = \lim_{R\to 0} U_R.$$

(It is assumed that, throughout this process of limit, the point w, *i.e.* the pole of the Green's function, remains fixed.) Then one has the following result.

Theorem 1. *In the notation just introduced holds*

$$U_0 = V - (1 - |z|^2 + |z|^2\log|z|^2)\cdot(1 - |w|^2 + |w|^2\log|w|^2). \tag{4}$$

Remark 3. For $R \to \infty$, one can similarly obtain

$$U_\infty = V - (1 - |z|^2 + \log|z|^2)\cdot(1 - t^2 + \log t^2).$$

This can also be inferred directly from the reflection principle (Corollary to Lemma 3 in Appendix I). $\square$

Proof. Let $|w| = t$. With no loss of generality we may assume that w lies on the positive real axis, in other words, that $w = t$. Let us begin by writing V in the form (cf. (3))

$$\begin{aligned} V &= \frac{1}{16\pi}\left(|z-w|^2\log\left|\frac{z-w}{1-z\bar{w}}\right|^2 + (1-|z|^2)(1-|w|^2)\right) = \\ &= \frac{1}{16\pi}(|z|^2 + t^2 - tz - t\bar{z})\Big(\log|z|^2 + 2\operatorname{Re}\Big[\log(1-\frac{z}{t}) - \log(1-tz)\Big]\Big). \end{aligned}$$

Assuming that $t < |z| < 1$ and using (2), we obtain from this the series expansion (we omit for a moment the additional term $(1-|z|^2)(1-t^2)$)

$$\begin{aligned} V = \frac{1}{16\pi}\Big(&|z|^2\log|z|^2 + t^2\log|z|^2 - 2\operatorname{Re}(tz\log|z|^2) + \\ &+ 2\operatorname{Re}\Big[-tz - \sum_{n=2}^{\infty}\tfrac{1}{n}t^n z^{-n}|z|^2 - \frac{t^3}{z} - \sum_{n=2}^{\infty}\tfrac{1}{n}t^{2+n}z^{-n} + \\ &+ t^2 + \tfrac{1}{2}t^3 z^{-1} + \sum_{n=2}^{\infty}\tfrac{1}{n+1}t^{2+n}z^{-n} + \sum_{n=2}^{\infty}\tfrac{1}{n-1}t^n z^{-n}|z|^2 + \\ &+ tz|z|^2 + \sum_{n=2}^{\infty}\tfrac{1}{n}t^n z^n|z|^2 + t^3 z + \sum_{n=2}^{\infty}\tfrac{1}{n}t^{2+n}z^n - \\ &- \sum_{n=2}^{\infty}\tfrac{1}{n-1}t^n z^n - t^2|z|^2 - \tfrac{1}{2}t^3 z|z|^2 - \sum_{n=2}^{\infty}\tfrac{1}{n-1}t^{2+n}z^n|z|^2\Big]\Big). \end{aligned} \tag{5}$$

In particular, the constant term ($n = 0$) gives the contribution

$$|z|^2 \log |z|^2 + t^2 \log |z|^2 + 2t^2 - 2t^2|z|^2. \tag{6}$$

On the other hand, we find using (3)-(5) in Section 3

$$\begin{aligned} \lim_{R\to 0} A_0^* &= -\lim_{R\to 0} B_0^* = \tfrac{1}{16\pi}(2t^2 - t^2 \log t^2); \\ \lim_{R\to 0} C_0^* &= \tfrac{1}{16\pi} t^2; \\ \lim_{R\to 0} D_0^* &= \tfrac{1}{16\pi}(t^2 - t^2 \log t^2). \end{aligned}$$

Here we have also taken into account that

$$\lim_{R\to 0} \frac{M_0}{R^{-2}} = -1.$$

and we have further remembered to change the sign twice (*sic!*). It follows that the contribution of these terms to the corresponding expansion U_0 is

$$(2t^2 - t^2 \log t^2)(1 - |z|^2) + t^2 \log |z|^2 + (t^2 - t^2 \log t^2)|z|^2 \log |z|^2. \tag{7}$$

Forming the difference of (7) and (6) yields

$$\begin{gathered} 2t^2 - t^2 \log t^2 - 2t^2|z|^2 + t^2 \log t^2|z|^2 + t^2 \log |z|^2 + t^2|z|^2 \log |z|^2 - \\ -t^2 \log t^2|z|^2 \log |z|^2 - |z|^2 \log |z|^2 - t^2 \log |z|^2 - 2t^2 + 2t^2|z|^2 \end{gathered}$$

which simplifies to

$$-t^2 \log t^2|z|^2 \log |z|^2 - t^2 \log t^2(1 - |z|^2) - (1 - t^2)|z|^2 \log |z|^2.$$

If we restore the missing term $(1 - |z|^2)(1 - t^2)$ we obtain the expression

$$-(1 - |z|^2 + |z|^2 \log |z|^2) \cdot (1 - t^2 + t^2 \log t^2).$$

We see that as regards the constant order terms the difference $U_0 - V$ agrees with the formula that we set out to prove, viz. (4).

In the same way as we determined the limits (7) we find now using (7)-(10) in Section 3

$$\begin{aligned} \lim_{R\to 0} A_1^* &= \tfrac{1}{16\pi}(-t + t^3); \\ \lim_{R\to 0} B_1^* &= \tfrac{1}{16\pi}(t - \tfrac{1}{2}t^3); \\ \lim_{R\to 0} C_1^* &= \tfrac{1}{16\pi}(-\tfrac{1}{2}t^3); \\ \lim_{R\to 0} D_1^* &= \tfrac{1}{16\pi}(-t). \end{aligned} \tag{8}$$

Here we have also taken into account that

$$\lim_{R\to 0} \frac{M_1}{R^{-2} \log R^2} = -1.$$

Now a pleasant discovery lies ahead! We see that the terms in the expansion of U_0 with $n = \pm 1$, arising from (8) and its counter-part with -1, are balanced by the corresponding terms coming from the expansion (5) of V.

In the same way we treat the case $|n| > 1$, which is actually already implicit in what we did in the beginning of this section. We find, e.g. for $n > 1$,

$$\begin{aligned} \lim_{R\to 0} A_n^* &= \tfrac{1}{16\pi}(\tfrac{1}{n} t^{2+n} - \tfrac{1}{n-1} t^n); \\ \lim_{R\to 0} B_n^* &= \tfrac{1}{16\pi}(\tfrac{1}{n} t^n - \tfrac{1}{n+1} t^{2+n}); \\ \lim_{R\to 0} C_n^* &= \tfrac{-1}{16\pi} \cdot \tfrac{1}{n(n+1)} t^{2+n}; \\ \lim_{R\to 0} D_n^* &= \tfrac{1}{16\pi} \cdot \tfrac{1}{n(n-1)} t^n, \end{aligned}$$

which is again balanced against the corresponding terms in (5). Alternatively, we can base the proof on the following purely conceptual argument: The expression of a term with $|n| > 1$ in the expansion for the difference $U_0 - V$ – note that this is a biharmonic function – depends on four coefficients. But the corresponding terms cannot blow up as we make $z = 0$. Therefore only two non-zero coefficients remain. But, if we take into account the boundary conditions for $|z| = 1$, we see that the latter must vanish too.

So in any case, as only the constant term remains, we have proved that (4) must hold. □

Let us give a simple application of the above result to the Boggio-Hadamard conjecture (the question of the sign of the biharmonic Green's function). However, a much stronger assertion will be proved in the next section.

Corollary. *If the inner* (*outer*) *radius* R *of the annulus* $\Omega = \Omega_R$ *is sufficiently small* (*big*) *then the Green's function cannot have constant sign.*

Proof. To fix the ideas let us again assume that $R < 1$ making eventually R tend to 0. It suffices it show that, for fixed $w \in \Omega_0$, the difference

$$V - (1 - |z|^2 + |z|^2 \log |z|^2) \cdot (1 - |w|^2 + |w|^2 \log |w|^2)$$

takes negative values for a suitable choice of z. We may assume again that w is on the positive real axis, $w = t$ with $0 < t < 1$. We take z too real but not necessary positive, writing $z = x$ with $-1 < x < 1$, and consider the real valued function

$$f(x) = (x-t)^2 \log\left(\frac{t-x}{1-tx}\right)^2 + (1-x^2)(1-t^2) - (1-x^2+x^2\log x^2)(1-t^2+t^2\log t^2),$$

treating t as a constant. We see at once that

$$f(0) = t^2 \log t^2 + 1 - t^2 - (1 - t^2 + t^2 \log t^2) = 0.$$

Differentiating yields

$$\begin{aligned} f'(x) = 2(x-t)\log\left(\frac{t-x}{1-tx}\right)^2 + (x-t)^2\left[\frac{-2}{t-x} + \frac{2t}{1-tx}\right] - 2x(1-t^2) - \\ -(-2x + 2x\log x^2 + 2x)(1-t^2+t^2\log t^2), \end{aligned}$$

whence

$$f'(0) = -2t\log t^2 + t^2\left[-\frac{2}{t} + 2t\right] = 2t[-\log t^2 + t^2 - 1].$$

It is easy to see that this is a positive number provided $0 < 1 < t$. Hence, by elementary calculus, f is increasing in a neighborhood of 0 and thus takes negative values in an interval $(-\varepsilon, 0)$, $\varepsilon > 0$. Therefore also U_R for R sufficiently small is susceptible of negative values. □

Remark 4. The above counter-example to the Boggio-Hadamard conjecture is not entirely new. We owe this information to Mark Ashbaugh and we are very grateful to him for this. For the sake of completeness we quote him *verbatim* [2]:

> The story of the Green's function for annular regions is somewhat more complicated than you may have been led to believe. First of all, the proof is indirect. It works off the fact that the first eigenfunction of the annular region is not of one sign (and is, in fact, doubly degenerate), if the ratio of the inner radius to the outer radius is sufficiently small. For such regions the Green's function could not possibly be of one sign since by Perron-Frobenius type arguments this would imply the constancy of the sign of the first eigenfunction. The argument showing that the first eigenfunction is not of one sign is due to Duffin and Shaffer, with later related papers by Coffman, Duffin, and/or Shaffer" (see [10], [6] [7]).

It is apparently a matter of taste which one of the two approaches is to be preferred. Let us remark that one virtue of our method is that it easily lends itself to a somewhat more precise statement: the existence of an "island" of negativity situated on the diametrically opposite side of the annulus to the point w.

Prof. Ashbaugh has also kindly directed our attention to the importance in this connection of the work of Gabor Szegö (see [19], Vol. 3). We take the liberty to quote him once more [3]:

> In the comments to Szegö's paper 'On Membranes and Plates' (paper 50-2, in the notation of the Collected Papers), Askey says "When the Green's function or some iterate is positive, the hypothesis Szego assumed is satisfied, as he remarked in 62-1". This is on page 194 of the Collected Papers, Vol. 3. The reference here to paper 62-1 is actually a misprint; the correct reference is to paper 52-1 (to be quoted from below, and which may perhaps be of greatest interest

to you for its discussion of Hadamard's conjecture). In paper 52-1, Szegö ends his paragraph discussing Hadamard's conjecture with, "Needless to say, the question of the first non-vanishing eigenfunction is not decided by these considerations. It would follow for instance from the positivity of any kernel arising from $\Gamma(p,q)$ by repeated iteration".

Finally, in paper 53-2 (On the Vibrations of a Clamped Plate), Szegö says, "According to a theorem of Jentzsch on integral equations, the positivity of the kernel G implies the lack of sign variations for the first characteristic function. This sufficient condition is of course very restrictive. Indeed, if we form the so-called iterated kernels, the characteristic functions remain all the same. Consequently the positivity of any iterated kernel implies just as well the lack of sign variations for the first characteristic function."

□

Remark 5. Let us likewise point out that the function U_0 has an interpretation as a Green's function for the punctured disc $\Omega_0\backslash\{0\}$. Namely, as the boundary of the latter consist of two components, a circle and a point, we can as boundary conditions take the usual Dirichlet conditions on $|z|=1$ and impose an additional condition(s) on the growth rate of the function (or of its normal derivative) at the origin. Note that this is something which is typical for higher order elliptic operators; for Laplace operator Δ this does not make sense. □

We end this section by a general result for the biharmonic equation in a punctured disk. Both the limit function U_0 and its normal derivative $\partial U_0/\partial N$ vanish on the unit circle $\mathbb{T}$, and U_0 moreover vanishes at the origin. Naïvely, one would like $\partial U_0/\partial N$ to vanish at the origin too. This is easily seen not to be possible, as follows from the following theorem.

Theorem 2. *Assume that $\Delta^2 u = 0$ in $\mathbb{D}\setminus\{0\}$, $u=\partial u/\partial N = 0$ on $\mathbb{T}$, and $u(0)=0$. Then*

$$u(z) = (Cz + D\overline{z})(1-|z|^2+\log|z|^2) \tag{9}$$

for some complex numbers C and D. In particular, if either $u(z)=o(|z|\log|z|^2)$ as $z\to 0$ or the radial derivative $\partial u/\partial r$ stays bounded near the origin, then $u\equiv 0$.

Proof. By Almansi's theorem [1] (see Theorem 1 of Appendix I), we have

$$u(z) = \sum_n a_n z^n + \sum_n b_n \overline{z}^n + A\log|z|^2 + \\ + \sum_n c_n|z|^2 z^n + \sum_n d_n |z|^2\overline{z}^n + B|z|^2\log|z|^2 + Cz\log|z|^2 + D\overline{z}\log|z|^2.$$

To avoid duplicity, we set $b_0=d_0=c_{-1}=d_{-1}=0$. The condition $u(0)=0$ implies that $A=0$ and

$$a_n e^{ni\theta} + b_n e^{-ni\theta} + c_{n-2}e^{(n-2)i\theta} + d_{n-2}e^{-(n-2)i\theta} = 0 \qquad \text{for all } n\le 0 \text{ and all } \theta,$$

or $a_n=b_n=c_{n-2}=d_{n-2}=0$ for all $n\le 0$. The conditions $u|_{\mathbb{T}}=0$ and $\frac{\partial u}{\partial N}|_{\mathbb{T}}=0$ then give

$$\sum_1^\infty a_n e^{ni\theta} + \sum_1^\infty b_n e^{-ni\theta} + \sum_0^\infty c_n e^{ni\theta} + \sum_1^\infty d_n e^{-ni\theta} = 0,$$

$$\sum_1^\infty n a_n e^{ni\theta} + \sum_1^\infty n b_n e^{-ni\theta} \sum_0^\infty (n+2)c_n e^{ni\theta} + \\ + \sum_1^\infty (n+2) d_n e^{-ni\theta} + 2B + 2Ce^{i\theta} + 2De^{-i\theta} = 0.$$

Comparing the coefficients at $e^{ni\theta}$, we see that $a_n=b_n=c_n=d_n=0$ for $n\ge 2$, $c_0=B=0$, and

$$a_1+c_1 = 0 = a_1 + 3c_1 + 2C,$$
$$b_1 + d_1 = 0 = b_1 + 3d_1 + 2D,$$

or $a_1=C$, $c_1=-C$, $b_1=D$, $d_1=-D$. This proves (9). The radial derivative then equals

$$\frac{\partial u}{\partial r} = (Ce^{i\theta} + De^{-i\theta})(3-3r^2+\log r^2)$$

which blows up at the origin unless $C = D = 0$. □

Corollary. *Suppose that* $\Delta^2 g = \delta_t$ *in* $\mathbb{D}\setminus\{0\}$, $g = \partial g/\partial N = 0$ *on* $\mathbb{T}$, $g(0) = 0$, *and* $\partial g/\partial r$ *stays bounded near the origin. Then* $g(z) = U_0(z,t)$.

Proof. It follows from (3) that U_0 is C^1 in a neighbourhood of the origin. Hence the function $u(z) = g(z) - U_0(z,t)$ satisfies the hypotheses of Theorem 2. □

7. Applications to the Boggio-Hadamard conjecture. In this section we apply our main result, viz. the explicit formula for $U = U_R$, to disprove the Boggio-Hadamard conjecture for the case of annuli.

Theorem 1. *For each* $R > 1$, *the Green function* $U = U_R$ *is not positive: there exist points* z *and* w *in* $\Omega = \Omega_R$ *such that* $U(z,w) < 0$.

Proof. Without loss of generality we can of course, as always, assume that the point w sits on the positive real axis, writing $w = t$ with $1 < t < R$. We observe that since both U and the normal derivative $\partial U/\partial N$ vanish on the boundary, it suffices to find a point $e^{i\theta}$ on the unit circle at which $\partial^2 U/\partial N^2$ is negative – $u(re^{i\theta})$ will then be negative for r close enough to 1. We have

$$\frac{\partial^2 U}{\partial N^2}\bigg|_{r=1} = \sum_n F_n(t)\, e^{ni\theta},$$

where

$$\begin{aligned} F_n &= n(n-1)A_n^* + (n+2)(n+1)B_n^* + (-n)(-n-1)C_n^* + \\ &\quad + (2-n)(1-n)D_n^*, \quad |n|>1, \\ F_1 &= 6B_1^* + 2C_1^* + 2D_1^*, \\ F_{-1} &= 2A_{-1}^* + 2B_{-1}^* + 6D_{-1}^* = F_1, \\ F_0 &= 2B_0^* - 2C_0^* + 6D_0^*. \end{aligned} \tag{1}$$

The relations $A_{-n}^* = C_n^*$ etc. imply that $F_n = F_{-n}$. Thus, we can write

$$\frac{\partial^2 U}{\partial N^2}\bigg|_{z=e^{i\theta}} = 2\,\mathrm{Re}\, f(e^{i\theta}),$$

where the function

$$f(z) =: \sum_{n=0}^{\infty} f_n z^n, \qquad f_0 = F_0/2, f_n = F_n \text{ for } n \geq 1,$$

is holomorphic in the unit disc $\mathbb{D}$. Of course, f depends on R and t. We need to show that for any R, there is always a t for which $\mathrm{Re}\, f(e^{i\theta}) < 0$ at some θ, i.e. *for which the image under* f *of* $\mathbb{D}$ *does not lie wholly in the* (*closed*) *right half-plane.* Equivalently, the function

$$g(z) = \frac{1 - f(z)}{1 + f(z)}, \qquad z \in \mathbb{D}, \tag{2}$$

should not map $\mathbb{D}$ wholly into the closed unit disc $\overline{\mathbb{D}}$.

Let us now recall the famous algorithm of I. Schur [S]. Suppose that g maps $\mathbb{D}$ into $\overline{\mathbb{D}}$. Then $\gamma_0 =: g(0)$ satisfies $|\gamma_0| \leq 1$. If $|\gamma_0| = 1$, $g \equiv \gamma_0$ identically by the maximum principle. If $|\gamma_0| < 1$, then the holomorphic function

$$g^\sharp(z) =: \frac{1}{z}\cdot\frac{g(z) - \gamma_0}{1 - \overline{\gamma}_0 g(z)}, \qquad z \in \mathbb{D},$$

also maps $\mathbb{D}$ into $\overline{\mathbb{D}}$, by the Schwarz lemma. This again means that $\gamma_1 =: g^\sharp(0)$ is of modulus at most one, and either $g^\sharp \equiv \gamma_1$ (when $|\gamma_1| = 1$) or the function

$$g^{\sharp\sharp}(z) =: \frac{1}{z}\cdot\frac{g^\sharp(z) - \gamma_1}{1 - \overline{\gamma}_1 g^\sharp(z)}, \qquad z \in \mathbb{D},$$

maps $\mathbb{D}$ holomorphically into $\overline{\mathbb{D}}$ (when $|\gamma_1| < 1$). Thus $\gamma_2 =: g^{\sharp\sharp}(0)$ is of modulus at most one, etc. The argument can plainly be iterated *ad infinitum*, but we shall not need that: we are going to show that for our function g given by (2) and for t close to 1, one has

$$|\gamma_0| < 1, \quad |\gamma_1| < 1, \quad \text{but} \quad |\gamma_2| > 1. \tag{3}$$

Hence g cannot map $\mathbb{D}$ into $\overline{\mathbb{D}}$, and f cannot map $\partial\mathbb{D}$ into the closed right half-plane, and we will be done.

It is easy to see that the Schur parameters γ_0, γ_1, and γ_2 can be expressed in terms of the Taylor coefficients of g: if $g(z) = g_0 + g_1 z + g_2 z^2 + \dots$, then

$$\begin{aligned} \gamma_0 &= g_0, \\ \gamma_1 &= \frac{g_1}{1-|\gamma_0|^2}, \\ \gamma_2 &= \frac{1}{1-|\gamma_1|^2}\left[\frac{g_2}{1-|\gamma_0|^2} + \gamma_1^2\overline{\gamma}_0\right]. \end{aligned} \tag{4}$$

The Taylor coefficients of our function g can in turn be expressed in terms of those of f:

$$\begin{aligned} g_0 &= \frac{1-f_0}{1+f_0}, \\ g_1 &= \frac{-2f_1}{(1+f_0)^2}, \\ g_2 &= \frac{2{f_1}^2 - 2f_2(1+f_0)}{(1+f_0)^3} \end{aligned} \tag{5}$$

where thus (in our case)

$$f_0 = F_0/2, \quad f_1 = F_1, \quad f_2 = F_2.$$

Substituting (5) into (4) gives – this calculation is valid also if f is a general function –

$$\begin{aligned} \gamma_0 &= \frac{1-f_0}{1+f_0}, \\ \gamma_1 &= \frac{\overline{1+f_0}}{1+f_0} \cdot \frac{-f_1}{2\operatorname{Re} f_0}, \\ \gamma_2 &= \frac{\overline{1+f_0}}{1+f_0} \cdot \frac{f_1^2 - 2f_2 \cdot \operatorname{Re} f_0}{4(\operatorname{Re} f_0)^2 - |f_1|^2}. \end{aligned} \tag{6}$$

Remark 1. Let us note that if we formally replace f by a multiple, say, μf where μ is any positive real number, then only the phase of the Schur parameters is changed (from index 1 on). This has a nice group theoretic interpretation. Indeed, one sees that the function g is replaced by $\psi \circ g$ where ψ is a suitable Moebius selfmap of $\mathbb{D}$, given by a unimodular quasi-unitary matrix, an element of the group SU(2). Let us introduce the notation φ_ζ where $\zeta \in \mathbb{D}$ to denote the Moebius selfmap of $\mathbb{D}$ defined by

$$\varphi_\zeta(z) = \frac{z-\zeta}{1-z\overline{\zeta}} \text{ corresponding to the matrix } \begin{pmatrix} 1 & -\zeta \\ -\overline{\zeta} & 1 \end{pmatrix} : (1-|\zeta|^2)^{-\frac{1}{2}}.$$

It is well-known that this map φ_ζ is characterized up to phase by the property of mapping ζ onto the origin 0 and, in view of this uniqueness, one has for any $\psi \in \mathrm{SU}(2)$ the formula $\varphi_{\psi(\zeta)} \circ \psi = k \circ \varphi_\zeta$ where k denotes a suitable rotation about 0. So it follows that the Schur transform $(\psi \circ g)^\sharp$ of $\psi \circ g$ is obtained from $g^\sharp$ by multiplication by a unimodular number. $\square$

Next, we seek expressions for the coefficients F_0, F_1, F_n $(n > 2)$ using the formulas (1)–(4) in Section 2 (for F_n) and analogous formulas in Section 3 (for $n = 0, 1$). From the formula (1), we find

$$\begin{aligned} F_0(t) &= \frac{1}{16\pi M_0(R)}\big[-4(\log R^2)^2 + 4(\log R^2)^2 t^2 + \\ &\quad + 4(1 - R^2 + \log R^2)\log t^2 + 4(1 - R^{-2} - \log R^2)\, t^2 \log t^2\big]; \\ F_1(t) &= \frac{1}{16\pi M_1(R)}\big[4(R^{-2} - R^2 + +(R^2 + R^{-2})\log R^2)\, t + \\ &\quad + 4(1 - R^{-2} - R^{-2}\log R^2)\, t^3 + \\ &\quad + 4(-1 + R^2 - R^2\log R^2)\, t^{-1} + 4(2 - R^2 - R^{-2})\, t \log t^2\big]; \\ F_n(t) &= \frac{1}{16\pi M_n(R)}\big[4(1 + n - nR^2 - R^{-2n})\, t^n + 4(n - 1 - nR^{-2} + R^{-2n})\, t^{n+2} + \\ &\quad + 4(1 - n + nR^2 - R^{2n})\, t^{-n} + 4(-1 - n + nR^{-2} + R^{2n}) t^{2-n}\big] \\ &\qquad \text{(if } |n| > 1). \end{aligned} \tag{7}$$

Here M_0, M_1 and M_n have the same meaning as in Sections 1 and 3:

$$\begin{aligned} M_0(R) &= (\log R^2)^2 - (R - R^{-1})^2, \\ M_1(R) &= (R^2 - R^{-2}) \log R^2 - 2(R - R^{-1})^2, \\ M_n(R) &= (R^n - R^{-n})^2 - n^2(R - R^{-1})^2. \end{aligned} \tag{8}$$

Note that $M_0 < 0$, while $M_1, M_2 > 0$. Observe also that if $n = 2$ the last line in (7) can also be written as

$$M_2(R) = (R - R^{-1})^4, \tag{9}$$

which observation will be used below.

From now on, we fix R and regard $F_n(t)$ solely as functions of t. Let us look more closely at the case when t is close to one.

Claim 1. *We have*

$$\begin{aligned} &F_n(1) = 0, \\ &F_n'(1) = \frac{1}{2\pi}, \\ &F_n''(1) = \frac{1}{2\pi} \begin{cases} 1 + 2\dfrac{(R^2 - R^{-2}) - 2\log R^2}{M_0(R)} & \text{if } n = 0, \\ 1 + 2\dfrac{(R^2 - R^{-2}) - (R^2 + R^{-2})\log R^2}{M_1(R)} & \text{if } n = 1, \\ 1 - 4\dfrac{(R^2 - R^{-2})(R - R^{-1})^2}{M_2(R)} & \text{if } n > 1. \end{cases} \end{aligned} \tag{10}$$

Proof of Claim 1. Again let us indicate the proof in the case $n = 0$. Put

$$F_0(t) = \frac{1}{16\pi M_0}\left[\alpha_0 + \beta_0 t^2 + \gamma_0 \log t^2 + \delta_0 t^2 \log t^2\right]$$

where the values of the coefficients α_0 etc. can be taken from the formula (7). Differentiating twice and putting $t = 1$ yields

$$\begin{aligned} F_0(1) &= \frac{1}{16\pi M_0}\left[\alpha_0 + \beta_0\right]; \\ F_0'(1) &= \frac{1}{16\pi M_0}\left[2\beta_0 + 2\gamma_0 + 2\delta_0\right]; \\ F_0''(1) &= \frac{1}{16\pi M_0}\left[2\beta_0 - 2\gamma_0 + 6\delta_0\right]. \end{aligned} \tag{11}$$

From the said formula we see at once that $\alpha_0 + \beta_0 = 0$ and likewise that $2\beta_0 + 2\gamma_0 + 2\delta_0 = 8M_0$, proving the two first lines in (10). Using the last identity we see that the last (third) line in (10) again can be rewritten as

$$F_0''(1) = \frac{1}{16\pi M_0}\left[8M_0 - 4\gamma_0 + 4\delta_0\right].$$

Using the values of β_0 and γ_0 the sought expression for $F_0''(1)$ follows readily. The proof in the cases $n = 1$ and $n > 1$ goes along similar lines. $\square$

In what follows only F_0, F_1 and F_2 will matter (and f_0, f_1 and f_2). It will be convenient to have a special notation for the second Taylor coefficients of these functions about the point $t = 1$, so we put

$$\begin{aligned} &F_0(t) = \tfrac{1}{2\pi}(h + ah^2 + O(h^3)) \quad \text{or} \quad f_0(t) = \tfrac{1}{2\pi}(\tfrac{1}{2}h + \tfrac{a}{2}h^2 + O(h^3)); \\ &F_1(t) = f_1(t) = \tfrac{1}{2\pi}(h + bh^2 + O(h^3)); \\ &F_2(t) = f_2(t) = \tfrac{1}{2\pi}(h + ch^2 + O(h^3)), \end{aligned}$$

where we have written $t = 1 + h$ and where the values of a, b and c can be readily inferred from formula (10). Using (6) above, we now see that

$$\gamma_0 = \frac{1 - \frac{1}{2\pi}(\frac{1}{2}h + O(h^2))}{1 + \frac{1}{2\pi}(\frac{1}{2}h + O(h^2))}$$

and

$$\gamma_1 = \frac{-(1 + bh + O(h^2))}{1 + ah + O(h^2)}; \qquad \gamma_2 = \frac{2b - c - a + O(h)}{2(a - b) + O(h)}.$$

Remark 2. Notice that in full agreement with Remark 1 the factor $\frac{1}{2\pi}$ has no influence here. Since all our quantities are real the phase factor too has disappeared. □

It follows from here that (3) *will follow if we can show that*

$$a > b \text{ and } 4b > 3a + c. \tag{12}$$

Let us first turn to the first inequality in (12). Using (10), we have

$$M_0M_1(a - b) = (R^2 - R^{-2} - 2\log R^2)M_1 - (R^2 - R^{-2} - (R^2 + R^{-2})\log R^2)M_0. \tag{13}$$

Substituting for M_0 and M_1 the expressions (8), we obtain (the proof is indicated in the next paragraph)

$$\begin{aligned} M_0M_1(a - b) = {} & (R^{-4} - R^4) + 2(R^2 - R^{-2}) - \\ & - 12\log R^2 + 6(R^2 + R^{-2})\log R^2 - 3(R^2 - R^{-2})(\log R^2)^2 + \\ & + (R^2 + R^{-2})(\log R^2)^3. \end{aligned} \tag{14}$$

Now make the substitution $R^2 = e^v$ (so $\log R^2 = v$). Then we can rewrite (14) in terms of hyperbolic sine and cosine

$$M_0M_1(a - b) = 2\cosh v \cdot v^3 - 6\sinh v \cdot v^2 + 12(\cosh v - 1)v - 2\sinh 2v + 4\sinh v. \tag{15}$$

Proof of (14) *and/or* (15). In order to obtain a streamlined proof of these formulae it will be convenient to introduce the *ad hoc* notation $S = 2\cosh\frac{v}{2} = R + R^{-1}$ (sum) and $D = 2\sinh\frac{v}{2} = R - R^{-1}$ (difference). (This will be used also below in connection with the proof of the second inequality (12).) Notice that $S^2 - D^2 = 4$, which is the well-known formula $\cosh^2 v - \sinh^2 v = 1$ in slight disguise. In this notation we have (see (8) and (9))

$$M_0 = v^2 - D^2; \quad M_1 = D(Sv - 2D); \quad M_2 = D^4. \tag{16}$$

In particular, the right hand side of (13) can now be written as

$$(SD - 2v)D(Sv - 2D) - (SD - (S^2 - 2)v)(v^2 - D^2),$$

which after expanding is

$$(S^2 - 2)v^3 - 3SDv^2 + 6D^2v - SD^3. \tag{17}$$

Note that this a cubic polynomial in v. Reinstating to the hyperbolic functions gives (15). □

If we now use the well-known Taylor expansions of sinh and cosh, we can expand the right hand side of (15) as

$$2\sum_{k=0}^{\infty}\frac{v^{2k}\cdot v^3}{(2k)!} - 6\sum_{k=0}^{\infty}\frac{v^{2k+1}\cdot v^2}{(2k+1)!} + 12\sum_{k=1}^{\infty}\frac{v^{2k}\cdot v}{(2k)!} - 2\sum_{k=0}^{\infty}\frac{(2v)^{2k+1}}{(2k+1)!} + 4\sum_{k=0}^{\infty}\frac{v^{2k+1}}{(2k+1)!}.$$

This sum can be rewritten as a single series:

$$M_0M_1(a - b) = -8\sum_{k=4}^{\infty}\frac{v^{2k+1}}{(2k+1)!}\cdot[2^{2k-1} - 2k^3 + 3k^2 - k - 2]. \tag{18}$$

Notice that the terms of index up to $k = 3$ drop out, in accordance with what can be inferred already from (15). In order to establish the left inequality in (12) it suffices thus, as $M_0 < 0$ and $M_1 > 0$, to show that the expression within brackets [] in the general term of the series in (18) is positive. This is an elementary number theoretic fact.

Claim 2. *We have* $2^{2k-1} \geq 2k^3 - 3k^2 + k + 2$ *for all positive integers with equality if and only if* $k = 1, 2, 3$.

Proof. That equality holds for $k = 1, 2, 3$ is trivial to check (and, by the way, we know it already). So factoring the polynomial part we see that it suffices to show that $2^{2k-1} > k(k - 1)(2k - 1)$. (Note that

$\frac{1}{2}k(k-1)$ is an integer!) We now just have to use the two more elementary inequalities $2^k > k(k-1)$ and $2^{k-1} > 2k-1$, valid for all positive integers k and $k > 3$ respectively, and multiply them together. For instance the former can be proved for $k > 4$ using the binomial expansion $2^k = (1+1)^k = 1+\binom{k}{1}+\binom{k}{2}+\cdots+1$ (and, for $k = 4$, by inspection). The proof of the latter is similar. □

The second inequality (12) can be proved along similar lines. By (10) we have

$$(4b-3a-c)M_0M_1M_2 = 2(R^2-R^{-2})(R-R^{-1})^2M_0M_1 - 3(R^2-R^{-2}-2\log R^2)M_1M_2+ \\ + 4(R^2-R^{-2}-(R^2+R^{-2})\log R^2)M_0M_2.$$

In order to expand this expression we use the above method. In terms of the quantities v, S and D the right hand side can be written as

$$2SD\cdot D^2M_0M_1 - 3(SD-2v)M_1M_2 + 4(SD-(S^2-2)v)M_0M_2$$

or again, using the formulae for the M's (16), as

$$2SD\cdot D^2(v^2-D^2)\cdot D(Sv-2D) - 3(SD-2v)\cdot D(Sv-2D)\cdot D^4 + 4(SD-(S^2-2)v)\cdot(v^2-D^2)\cdot D^4.$$

Expanding this yields the expression

$$D^4\{-2D^2v^3+6SDv^2-D^2(D^2+24)v+6SD^3\}.$$

It is easily seen from this that this quantity behaves as $O(v^7)$ at the origin. In particular, the fact that we have isolated a factor D^4 is conspicuous, and is of great service to us: as D^4 is always positive, we need to worry only about the expression within the curly brackets.

Now, remembering the meaning of S and D, we reintroduce the hyperbolic functions. We find that the said expression inside the curly brackets equals to

$$-4(\cosh v-1)v^3 + 12\sinh v\cdot v^2 - (2\cosh 2v + 40\cosh v - 42)v + (12\sinh 2v - 24\sinh v).$$

Following the same strategy as in the previous case, we use Taylor expansions for sinh and cosh to rewrite this as

$$-4\sum_{k=1}^{\infty}\frac{v^{2k}\cdot v^3}{(2k)!} + 12\sum_{k=0}^{\infty}\frac{v^{2k+1}\cdot v^2}{(2k+1)!} - 2\sum_{k=0}^{\infty}\frac{(2v)^{2k}\cdot v}{(2k)!} - 40\sum_{k=0}^{\infty}\frac{v^{2k}\cdot v}{(2k)!} + 42v+ \\ + 12\sum_{k=0}^{\infty}\frac{(2v)^{2k+1}}{(2k+1)!} - 24\sum_{k=0}^{\infty}\frac{v^{2k+1}}{(2k+1)!}$$

and then combine everything into a single series:

$$\frac{(4b-3a-c)M_0M_1M_2}{D^4} = -16\sum_{k=5}^{\infty}\frac{v^{2k+1}}{(2k+1)!}[(2k-11)2^{2k-3}+2k^3-3k^2+3k+4].$$

Again, the terms up to $k = 4$ have cancelled out. As before, in order to establish the second inequality in (12), it suffices to show that the expression inside the last square brackets is always positive, for any $k \geq 5$. This time the situation turns out to be even more elementary: since $2k^3 - 3k^2 = k^2(2k-3) > 0$, it follows that the said expression is positive for $k \geq 6$, while a direct calculation reveals that it is positive for $k = 5$ too (and, in fact, vanishes for k between 2 and 4). This completes the proof of nonpositivity of the biharmonic Green's function. □

Remark 3. In view of the above proof one is tempted to make the conjecture that the Green's function of a clamped plate takes negative values whenever the underlying planar domain is of higher connectivity. At least we are nor aware of any counter-example to such a hypothesis. □

Remark 4. Most of the calculations above (as well as in much of the rest of this paper) were checked by the W. R. I. program `Mathematica`. □

Remark 5. The method above is not constructive in the sense that it does not tell at which point on the unit circle the second normal derivative is negative. Taking guidance from the limiting case $R \to 0$ (or $R \to +\infty$) in Section 6, one can expect negative values when z lies "opposite" t, *i.e.* when $z/t < 0$. It would certainly be desirable to have some numerical evidence in this matter. □

8. Discussion of some transcendental functions. In this section, which may be read independently of the rest of the paper, we study in some detail the function $\mathbf{X}$ as well as the related functions $\mathbf{Y}$, $\mathbf{Y}_+$, $\mathbf{Y}_-$, $\mathbf{Z}_+$, $\mathbf{Z}_-$ introduced in Section 4, and used there and in Section 5.

We shall establish a result on the meromorphic continuation of $\mathbf{X}(\lambda)$ already mentioned there (see Remark 1 of Section 4). In order to formulate it we introduce for each integer $k = 0, 1, 2, \dots$ the following function for $|\lambda| < 1$ given by the expansion

$$H_k(\lambda) = \sum_{n=2}^{\infty} n^{2k} \lambda^n. \tag{1}$$

It is clear that $H_k(\lambda)$ is a rational function with a pole of order $2k+1$ at $\lambda = 1$. Indeed, we have

$$H_0(\lambda) = \frac{1}{1-\lambda} - 1 - \lambda; \qquad H_k(\lambda) = \left(\lambda \frac{d}{d\lambda}\right)^{2k} \left(\frac{1}{1-\lambda} - \lambda\right) \quad (k = 1, 2, \dots). \tag{2}$$

Remark 1. Consider quite generally

$$G_j(\lambda) = \mathfrak{E}^j \left(\frac{1}{1-\lambda}\right) \quad (j = 1, 2, \dots)$$

where we have introduced the notation (Euler operator)

$$\mathfrak{E} = \lambda \frac{d}{d\lambda}.$$

Then one has

$$G_j(\lambda) = \sum_{p=1}^{j} \frac{b_{jp} \lambda^p}{(1-\lambda)^{p+1}} \quad (j = 1, 2, \dots),$$

where the coefficients b are in a simple way related to Stirling's numbers of the second kind, $b_{jp} = p! \mathcal{S}_j^{(p)}$. $\square$

Below we use Pochhammer's notation:

$$(a)_N = a(a+1)(a+2)\dots(a+N-1).$$

Theorem 1. *Consider the function* $\mathbf{X}(\lambda)$ *defined for* $R^{-2} < |\lambda| < R^2$ *by the series development*

$$\mathbf{X}(\lambda) = \sum_{|n|>1} \frac{\lambda^n}{M_n}.$$

Here, as before (see (8) in Section 1) $M_n = (R^n - R^{-n})^2 - n^2(R - R^{-1})^2$. *Then* $\mathbf{X}(\lambda)$ *can be continued to a meromorphic function in* $\mathbb{C}\backslash\{0\}$ *with poles at the points* $R^{\pm 2}, R^{\pm 4}, R^{\pm 6}, \dots$ *of order* $1, 3, 5, \dots$. *Indeed, one has the* <u>*partial fraction*</u> *expansion*

$$\begin{aligned}\mathbf{X}(\lambda) = &\sum_{N=0}^{\infty} \sum_{k=0}^{N} (R - R^{-1})^{2k} \frac{(2k+2)_{N-k}}{(N-k)!} H_k\left(\frac{\lambda}{R^{2(N+1)}}\right) + \\ &+ \sum_{N=0}^{\infty} \sum_{k=0}^{N} (R - R^{-1})^{2k} \frac{(2k+2)_{N-k}}{(N-k)!} H_k\left(\frac{1}{R^{2(N+1)}\lambda}\right),\end{aligned} \tag{3}$$

where H_k *is given by* (2). *We have furthermore*

$$\mathbf{X}\left(\frac{1}{\lambda}\right) = \mathbf{X}(\lambda). \quad \square \tag{4}$$

Proof: It suffices to consider separately each of the series

$$\mathbf{X}^+(\lambda) = \sum_{n=2}^{\infty} \frac{\lambda^n}{M_n} \text{ and } \mathbf{X}^-(\lambda) = \sum_{-\infty}^{n=-2} \frac{\lambda^n}{M_n}.$$

As $M_{-n} = M_n$, we clearly have

$$\mathbf{X}_-(\lambda) = \mathbf{X}^+\left(\frac{1}{\lambda}\right). \tag{5}$$

Therefore it suffices to consider $\mathbf{X}^+$ only. With no loss of generality we may assume that $R > 1$.

Let us write for $n > 1$

$$\begin{aligned}\frac{1}{M_n} &= \frac{1}{(R^n - R^{-n})^2} \frac{1}{1 - n^2\left(\dfrac{R - R^{-1}}{R^n - R^{-n}}\right)^2} = \\ &= \sum_{k=0}^{\infty} \frac{n^{2k}(R - R^{-1})^{2k}}{(R^n - R^{-n})^{2(k+1)}}\end{aligned} \tag{6}$$

As

$$0 < \frac{R - R^{-1}}{R^n - R^{-n}} < 1 \text{ for } n > 1 \text{ (or } n < -1)$$

it is clear that this series is convergent. (We have assumed that $R > 1$.) We note that all the series encountered in this context are absolutely convergent so that all manipulations involved are justified. Thus, interchanging the order of summation we obtain from(6)

$$\mathbf{X}_+(\lambda) = \sum_{k=0}^{\infty}\sum_{n=2}^{\infty} \frac{n^{2k}(R - R^{-1})^{2k}}{(R^n - R^{-n})^{2(k+1)}}\lambda^n.$$

Next we write

$$\frac{1}{(R^n - R^{-n})^{2(k+1)}} = \frac{1}{R^{2n(k+1)}} \frac{1}{(1 - R^{-2n})^{2(k+1)}} = \sum_{\nu=0}^{\infty} \frac{(2k+2)_\nu}{\nu!} \frac{1}{R^{2n(k+\nu+1)}},$$

where the series converges as $n > 1$ and $R > 1$. This gives

$$\begin{aligned}\mathbf{X}^+(\lambda) &= \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{(2k+2)_\nu}{\nu!}(R - R^{-1})^{2k} \sum_{n=2}^{\infty} n^{2k}\left(\frac{\lambda}{R^{2(k+\nu+1)}}\right)^n = \\ &= \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{(2k+2)_\nu}{\nu!}(R - R^{-1})^{2k} H_k\left(\frac{\lambda}{R^{2(k+\nu+1)}}\right).\end{aligned}$$

Putting $N = k + \nu$ and rearranging terms gives

$$\mathbf{X}^+(\lambda) = \sum_{N=0}^{\infty}\sum_{k=0}^{N} \frac{(2k+2)_{N-k}}{(N-k)!}(R - R^{-1})^{2k} H_k\left(\frac{\lambda}{R^{2(N+1)}}\right).$$

As this is the analogue of (3) for the function $\mathbf{X}^+$, this proves also formula (3) itself for the function $\mathbf{X}$ itself. □

Remark 2. An alternative approach can be based on first writing

$$\frac{1}{M_n} = \frac{1}{2n(R - R^{-1})} \cdot \left(\frac{1}{R^n - R^{-n} - n(R - R^{-1})} - \frac{1}{R^n - R^{-n} + n(R - R^{-1})}\right).$$

This suggests to consider the series

$$\sum_{|n|>1} \frac{1}{n} \frac{1}{R^n - R^{-n} \pm n(R - R^{-1})}. \tag{7}$$

They may be treated in an analogous manner. Note, however, that owing to the factor $\frac{1}{n}$ we obtain <u>multivalued</u> functions with logarithmic singularities. Again this can be evaded by instead taking

$$\sum_{|n|>1} \frac{1}{R^n - R^{-n} \pm n(R - R^{-1})}, \tag{8}$$

without this unpleasant factor. Cf. also the analogous computation connected with the series

$$\sum_{|n|>0} \frac{1}{n} \frac{\lambda^n}{R^n - R^{-n}},$$

in Appendix V. It is not clear if it is possible to obtain product representations of the type encountered there in the present situation. We note also that series involving a divisor of the type $R^n - R^{-n} + n(R - R^{-1})$ occur in [12], formula (36), p. 512, as we alluded to already in Remark 3 in the Introduction. □

Remark 3. It is easy to see now that the function $\mathbf{X}$ satisfies the following functional equation:

$$\mathbf{X}(R^2\lambda) + \mathbf{X}(R^{-2}\lambda) - 2\mathbf{X}(\lambda) - (R - R^{-1})^2\mathfrak{E}^2\mathbf{X}(\lambda) = -(\lambda + \lambda^{-1} + 1).$$

Thus our theory is connected with the difference-differential operator:

$$f(\lambda) \mapsto f(R^2\lambda) + f(R^{-2}\lambda) - 2f(\lambda) - (R - R^{-1})^2\mathfrak{E}^2 f(\lambda),$$

which may be viewed as a natural generalization of the operator

$$f(\lambda) \mapsto f(R\lambda) - f(R^{-1}\lambda),$$

which is basic for quantum- or q-function theory.[4] It is however not clear at this stage how far this analogy can be carried. In case of the series (7) and (8) we encounter the somewhat simpler operator

$$f(\lambda) \mapsto f(R\lambda) - f(R^{-1}\lambda) - (R - R^{-1})\mathfrak{E} f(\lambda). \quad \square$$

Now we turn our attention to the remaining functions $\mathbf{Y}$, $\mathbf{Y}_+$, $\mathbf{Y}_-$, $\mathbf{Z}_+$, $\mathbf{Z}_-$. A glance at how they were defined (see (1) in Section 4) reveals that they arise essentially by integration from the function $\mathbf{X}$. Due to the residues at the points $\lambda = R^{\pm 2(N+1)}$ $(N = 0, 1, 2, \dots)$ they display however a logarithmic singularity a these points.

We shall limit ourselves to writing down a number of functional relations connecting them. In order to indicate the dependence on R we shall write $\mathbf{X}(\lambda) = \mathbf{X}(\lambda, R)$ etc. Then it is easy to see that one has the following symmetries:

$$\begin{aligned} \mathbf{X}(\lambda, R) &= \mathbf{X}\big(\lambda, \frac{1}{R}\big) = \mathbf{X}\big(\frac{1}{\lambda}, R\big) = \mathbf{X}\big(\frac{1}{\lambda}, \frac{1}{R}\big); \\ \mathbf{Y}(\lambda, R) &= -\mathbf{Y}\big(\frac{1}{\lambda}, \frac{1}{R}\big); \\ \mathbf{Y}_-(\lambda, R) &= -\mathbf{Y}_+\big(\frac{1}{\lambda}, \frac{1}{R}\big); \\ \mathbf{Z}_-(\lambda, R) &= \mathbf{Z}_+\big(\frac{1}{\lambda}, \frac{1}{R}\big). \end{aligned} \tag{9}$$

Moreover, one can prove that

$$\mathbf{Z}_+(\lambda) = -(R^2 - 1)\mathbf{X}(\lambda) + \mathbf{Y}_+(\frac{1}{\lambda}) - \mathbf{Y}(\frac{1}{\lambda}) \tag{10}$$

and, similarly,

$$\mathbf{Z}_-(\lambda) = -(R^{-2} - 1)\mathbf{X}(\lambda) - \mathbf{Y}_-(\frac{1}{\lambda}) + \mathbf{Y}(\frac{1}{\lambda}). \tag{11}$$

Thus one can in principle dispense with the two functions $\mathbf{Z}_\pm$. Finally, one has

$$\begin{aligned} \mathfrak{E}\mathbf{Y}(\lambda) &= \mathbf{X}\big(\frac{\lambda}{R^2}\big) - \mathbf{X}(\lambda); \\ \lambda\mathfrak{E}\left(\frac{\mathbf{Y}_+(\lambda)}{\lambda}\right) &= \mathbf{X}\big(\frac{\lambda}{R^2}\big) - R^2\mathbf{X}(\lambda); \\ \frac{1}{\lambda}\mathfrak{E}\big(\mathbf{Y}_-(\lambda) \cdot \lambda\big) &= \mathbf{X}\big(\frac{\lambda}{R^2}\big) - R^{-2}\mathbf{X}(\lambda). \end{aligned} \tag{12}$$

We see that the functions $\mathbf{Y}$, $\mathbf{Y}_\pm$ arise from $\mathbf{X}$ via a process of integration. Due to this we see also that these are not meromorphic (single valued) functions but are multivalued with logarithmic singularities at the points $R^{\pm 2}, R^{\pm 4}, R^{\pm 6}, \dots$.

[4] As indicated in a previous footnote (in the Introduction), one usually puts $q = R^2$ and then the operator considered is $f(\lambda) \mapsto f(q\lambda) - f(\lambda)$.

Appendices

Appendix I. Biharmonic continuation and related issues. In this appendix we have collected some salient facts about biharmonic functions in general. Much of this is probably known but perhaps not so readily accessible[5].

We begin by putting into play the *Euler operator*

$$\mathfrak{E} = r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}.$$

We recall also the operational formula

$$\Delta\varphi = \varphi\Delta + 2\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial x} + 2\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial y} + \Delta\varphi, \tag{1}$$

where φ stands for any function φ (acting as a multiplication operator). With the aid of (1) it is easy to establish the following lemmata.

Lemma 1. $\Delta\mathfrak{E} = \mathfrak{E}\Delta + 2\Delta$.

Proof. Using (1) we find

$$\Delta(x\frac{\partial}{\partial x}) = x\Delta\frac{\partial}{\partial x} + 2\frac{\partial^2}{\partial x^2} = x\frac{\partial}{\partial x}\Delta + 2\frac{\partial^2}{\partial x^2};$$
$$\Delta(y\frac{\partial}{\partial y}) = y\Delta\frac{\partial}{\partial y} + 2\frac{\partial^2}{\partial y^2} = y\frac{\partial}{\partial y}\Delta + 2\frac{\partial^2}{\partial y^2},$$

where we used also in the last link the fact that Δ commutes with the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Adding up gives the desired result. □

Corollary. *If u is harmonic so is the function $\mathfrak{E}u$.* □

Lemma 2. $\Delta r^2 = r^2\Delta + 4\mathfrak{E} + 4$.

Proof. The proof of this lemma is even simpler. Indeed, the result follows directly from (1) applied to the function $\varphi = r^2 = x^2 + y^2$, noting that in this case

$$\frac{\partial\varphi}{\partial x} = 2x,\ \frac{\partial\varphi}{\partial y} = 2y,\ \Delta\varphi = 4. \quad \square$$

Next we provide a self-contained of Almansi's theorem [1] (already referred to in the Introduction).[6]

Theorem 1 (Almansi [1])**.** *Let u be biharmonic in the annulus $\Omega = \{1 < |z| < R\}$. Then u can be written in the form*

$$u = h_0 + r^2 h_1 + B\bar{z}\log r^2 + Dz\log r^2 \tag{2}$$

where h_0 and h_1 are harmonic functions in Ω and B and D are complex numbers. The numbers B and D are uniquely determined but not the functions h_0 and h_1: any other representation of the type (2) *is obtained by replacing h_0 and h_1 by harmonic functions h_0' and h_1' of the form*

$$h_0' = h_0 + Az + C\bar{z}, \quad h_1' = h_1 - A\frac{1}{\bar{z}} - C\frac{1}{z}$$

where A and C are arbitrary complex numbers. Conversely, every such function u is biharmonic.

Remark 1. If u is real valued we can take h_0 and h_1 real in (2), and $D = \bar{B}$. □

[5] We refer, in particular, to the monograph [4]; although the bulk of this book is devoted to polyharmonic functions of infinite order, Chap. 1 lists many references of interest from our point of view.

[6] Although Almansi's theorem is often quoted in the literature, not many people seem to have read his memoir, as it is seldom mentioned that this author actually considered not only the case of the disk but the much harder case of the annulus (and several other things too). In our case we read [1], regretfully, only at a rather late stage, and likewise we did with [12], another classic in this area.

Proof. We begin by establishing the converse. Assume thus that the function u admits a representation of the type (2) with h_0 and h_1 and certain constants B and D. Let us set

$$s = B\bar{z}\log r^2 + Dz\log r^2; \tag{3}$$

we think of s as the "singular" part of u. We have

$$\Delta s = B\frac{4}{z} + D\frac{4}{\bar{z}}, \tag{4}$$

implying that s is biharmonic. Using Lemma 2 we then obtain

$$\Delta u = \Delta h_0 + r^2\Delta h_1 + 4\mathfrak{E}h_1 + 4h_1 + \Delta s = 4\mathfrak{E}h_1 + 4h_1 + B\frac{4}{z} + D\frac{4}{\bar{z}}, \tag{5}$$

where we used $\Delta h_0 = \Delta h_1 = 0$ in the last step, along with (4). Using now the Corollary of Lemma 1 we find that u is indeed biharmonic.

It is clear that the sum $h_0 + r^2 h_1$ remains unaffected if we replace h_0 and h_1 by $h_0' = h_0 + Az + C\bar{z}$ and $h_1' = h_1 - A\frac{1}{\bar{z}} - C\frac{1}{z}$ with arbitrary constants A and C.

In order to prove the converse we prove first that, given a biharmonic function u, there exist a harmonic function h_1 and suitable constants B and D such that

$$\mathfrak{E}h_1 + h_1 = \tfrac{1}{4}\Delta\left(u - B\bar{z}\log r^2 + Dz\log r^2\right) = \tfrac{1}{4}\Delta(u - s).$$

Writing $v = \frac{1}{4}\Delta(u - s)$ we see that we are faced with an equation of the type

$$\frac{d(rh)}{dr} = v \tag{6}$$

with v harmonic in Ω. Being harmonic the function v admits an expansion of the type

$$v = a + b\log r + \sum{}'(a_n z^n + b_n \bar{z}^{-n}), \tag{7}$$

where the single stroke $'$ indicates that we take the summation over all integers $n \neq 0$. We have the following general result, the proof of which will be given below.

Lemma 3. *The differential equation* (7) *has a solution h which is a harmonic function if and only if $a_{-1} = b_{-1} = 0$. The solution is unique up to a term $A\frac{1}{\bar{z}} + C\frac{1}{z}$.*

This lemma clearly is applicable in our special case, viz. $v = \frac{1}{4}\Delta(u-s)$, because we can adjust the constants B and D occurring in the expression of s (see (3)) in such a way that the hypothesis $a_{-1} = b_{-1} = 0$ is fulfilled.

Finally, we put $h_0 = u - r^2 h_1 - s$. By the computation in the first half of the proof we see that h_0 is harmonic. This gives the representation (2). As h_0 is unique up to a linear function of the form $Az + C\bar{z}$, this completes the proof. □

Proof of Lemma 3. If $a_{-1} = b_{-1} = 0$, direct integration of (7) gives

$$h = a + b(\log r - 1) + \sum{}'\tfrac{1}{n+1}(a_n z^n + b_n \bar{z}^n) + r^{-1}f(\theta),$$

$f(\theta)$ being an arbitrary function of θ. As $\Delta v = (f + f'')/r^3$, h is harmonic if and only if $f + f'' = 0$, or $f = r(\frac{A}{\bar{z}} + \frac{C}{z})$. On the other hand, if $a_k = b_k = 0$ for all $k \neq -1$, we find in the same way that the only harmonic solution is

$$h = a_{-1}\frac{\log z}{z} + b_{-1}\frac{\log \bar{z}}{\bar{z}} + \frac{A}{\bar{z}} + \frac{C}{z},$$

which is not single-valued in the annulus unless $a_{-1} = b_{-1} = 0$. □

As a simple application of Theorem 1 we have the following result.

Corollary. *Let u be biharmonic in a neighborhood of the circle $|z| = 1$. Then the function $u^\spadesuit$ defined by*

$$u^\spadesuit(z) = |z|^2 u\left(\frac{1}{\bar{z}}\right),$$

likewise defined in a neighborhood of the $|z| = 1$, but perhaps a different one, is biharmonic too.

Proof. By rescaling Theorem 1 is applicable to any annulus, so we may assume that u admits a representation of the type (2). Then we obtain

$$u^{\spadesuit} = h_0^{\spadesuit} + r^2 h_1^{\spadesuit} - Dz \log r^2 - B\bar{z} \log r^2,$$

where $h_0^{\spadesuit}$ and $h_1^{\spadesuit}$ are given by

$$h_0^{\spadesuit} = h_1\left(\frac{1}{\bar{z}}\right); \quad h_1^{\spadesuit} = h_0\left(\frac{1}{\bar{z}}\right);$$

by Kelvin's theorem (reflection) they are again harmonic functions. The conclusion follows now by the reverse part of Theorem 1. □

Remark 2. The condition that u be defined in a neighborhood of a circle is superfluous. Indeed, the conclusion of the corollary remains in force for biharmonic functions defined in an arbitrary open set not containing the origin. This again is but a very special case of a general theorem due to Bojarski [5] concerning conformal or Moebius invariance of the iterated operators Δ^p $(p = 1, 2, \dots)$, not only in two but in any number of dimensions. □

We now come to the question of biharmonic continuation. What we have in mind is an extension of Kelvin's reflection for harmonic functions to the biharmonic case. So let u be biharmonic in the annulus Ω and assume that it satisfies Dirichlet boundary conditions on the inner circle:

$$u = \frac{\partial u}{\partial N} = 0 \text{ for } |z| = 1, \tag{8}$$

where N denotes the normal. Note that the second equality in (8) can also be written as $\mathfrak{E}u = 0$.

Theorem 2. *The above function u has a biharmonic continuation $\tilde{u}$ to the annulus $\tilde{\Omega} = \{R^{-1} < |z| < 1\}$.*

Proof. Let us begin by rewriting the representation formula (2) in Theorem 1 in a form more suitable for the present purpose. Instead of s we use as singular part the function S,

$$S(z) = B\bar{z}(\log r^2 + 1 - r^2) + Dz(\log r^2 + 1 - r^2). \tag{9}$$

Clearly S is biharmonic too and it vanishes for $r = |z| = 1$. To see that also the normal derivative vanishes we compute $\mathfrak{E}S$. We find

$$\mathfrak{E}u = B\bar{z}(\log r^2 + 1 - r^2) + B\bar{z}r(\frac{2}{r} - 2r) + Dz(\log r^2 + 1 - r^2) + Dzr(\frac{2}{r} - 2r).$$

¿From this formula it is clear that $\mathfrak{E}S = 0$ for $r = |z| = 1$.

Next we modify h_0 and h_1 replacing them by the harmonic functions $h_0^{\dagger}$, $h_1^{\dagger}$,

$$h_0^{\dagger} = h_0 + h_1, \qquad h_1^{\dagger} = h_1 + B\bar{z} + Dz.$$

So in place of (2) we have now the formula

$$u = h_0^{\dagger} + (r^2 - 1)h_1^{\dagger} + S. \tag{10}$$

We have not yet utilized that u satisfies the boundary condition. From (10) we see directly that $h_0^{\dagger} = 0$ if $|z| = 1$. Differentiating yields

$$\mathfrak{E}u = \mathfrak{E}h_0^{\dagger} + 2rh_1^{\dagger} + (r^2 - 1)\mathfrak{E}h_1^{\dagger} + \mathfrak{E}S.$$

Hence $\mathfrak{E}h_0^{\dagger} + 2h_1^{\dagger} = 0$ if $|z| = 1$. This suggests that we change our notation once more, putting

$$H = h_0^{\dagger}, \quad K = h_1^{\dagger} + \tfrac{1}{2}\mathfrak{E}h_0^{\dagger}.$$

Then (10) can be stated as

$$u = H + (r^2 - 1)(-\tfrac{1}{2}\mathfrak{E}H + K) + S. \tag{11}$$

We summarize: *In this formula H and K are harmonic in Ω and both vanish if $|z| = 1$, and S, given by* (9), *is biharmonic and satisfies the boundary condition* (8).

Now it is easy to perform the continuation. The functions H and K are continued to harmonic functions $\tilde{H}$ and $\tilde{K}$ in $\tilde{\Omega}$ by reflection,

$$\tilde{H}(z) = -H(\frac{1}{\bar{z}}),\ \tilde{K}(z) = -K(\frac{1}{\bar{z}}) \text{ for } z \in \tilde{\Omega}.$$

Finally, we set

$$\tilde{u} = \tilde{H} + (r^2 - 1)(-\tfrac{1}{2}\mathfrak{E}\tilde{H} + \tilde{K}) + S.$$

It is clear that $\tilde{u}$ is biharmonic in $\tilde{\Omega}$ and extends u (as both functions satisfy the Dirichlet boundary condition (8) on $|z| = 1$). □

Let us also have a look at the more general situation when u has isolated singularities in Ω. To fix the ideas let us assume that u is biharmonic but for a single pole of strength one at the point t of the positive halfaxis ($1 < t < R$), in other words, that u satisfies the equation $\Delta^2 u = \delta_t$, where δ_t is the Dirac delta function at the point t; it is still assumed that the boundary condition (8) is fulfilled.

Theorem 3. *Now u has a continuation $\tilde{u}$ which is biharmonic in $\tilde{\Omega}$ but for a triple pole at the point $\frac{1}{t}$.*

Proof. Let V be Green's function for the exterior disc $\{1 < |z| \leq \infty\}$ with pole at t. This function will be discussed in Appendix IV; in particular, it will be seen there that it has the same type of singularities. So it suffices to apply Theorem 2 to the difference $u - V$. □

Appendix II. On Hedenmalm's weighted bi-Laplace operator. Now we extend our results for Δ^2 to the case of the more general operator $\Delta|z|^{-2\alpha}\Delta$ (where $\alpha > -1$) considered by Hedenmalm [13]. It will be convenient to put $\beta = \alpha + 1$, so that $\beta > 0$ while the case $\beta = 1$ corresponds to the initial case of the operator Δ^2. Let us refer to null solutions of this operator as *β-biharmonic functions.*

It is easy to extend Almansi's theorem, even for the annulus (cf. Appendix I), the case of the disc having been treated by Hedenmalm himself ([13], Lemma 3.1): in place of $r^2 h_1$ we must write $r^{2\beta} h_1$ and, if β is an integer ($\beta = 1, 2, \dots$), we must modify the "singular" part taking $z^\beta \log r^2$ and $\bar{z}^\beta \log r^2$ instead of $z \log r^2$ and $\bar{z} \log r^2$; if β is not an integer there will be no singular part.

Similarly, one can show that β-biharmonic functions are invariant under the transformation

$$u(z) \mapsto |z|^{2\beta} u\left(\frac{1}{\bar{z}}\right);$$

of course, we cannot expect Moebius invariance unless $\beta = 1$.

Now we indicate the computations of the Fourier coefficients of the β-biharmonic Green's functions for the annulus $\Omega = \{1 < |z| < R\}$. In fact, a pleasant surprise lies ahead, as it turns out that the result now becomes much more symmetric.[7] The general framework set up in Section 1 is applicable with the multipliers x given by (cf. (1) in Section 1)

$$x_1 = n, \quad x_2 = 2\beta + n, \quad x_3 = -n \quad x_4 = 2\beta - n.$$

In this case the sums of the multipliers are determined by the matrices (cf. Example in Section 1)

$$(x_i - x_k) = \begin{pmatrix} 0 & -2\beta & 2n & -2\beta + 2n \\ 2\beta & 0 & 2\beta + 2n & 2n \\ -2n & -2\beta - 2n & 0 & -2\beta \\ 2\beta - 2n & -2n & 2\beta & 0 \end{pmatrix}$$

and

$$(x_i + x_k) = \begin{pmatrix} \bullet & 2\beta + 2n & 0 & 2\beta \\ 2\beta + 2n & \bullet & 2\beta & 4\beta \\ 0 & 2\beta & \bullet & 2\beta - 2n \\ 2\beta & 4\beta & 2\beta - 2n & \bullet \end{pmatrix}$$

respectively. This gives in the first place the determinant $\Lambda = 4R^{2\beta} M_{n\beta}$ with (cf. (8) in Section 1)

$$M_{n\beta} = M_{n\beta}(R) = \beta^2 (R^n - R^{-n})^2 - n^2 (R^\beta - R^{-\beta})^2.$$

[7]This is another instance of an often observed fact that, in mathematics, complicated things sometimes become much more transparent when looked upon from a sufficiently general angle.

We observe right away that this expression is skew-symmetric in n and β; furthermore, it is an even function in each of these variables. For the Fourier coefficients of the Green's function we find now e.g. that

$$(1)\qquad \begin{aligned} A_n^* = \frac{1}{16\pi M_{n\beta}} \Bigg[& \frac{1}{\beta n(n-\beta)}\{\beta^2 R^{2n} - n^2 R^{-2\beta} + (n^2-\beta^2)\}t^{2\beta-n} - \\ & - \frac{R^{2\beta}-1}{\beta} t^{-n} + \frac{R^{-2n}-1}{-n} t^{2\beta+n} + \frac{R^{-2n}-R^{2\beta}}{n-\beta} t^n \Bigg] \end{aligned}$$

and

$$\begin{aligned} B_n^* = \frac{1}{16\pi M_{n\beta}} \Bigg[& -\frac{1}{\beta n(n+\beta)}\{\beta^2 R^{2n} - n^2 R^{2\beta} + (n^2-\beta^2)\}t^{-n} - \\ & + \frac{R^{-2\beta}-1}{\beta} t^{2\beta-n} + \frac{R^{-2n}-R^{-2\beta}}{n+\beta} t^{2\beta+n} - \frac{R^{-2n}-1}{n} t^n \Bigg]. \end{aligned}$$

So far we have not investigated the corresponding series.

Remark 1. As a possible higher order generalization of the Hedenmalm operator considered above one may conceive the operator

$$\Delta|z|^{2\beta_1}\Delta|z|^{2\beta_2}\Delta\ldots\Delta|z|^{2\beta_{m-1}}\Delta$$

of order $2m$, where the β's are given numbers > 0. A basis of "holomorphic" solutions of the corresponding homogeneous partial differential equation (in a circular region) is given by the functions

$$z^n, |z|^{2\gamma_1}z^n, |z|^{2\gamma_2}z^n, \ldots, |z|^{2\gamma_{m-1}}z^n \qquad (n \in \mathbb{Z}),$$

where we have written $\gamma_1 = \beta_1, \gamma_2 = \beta_1 + \beta_2, \ldots, \gamma_{m-1} = \beta_1 + \beta_2 + \ldots \beta_{m-1}$; it is understood that if any on the numbers γ is of the form $\pm n$ these expressions have to be conveniently modified by introducing logarithms. It seems that the special case $\beta_1 = \cdots = \beta_{m-1} = \beta$ is the most productive one. In particular, we expect that the above symmetry of the Fourier coefficients of the Green's function recurs once more.

Appendix III. The case of a strip. The strip enters in a dual way. On the one hand, by Moebius invariance we could have considered in principle, instead of the annulus, more generally domains bounded by any two circles. So as a limiting case we have the case of two tangent circles. Performing a suitable Moebius transformation we can, in view of Bojarski's theorem [5], always put ourselves in the situation of a strip, say, the standard strip $\{0 < \operatorname{Re} z < 1\}$. Again the Green's function U for Δ^2 can be found using Fourier methods. Only instead of Fourier series one encounters now Fourier integrals. We defer the detailed discussion to the end of this appendix.

Remark 1. (An even more general case.) What is common between these two cases? Well, both the annulus and the strip admit a one parameter group of Moebius transformations. So one can ask in what happens if we have a general domain with the said property. (For a similar point of view in a different context, see [17].) In particular, we have in mind the case of a domain bounded by two circular arcs making non-zero angles with each other – a lunula. We have not investigated this situation so far. □

On the other hand, the strip arises via uniformization. It is clear that the universal covering space of the annulus (in the sense of topology) is the strip. In order to get a suitable uniformizing parameter we recall that we have written for the generic point $z = re^{i\theta}$ where r and θ are the usual polar coordinates, with $r > 0$ and θ being counted modulo 2π. This suggests to set $r = e^{\sigma}$ at the same time dropping the restriction on θ. Let us write $s = \sigma + i\theta$. Then we obtain a strip of width $\Lambda = \log R$ in the s-plane and lying over the annulus, while the operator Δ is replaced by

$$(1)\qquad e^{-2\sigma}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{\partial}{\partial\sigma} + \frac{\partial^2}{\partial\theta^2}\right],$$

its square Δ^2 by

$$(2)\qquad e^{-2\sigma}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{\partial}{\partial\sigma} + \frac{\partial^2}{\partial\sigma^2}\right]e^{-2\sigma}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{\partial}{\partial\sigma} + \frac{\partial^2}{\partial\theta^2}\right].$$

A basis of null-solutions for the operator in (2) is given by the quadruple family of functions

$$e^{\pm\xi\sigma + i\xi\theta}; \quad e^{(2\pm\xi)\sigma + i\xi\theta},$$

corresponding to the functions $R^{\pm n}e^{in\theta}$, $R^{2\pm n}e^{in\theta}$ down on the annulus. It follows that we obtain general solutions u given by the Fourier integral:

$$u(\sigma) = \int_{-\infty}^{\infty} [a(\xi)e^{\xi\sigma} + b(\xi)e^{(2+\xi)\sigma} + c(\xi)e^{-\xi\sigma} + d(\xi)e^{(2-\xi)\sigma}]e^{i\sigma\xi}\, d\xi$$

with essentially arbitrary functions $a(\xi)$ etc. When taking account of boundary conditions we obtain linear equations for these coefficients which are analogous to those encountered in Section 1 in the case of the annulus. Therefore we can, in principle, carry over our previous results to obtain a formula for the corresponding Green's function U^{str}, say, also in this case. There is, of course, also the additional difficulty, to be taken care of, that the functions in (3) are not linearly independent if $\xi = 0, \pm 1$. It is not clear that the resulting integrals are any easier to handle than the previous infinite series.

Let us note that if we know the Green's function in the case of the strip, U^{str}, then the one for the annulus U^{ann} (previously written just U) can be obtained simply by averaging:

$$U^{\mathrm{ann}}(z) = U^{\mathrm{per}}(s) =: \sum_{m\in\mathbf{Z}} U^{\mathrm{str}}(s + 2\pi m) \qquad (z = e^s = e^{\sigma+i\theta}).$$

This is, formally speaking, a consequence of Poisson's summation formula. By virtue of the results in Section 6, we immediately get as a corollary the following result due to Duffin [9] (also referred to in [12], p. 510).

Corollary. *The Green function U^{str} for the operator* (2) *on the strip is not of one sign.* □

The above can be given yet another twist, namely, we can pass to the limit $\Lambda \to 0$. Indeed, making the substitution $s \mapsto \Lambda s$, that is, $\sigma \mapsto \Lambda\sigma$, $\theta \mapsto \Lambda\theta$, we get the normalized strip $0 < \sigma < 1$ and, instead of (1), the partial differential operator

$$\Lambda^{-2}e^{-2\Lambda\sigma}\left[\frac{\partial^2}{\partial\sigma^2} + \Lambda\frac{\partial}{\partial\sigma} + \frac{\partial^2}{\partial\theta^2}\right].$$

So in the limit (ignoring the factor Λ^{-2}) we get back the operator Δ^2, thus the case with which we set out in the beginning of this appendix. It is conceivable that the (renormalized) periodic Green's function

$$\frac{1}{\Lambda^2}U^{\mathrm{per}}(\frac{s}{\Lambda})$$

gives when $\Lambda \to 0$ the corresponding Green's function for Δ^2 in the normalized strip.

We say now a few words about the latter. Let us change notation writing $z = x + iy$ in place of $s = \sigma + i\theta$. Thus we seek our function U subject to the conditions

$$\begin{cases} \Delta^2 u = \delta_t & \text{for } 0 < x < 1; \\ U = \dfrac{\partial U}{\partial x} = 0 & \text{for } x = 0, 1, \end{cases}$$

where δ_t is the Dirac function placed at the point t on the unit interval, $0 < t < 1$. Then U must admit Fourier expansions of the form (cf. the Introduction in the case of the annulus)

$$U = \int_{-\infty}^{\infty}\left[A^*(\xi)e^{x\xi} + B^*(\xi)xe^{x\xi} + C^*(\xi)e^{-x\xi} + D^*(\xi)(-x)e^{-x\xi}\right]e^{i\xi y}\, d\xi \quad \text{if } x < t;$$
$$U = \int_{-\infty}^{\infty}\left[A^{**}(\xi)e^{x\xi} + B^{**}(\xi)xe^{x\xi} + C^{**}(\xi)e^{-x\xi} + D^{**}(\xi)(-x)e^{-x\xi}\right]e^{i\xi y}\, d\xi \quad \text{if } x > t,$$

where the coefficients A^* etc. and A^{**} etc. are determined from a certain system of linear equations.

Appendix IV. The singularities of Green's function. Let us return to a point left open in Appendix I. By inspection we see from the formula (already encountered in Section 6)

$$V(z,w) = \frac{1}{16\pi}\left(|z-w|^2\log\left|\frac{z-w}{1-z\bar{w}}\right|^2 + (1-|z|^2)(1-|w|^2)\right). \tag{1}$$

that the Green function V for the exterior disc $\{1 < |z| \le \infty\}$ admits a continuation to $\{|z| < 1\}$ which is biharmonic except at the point $\frac{1}{\bar{w}}$: The same expression can used for the entire punctured plane $\mathbb{C}\backslash\{\frac{1}{\bar{w}}\}$ so we are going to keep the notation V. It remains the investigate the nature of the singularity at the point $\frac{1}{\bar{w}}$.

Theorem 1. *The point $\frac{1}{\bar{w}}$ is a pole of order three. More precisely, we have the equation*

(2)
$$\Delta^2 V = \delta_z - \delta_{\frac{1}{\bar{w}}} - (1-|w|^2)\left(\frac{1}{\bar{w}}\frac{\partial}{\partial z} + \frac{1}{w}\frac{\partial}{\partial \bar{z}}\right)\delta_{\frac{1}{\bar{w}}} - \frac{1}{4}\left|\frac{1}{\bar{w}} - w\right|^2 \Delta\delta_{\frac{1}{\bar{w}}}.$$

Proof. For convenience let us take w on the positive halfaxis, writing $w = t$ (with $1 < t < \infty$) so that $\frac{1}{\bar{w}} = \frac{1}{t}$. Then V comes as the difference of two terms:

$$V = \tfrac{1}{16\pi}|z-t|^2 \log|z-t|^2 - \tfrac{1}{16\pi}|z-t|^2 \log|1-zt|^2.$$

(We can ignore the term $(1-|z|^2)(1-t^2)$ which is biharmonic in the whole plane.) As we are interested in what happens near $z = \frac{1}{t}$, we may concentrate on the second term, call it $\frac{1}{16\pi}H$. (The first term is biharmonic off the point $z = t$.) We have

$$\begin{aligned} H = |z-t|^2 \log|1-zt|^2 &= |z-t|^2 \log t^2 - |z-t|^2 \log|\frac{1}{t} - z|^2 = \\ &= |z-t|^2 \log t^2 - |z-\frac{1}{t}|^2 \log|z-\frac{1}{t}|^2 - \\ &- 2(\frac{1}{t} - t)\operatorname{Re}(z-\frac{1}{t})\log|z-\frac{1}{t}|^2 - (\frac{1}{t}-t)^2 \log|z-\frac{1}{t}|^2. \end{aligned}$$

The first term clearly is biharmonic and so can be disregarded. Shifting the origin to the point $\frac{1}{t}$ let us look at the three functions

$$H_1 = |z|^2 \log|z|^2,\ H_2 = x\log|z|^2 \text{ and } H_1 = \log|z|^2$$

and apply the operator Δ^2 to each of them.

<u>*Case i*</u>. Clearly $\Delta^2 H_1 = 16\pi\,\delta$.

<u>*Case ii*</u>. Using formula (1) in Appendix I we obtain

(3)
$$\Delta H_2 = 2\frac{\partial \log|z|^2}{\partial x} + x\Delta\log|z|^2.$$

Now recall that $\frac{1}{2\pi}\log r^2$ is the fundamental solution of the operator Δ, that is $\Delta\log|z|^2 = 4\pi\delta$. It follows that the second term in (2) vanishes: $x\Delta\log|z|^2 = 4\pi x\delta = 0$. Hence applying Δ to (3) we find

$$\Delta^2 H_2 = 2\frac{\partial\Delta\log|z|^2}{\partial x} = 8\pi\frac{\partial\delta}{\partial x}.$$

<u>*Case iii*</u>. By the same device $\Delta^2 H_3 = 4\pi\Delta\delta$.

Collecting all this information (cases i-iii), shifting the origin back to $z = 0$ and dividing by 16π, we obtain

$$\Delta^2\left(\tfrac{1}{16\pi}H\right) = -\delta_{1/t} + (t-1/t)\frac{\partial\delta_{1/t}}{\partial x} - \tfrac{1}{4}(t-1/t)^2\Delta\delta_{1/t}.$$

This establishes (2). □

Appendix V. Green's function for Laplace operator in the annulus. This appendix is written mainly for the benefit of the reader so that he or she can quickly see how the corresponding computations go in the case of Δ. (Recall that, as was related in the Introduction, the treatment in [8] is a different one.[8])

We seek to determine the Green's function U subject to the conditions

$$\begin{cases} \Delta U = \delta_t & \text{in } \{1 < |z| < R\}; \\ U = 0 & \text{for } |z| = 1 \text{ and } |z| = R, \end{cases}$$

where δ_t is the Dirac function placed at the point t on the real axis, $1 < t < R$. We have the Fourier expansion

$$U = A_0^* + B_0^*\log r + \sum_{|n|>1}(A_n^* r^n + B_n^* r^{-n})e^{in\theta} \qquad \text{in } \{1 < |z| < t\};$$
$$U = A_0^{**} + B_0^{**}\log r + \sum_{|n|>1}(A_n^{**} r^n + B_n^{**} r^{-n})e^{in\theta} \qquad \text{in } \{t < |z| < R\}.$$

[8]Yet another proof is indicated in [12], p. 497-498.

For $n \neq 0$ we have the system of linear equations

$$\begin{cases} A_n^* + B_n^* = 0; \\ R^n A_n^{**} + R^{-n} B_n^{**} = 0; \\ \Delta A_n t^n + \Delta B_n t^{-n} = 0; \\ \Delta A_n n t^n + \Delta B_n (-n) t^{-n} = \dfrac{1}{2\pi}, \end{cases}$$

where we have written $\Delta A_n = A_n^{**} - A_n^*$, $\Delta B_n = B_n^{**} - B_n^*$. It is readily seen that the solution is given by

$$A_n^* = -B_n^* = \frac{1}{4\pi}\frac{1}{n}\frac{-R^n t^{-n} + R^{-n} t^n}{R^n - R^{-n}};$$
$$A_n^{**} = -R^{-2n} B_n^{**} = \frac{1}{4\pi}\frac{1}{n}\frac{R^n t^{-n} - R^{-n} t^n}{R^n - R^{-n}}.$$

The case $n = 0$ is settled in a similar way and one finds

$$A_0^* = 0;\ B_0^* = \frac{1}{2\pi}\frac{\log t - \log R}{\log R};\ A_0^{**} = -\log R;\ B_0^{**} = \frac{1}{2\pi}\log t.$$

Inserting this into the series and making some formal manipulations one obtains the expression of the Green's function U in terms of (the logarithm of) Jacobi theta functions given in [8], p. 335-337.

Let us indicate the main idea of the "manipulations" just referred to at the hand of the model series (cf. the proof of Theorem 1 in Section 6)

$$\sum_{n=1}^{\infty} \frac{1}{n}\frac{\lambda^n}{R^n - R^{-n}}.$$

Let us write (assuming that $R > 1$)

$$\frac{1}{R^n - R^{-n}} = R^{-n}\frac{1}{1 - R^{-2n}} = \sum_{\nu=0}^{\infty} R^{-(2\nu+1)n}.$$

Hence, inserting and interchanging the order of the n and the ν summation, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n}\frac{\lambda^n}{R^n - R^{-n}} = \sum_{\nu=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}R^{-(2\nu+1)n}\lambda^n = -\sum_{\nu=0}^{\infty}\log(1 - \lambda R^{-(2\nu+1)}) =$$
$$= \log\prod_{\nu=0}^{\infty}(1 - \lambda R^{-(2\nu+1)})^{-1}.$$

One sees that the product is a product of the type that usually enters in the expansion of a theta function.

Appendix VI. On an interpolation problem. In the basic computation in Section 1 we encountered the problem of inverting the matrix

$$(1)\qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ R^{x_1} & R^{x_2} & R^{x_3} & R^{x_4} \\ x_1 R^{x_1} & x_2 R^{x_2} & x_3 R^{x_3} & x_4 R^{x_4} \end{pmatrix};$$

in particular, we evaluated the corresponding determinant. Indeed, (1) is a special case of more general matrices, for instance, matrices formed in an analogous way with arbitrary many exponents x. Matrices of the last type arise also in connection with the following interpolation problem: to reconstruct a function f of the type $f(x) = P(x) + e^{\mu x}Q(x)$, where P and Q are polynomials of fixed degrees, say, m and n, given its values at $m+n$ points $x_1, x_, \ldots, x_{m+n}$. This leads to a $m+n$ times $m+n$ matrix whose typical column has entries $1, x_i, \ldots, x_i^{m-1}, e^{\mu x_i}, x_i e^{\mu x_i}, \ldots, x_i^{n-1}e^{\mu x_i}$ $(i = 1, \ldots, m+n)$. Clearly, if $m = n = 2$ writing $R = e^{\mu}$ we are in the case of (1). Even more general matrices arise if we allow more general exponential-polynomials; for instance, $f(x) = P(x) + e^{\mu x}Q(x) + e^{\nu x}R(x)$ would be the next case in order of complexity. Finally, we remark that the matrices (or determinants) referred to here may be viewed as natural generalizations of Vandermonde matrices (or determinants); this corresponds to the case of interpolation of ordinary polynomials (Lagrange's interpolation formula etc.). Generalizing our previous terminology (see Remark 1 in Section 1) we should perhaps even speak here of Almansi matrices (and determinants).

Example 1. The main situation considered throughout this paper concerns the case $x_1 = n$, $x_2 = 2+n$, $x_3 = n$, $x_4 = 2-n$ (see (2) in Section 1). As a generalization let us take

$$x_1 = n, \quad x_2 = 2+n, \quad x_3 = 4+n, \quad x_4 = -n, \quad x_5 = 2-n, \quad x_6 = 4-n,$$

which amounts to passing to the cube Δ^3 of the Laplacean. Let us also again write $e^\mu = R$. Using `Mathematica` we found that the corresponding determinant (a 6×6 determinant) is, except for trivial factors, given by

$$\begin{aligned} M_n(R) = &-4\,R^{3\,n} + 4R^{-3\,n} + \\ &+ \left(n^2 - 2\,n^3 + n^4\right) R^{4+n} + \left(-n^2 - 2\,n^3 - n^4\right) R^{4-n} + \\ &+ \left(8\,n^2 + 4\,n^3 - 4\,n^4\right) R^{2+n} + \left(-8\,n^2 + 4\,n^3 + 4\,n^4\right) R^{2-n} + \\ &+ \left(12 - 18\,n^2 + 6\,n^4\right) R^n + (-12 + 18\,n^2 - 6\,n^4) R^{-n} + \\ &+ \left(8\,n^2 - 4\,n^3 - 4\,n^4\right) R^{-2+n} + \left(-8\,n^2 - 4\,n^3 + 4\,n^4\right) R^{-2-n} + \\ &+ \left(n^2 + 2\,n^3 + n^4\right) R^{-4+n} + \left(-n^2 + 2\,n^3 - n^4\right) R^{-4-n} \end{aligned}$$

and, moreover, that this expression has a factorization of the form

$$M_n(R) = R^{-3n} p_n(R) q_n(R)$$

where p_n and q_n are cubic polynomials in R^n. This should be compared to the factorizations (corresponding to Δ^p)

$$\begin{aligned} R^n - R^{-n} &= R^{-n}(R^n + 1)(R^n - 1) \qquad \text{— the case } p = 1; \\ (R^n - R^{-n})^2 - n^2(R^2 - R^{-2})^2 &= \left((R^n - R^{-n}) + n(R^2 - R^{-2})\right) \cdot \\ &\quad \cdot \left((R^n - R^{-n}) - n(R^2 - R^{-2})\right) \quad \text{— the case } p = 2, \end{aligned}$$

the present case being the case $p = 3$. Continuing we tried with the case $p = 4$ (the 8×8 case). However, in this situation we (or rather `Mathematica`) failed to detect a corresponding factorization.

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Abstract

In this paper we find an expression for Green's function for the operator Δ^2 in a planar circular annulus with Dirichlet boundary conditions (clamped elastic plate). We likewise determine the corresponding Poisson type kernels and the harmonic Bergman kernel. These results come in terms of certain new transcendental functions which in a natural way generalize the Weierstrass zeta function. They are analogous to the results of R. Courant - D. Hilbert (Methoden der Mathematischen Physik I (3. Aufl.), Springer-Verlag, Berlin - Heidelberg - New York, 1968, p. 335-337) and H. Villat (Rend. Circ. Mat. Palermo 33 (1912), 134-175) respectively. As an application we show that, regardless of the size of the ratio of the radii of the bounding circles, the Green's function always assumes negative values, which constitutes another rather striking counter-example to the well-known Boggio-Hadamard conjecture.

Classification: 35C10; 35J40

Keywords: biharmonic function; annulus; Green's function; Almansi theorem; Boggio-Hadamard conjecture

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