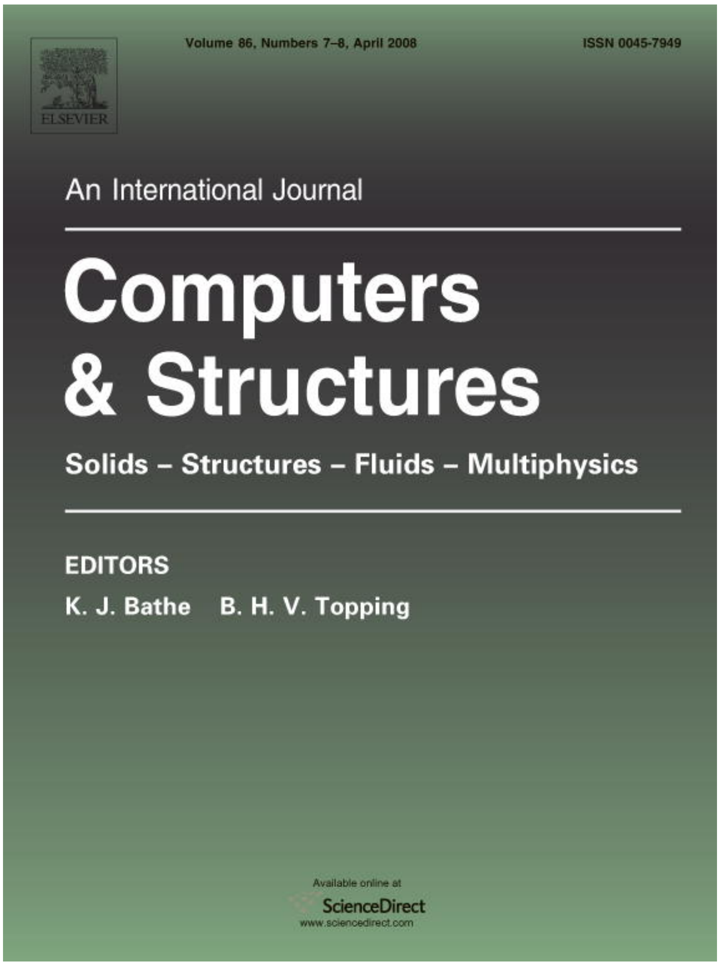
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A displacement solution for transverse shear loading of beams using the boundary element method

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Abstract

In this paper the boundary element method is employed to develop a displacement solution for the general transverse shear loading problem of prismatic beams of arbitrary simply or multiply connected cross section. The analysis of the beam is accomplished with respect to a coordinate system that has its origin at the centroid of the cross section, while its axes are not necessarily the principal ones. The transverse shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. A boundary value problem is formulated with respect to a warping function and solved employing a pure BEM approach requiring only a boundary discretization. The evaluation of the transverse shear stresses at any interior point is accomplished by direct differentiation of this function, while the coordinates of the shear center are obtained from this function using only boundary integration. The shear deformation coefficients are obtained from the solution of two boundary value problems with respect to warping functions appropriately arising from the aforementioned one, using again only boundary integration. Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The accuracy of both the thin tube theory and the engineering beam theory is examined through examples of practical interest.

Keywords: Shear; Transverse shear stresses; Shear center; Shear deformation coefficients; Beam; Principal shear axes; Boundary element method

1. Introduction

The problem of a prismatic beam subjected in shear torsionless loading has been widely studied from both the analytical and numerical point of view. Theoretical discussions concerning flexural shear stresses [1], or the problem of the center of shear [1,2] and text books giving detailed representations of these topics [3,4] are mentioned among the extended analytical studies. In all these studies which present a stress function formulation either the employed stress function is split into a primary part independent of the beam material describing the beam equilibrium and a secondary one dependent on the Poisson's ratio satisfying compatibility equations or the governing differential equation is split into two parts representing shear and torsion problems. Moreover, these studies are limited in the analysis based on the principal cross section system of axes.

Numerical methods have also been used for the analysis of the aforementioned problem. Among these methods the majority of researchers have employed the finite element method. Mason and Herrmann [5] based on assumptions for the displacement field and exploiting the principle of minimum potential energy developed triangular finite elements for a beam of arbitrary cross section and isotropic material subjected to bending. This method using triangular or quadrilateral finite elements has also been used for beams with orthotropic material [6] or anisotropic material [7]. Later, a finite element solution for the evaluation of the

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shear stresses [8,9] and the shear deformation coefficients [10] was developed formulating all basic equations to an arbitrary coordinate system, using isoparametric element functions and introducing a stress function which fulfils the equilibrium equations.

Moreover, boundary integral methods seem to be an alternative powerful tool for the solution of the aforementioned problem, having in mind that finite element methods require the whole cross section to be discretized into area elements and are also limited with respect to the shape (distortion) of the elements. BEM solutions require only boundary discretization, while a small number of boundary elements are required to achieve high accuracy. The boundary element procedure was first employed by Sauer [11] for the shear stresses calculation based on Weber analysis [1] and neglecting Poisson's ratio. BEM was also used for the calculation of the shear center location in an arbitrary cross section by Chou [12] and for the presentation of a solution to the general flexure problem in an isotropic only simply connected arbitrary cross section beam by Friedman and Kosmatka [13]. In this research effort the analysis is accomplished with respect only to the principal bending axes of the cross section restricting in this way its generality. Finally, Sapountzakis and Mokos in [14] and Mokos and Sapountzakis in [15] presented a stress function solution employing the BEM for the general transverse shear loading problem of homogeneous and composite prismatic beams of arbitrary cross section, respectively.

In this paper the boundary element method is employed to develop a displacement solution for the general transverse shear loading problem in prismatic beams of arbitrary simply or multiply connected cross section. The formulation of the problem follows the displacement field adopted in the FEM solutions presented in [8,9,16]. The shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. A boundary value problem is formulated with respect to a warping function and solved employing a pure BEM approach requiring only a boundary discretization. The evaluation of the transverse shear stresses at any interior point is accomplished by direct differentiation of this function, while the coordinates of the shear center are obtained from this function using only boundary integration. The shear deformation coefficients are obtained from the solution of two boundary value problems with respect to warping functions appropriately arising from the aforementioned one, using again only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

(i) The proposed displacement solution constitutes the first step to the solution of the nonuniform shear problem avoiding the use of stress functions.
(ii) All basic equations are formulated with respect to an arbitrary coordinate system, which is not restricted to the principal axes one.
(iii) The shear deformation coefficients are evaluated using an energy approach [17] instead of Timoshenko's [3] and Cowper's [18] definitions, for which several authors [19,20] have pointed out that one obtains unsatisfactory results or definitions given by other researchers, for which these factors take negative values.
(iv) The present formulation is also applicable to multiple connected domains without fulfillment of further constraints.
(v) The developed procedure retains the advantages of a BEM solution over a pure domain discretization method.

Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The accuracy of both the thin tube theory and the engineering beam theory is examined through examples of practical interest.

2. Statement of the problem

Consider a prismatic beam of length L with a cross section of arbitrary shape, occupying the two dimensional multiply connected region Ω of the y, z plane bounded by the $K+1$ curves Γ_1, $\Gamma_2, \ldots, \Gamma_K$, Γ_{K+1}, as shown in

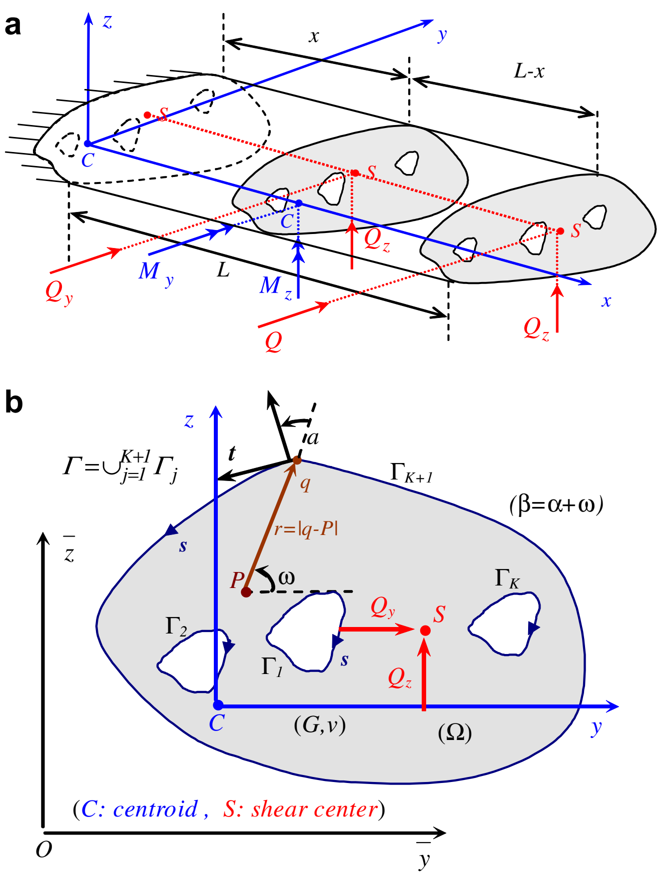


Fig. 1. Prismatic beam subjected to torsionless bending (a) with a cross-section of arbitrary shape occupying the two dimensional region Ω (b).

Fig. 1. These boundary curves are piecewise smooth, i.e. they may have a finite number of corners. The material of the beam, with shear modulus G and Poisson's ratio μ is assumed homogeneous, isotropic and linearly elastic. Without loss of generality, it may be assumed that the beam end with centroid at point C is fixed, while the x-axis of the coordinate system is the line joining the centroids of the cross sections.

When the beam is subjected to torsionless bending arising from a concentrated load Q having as Q_y, Q_z its components along y and z axes, respectively, applied at the shear center S of its free end cross section, the displacement components in the x, y and z directions are approximated as

$$u = -\frac{\partial w(x)}{\partial x} \cdot z - \frac{\partial v(x)}{\partial x} y + \varphi_c(y,z),$$
$$v = v(x), \quad w = w(x) \qquad (1a,b,c)$$

where $\varphi_c(y, z)$ is the warping function due to shear with respect to the centroid C of the cross section. From the above definition, it follows that this function is a parameter of the cross section assuming it independent of its x coordinate. However, in a more refined model the influence of this coordinate may also be considered.

Employing the strain – displacement equations of the three-dimensional elasticity the following nonzero strain components can be easily obtained

$$\varepsilon_x = -\frac{\partial^2 w}{\partial x^2} z - \frac{\partial^2 v}{\partial x^2} y, \quad \varepsilon_{xy} = \frac{1}{2}\frac{\partial \varphi_c}{\partial y},$$
$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial \varphi_c}{\partial z} \qquad (2a,b,c)$$

while the resulting from three-dimensional elasticity stress components in the region Ω are given as

$$\sigma_x = -\frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left(\frac{\partial^2 w}{\partial x^2} z + \frac{\partial^2 v}{\partial x^2} y\right), \qquad (3a)$$

$$\tau_{xy} = G\frac{\partial \varphi_c}{\partial y}, \quad \tau_{xz} = G\frac{\partial \varphi_c}{\partial z} \qquad (3b,c)$$

after setting equal to zero the rest of them σ_y, σ_z and τ_{yz}, according to the beam theory. Applying the stress components (3) in the first elasticity equation of equilibrium neglecting the body forces

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (4)$$

we obtain the following relation

$$\frac{\partial^2 \varphi_c}{\partial y^2} + \frac{\partial^2 \varphi_c}{\partial z^2} = \frac{(1-\mu)}{(1+\mu)(1-2\mu)}\frac{E}{G}\left(\frac{\partial^3 w}{\partial x^3} z + \frac{\partial^3 v}{\partial x^3} y\right) \qquad (5)$$

while the last two elasticity equations of equilibrium are identically satisfied. Substituting Eq. (3a) into the well known relations

$$M_y = \int_\Omega \sigma_x z \cdot d\Omega, \quad M_z = -\int_\Omega \sigma_x y \cdot d\Omega \qquad (6a,b)$$

we obtain the expressions for the bending moments as

$$M_y = -\frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left(\frac{\partial^2 w}{\partial x^2} \cdot I_y + \frac{\partial^2 v}{\partial x^2} \cdot I_{yz}\right) \qquad (7a)$$

$$M_z = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left(\frac{\partial^2 v}{\partial x^2} \cdot I_z + \frac{\partial^2 w}{\partial x^2} \cdot I_{yz}\right) \qquad (7b)$$

where

$$I_y = \int_\Omega z^2 d\Omega, \quad I_z = \int_\Omega y^2 d\Omega, \quad I_{yz} = \int_\Omega yz\, d\Omega \qquad (8a,b,c)$$

are the moments of inertia of the cross section with respect to y and z axes, respectively and its product of inertia. Differentiating these relations with respect to x we come to the expressions for the shear forces as

$$Q_z = \frac{\partial M_y}{\partial x} = -\frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left(\frac{\partial^3 w}{\partial x^3} \cdot I_y + \frac{\partial^3 v}{\partial x^3} \cdot I_{yz}\right) \qquad (9a)$$

$$Q_y = -\frac{\partial M_z}{\partial x} = -\frac{(1-\mu)E}{(1+\mu)(1-2\mu)}\left(\frac{\partial^3 v}{\partial x^3} \cdot I_z + \frac{\partial^3 w}{\partial x^3} \cdot I_{yz}\right) \qquad (9b)$$

Eq. (5), after eliminating the third derivatives of the deflections w''', v''' with respect to x employing Eqs. (9a,b), can be written as

$$\nabla^2 \varphi_c(y,z) = \frac{\partial^2 \varphi_c}{\partial y^2} + \frac{\partial^2 \varphi_c}{\partial z^2} = -\frac{[(Q_z I_z - Q_y I_{yz})z + (Q_y I_y - Q_z I_{yz})y]}{G(I_y I_z - I_{yz}^2)} \qquad (10)$$

The boundary condition of the aforementioned warping function with respect to the cross section's centroid C is derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ vanishes on the free prismatic surface of the beam, that is

$$\tau_{xn} = \tau_{xy} n_y + \tau_{xz} n_z = 0 \qquad (11)$$

where $n_y = \cos\beta$, $n_z = \sin\beta$ (with $\beta = \hat{y},\boldsymbol{n}$ as shown in Fig. 1b) are the direction cosines of the normal vector $\boldsymbol{n}$ to the boundary Γ. Substituting Eq. (3b,c) in Eq. (11), the Neumann type boundary condition for the warping function can be written as

$$\frac{\partial \varphi_c}{\partial n} = 0, \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \qquad (12)$$

where $\partial/\partial n \equiv n_y \partial/\partial y + n_z \partial/\partial z$ denotes the directional derivative normal to the boundary Γ. It is worth here noting that the evaluated warping function φ_c due to the solution of the Neumann problem contains an integration constant c (parallel displacement of the cross section along the beam axis), which can be obtained from the request that

$$\int_\Omega \bar{\varphi}_c d\Omega = 0 \qquad (13)$$

as

$$c = -\frac{1}{A}\int_\Omega \varphi_c d\Omega \qquad (14)$$

and the main warping function $\bar{\varphi}_c$ is given as

$$\bar{\varphi}_c = \varphi_c + c \tag{15}$$

Obviously, after the evaluation of this constant the displacement component of Eq. (1a)–(c) should be modified appropriately.

Finally, Eq. (3a), after eliminating the second derivatives of the deflections w'', v'' with respect to x employing the integrated Eqs. (9a,b), can be written as

$$\sigma_x = -\frac{[(Q_z I_z - Q_y I_{yz})z + (Q_y I_y - Q_z I_{yz})y](L-x)}{(I_y I_z - I_{yz}^2)} \tag{16}$$

Having in mind that the shear center S is defined as the point of the cross section at which the torsional moment arising from the transverse shear stress distribution vanishes, the coordinates $\{y_S, z_S\}$ of this point with respect to the system of axes with origin the cross section centroid can be derived from the condition

$$y_S Q_z - z_S Q_y = \int_\Omega (\tau_{xz} y - \tau_{xy} z) d\Omega \tag{17}$$

For $Q_y = 0$, after substituting Eq. (3b,c) in Eq. (17), the y_S coordinate of the shear center S can be obtained from

$$y_S = G \int_\Omega \left(y \frac{\partial \varphi_{cy}}{\partial z} - z \frac{\partial \varphi_{cy}}{\partial y} \right) d\Omega \tag{18}$$

while for $Q_z = 0$ the z_S coordinate is given as

$$z_S = G \int_\Omega \left(y \frac{\partial \varphi_{cz}}{\partial z} - z \frac{\partial \varphi_{cz}}{\partial y} \right) d\Omega \tag{19}$$

Eqs. (18) and (19) declare that the $\{y_S, z_S\}$ coordinates of the shear center S are independent from shear loading. Moreover, it can be shown that in the case of zero Poisson's ratio, the coordinates of the shear center S and the center of twist M coincide, that is

$$y_S = y_M, \quad z_S = z_M \tag{20a, b}$$

where the equations for the coordinates $\{y_M, z_M\}$ are given in Sapountzakis [21]. This coincidence of these centers was first recognized by Weber [1] applying the Betty-Maxwell reciprocal relations.

Furthermore, the shear deformation coefficients a_y, a_z and $a_{yz} = a_{zy}$, which are introduced from the approximate formula for the evaluation of the shear strain energy per unit length [19]

$$U_{\text{appr.}} = \frac{a_y Q_y^2}{2AG} + \frac{a_z Q_z^2}{2AG} + \frac{a_{yz} Q_y Q_z}{AG} \tag{21}$$

are evaluated equating this approximate energy with the exact one given from

$$U_{\text{exact}} = \int_\Omega \frac{\tau_{xz}^2 + \tau_{xy}^2}{2G} d\Omega \tag{22}$$

and are obtained for the cases $\{Q_y \neq 0, Q_z = 0\}$, $\{Q_y = 0, Q_z \neq 0\}$ and $\{Q_y \neq 0, Q_z \neq 0\}$, respectively, as

$$a_y = AG^2 \int_\Omega \left[\left(\frac{\partial \varphi_{cz}}{\partial y} \right)^2 + \left(\frac{\partial \varphi_{cz}}{\partial z} \right)^2 \right] d\Omega \tag{23a}$$

$$a_z = AG^2 \int_\Omega \left[\left(\frac{\partial \varphi_{cy}}{\partial y} \right)^2 + \left(\frac{\partial \varphi_{cy}}{\partial z} \right)^2 \right] d\Omega \tag{23b}$$

$$a_{yz} = -AG^2 \int_\Omega \left(\frac{\partial \varphi_{cy}}{\partial y} \frac{\partial \varphi_{cz}}{\partial y} + \frac{\partial \varphi_{cy}}{\partial z} \frac{\partial \varphi_{cz}}{\partial z} \right) d\Omega \tag{23c}$$

It is worth noting that the warping function φ_{cy} of Eq. (18,23b,23c), results from the solution of the boundary value problem

$$\nabla^2 \varphi_{cy}(y,z) = \frac{I_{yz} y - I_z z}{G(I_y I_z - I_{yz}^2)} \quad \text{in } \Omega \tag{24a}$$

$$\frac{\partial \varphi_{cy}}{\partial n} = 0 \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{24b}$$

and the warping function φ_{cz} of Eqs. (19) and (23a,c) from the boundary value problem

$$\nabla^2 \varphi_{cz}(y,z) = \frac{I_y y - I_{yz} z}{G(I_y I_z - I_{yz}^2)} \quad \text{in } \Omega \tag{25a}$$

$$\frac{\partial \varphi_{cz}}{\partial n} = 0 \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{25b}$$

Employing the shear deformation coefficients a_y, a_z, a_{yz} using Eqs. (23a,b,c) we can define the cross section shear rigidities of the Timoshenko's beam theory as

$$GA_{sy} = GA/a_y, \quad GA_{sz} = GA/a_z, \quad GA_{syz} = GA/a_{yz} \tag{26a, b, c}$$

It is worth here noting that in the case of an asymmetric cross section, the principal shear axes defined as [19]

$$\tan 2\varphi^S = \frac{2a_{yz}}{a_y - a_z} \tag{27}$$

do not coincide with the principal bending ones defined by the engineering beam theory. Due to this difference, the deflection components in the y and z directions are in general coupled, even if the system of axes of the cross section coincides with the principal bending one [17]. If the cross section is symmetric about an axis, the principal shear axes system coincides with the principal bending one. In this case, the deflection components with respect to the principal directions are not coupled any more ($a_{yz} = a_{zy} = 0$ and $I_{yz} = I_{zy} = 0$).

3. Integral representations – numerical solution

According to the precedent analysis, the shear problem of a beam reduces in establishing the warping functions $\varphi_c(y,z)$, $\varphi_{cy}(y,z)$, $\varphi_{cz}(y,z)$ having continuous partial derivatives up to the second order and satisfying the boundary value problems described by Eqs. (10, 12, 24a, 24b, 25a, 25b). The numerical solution of these problems is similar. For this reason, in the following we will analyze the solution of the problem of Eqs. (10) and (12).

The evaluation of the warping function φ_c is accomplished using BEM [22] as this is presented in Sapountzakis [21]. According to this method employing the Green identity and Eq. (10) the following integral representation for the warping function $\varphi_c(y,z)$ is obtained

$$\varepsilon\varphi_c(P) = \int_\Omega f(Q)\ln r \mathrm{d}\Omega_Q + \int_\Gamma \left(\varphi_c(q)\frac{\cos a}{r} - \frac{\partial\varphi_c(q)}{\partial n}\ln r\right)\mathrm{d}s_q \tag{28}$$

where $\alpha = \hat{r}, n$ (as shown in Fig. 1b); $r = |P - q|$, $P,Q \in \Omega$, $q \in \Gamma$ and $\varepsilon = 2\pi,\pi$ or 0 depending on whether the point P is inside the domain Ω, $P \equiv p$ on the boundary Γ or P outside Ω and the function f is defined as

$$f = -\frac{[(Q_zI_z - Q_yI_{yz})z + (Q_yI_y - Q_zI_{yz})y]}{G(I_yI_z - I_{yz}^2)} \tag{29}$$

Applying once more the Green identity for the function f the domain integral of Eq. (28) can be converted into a line integral along the boundary of the cross section and the integral representation Eq. (28) can be written as

$$\varepsilon\varphi_c(P) = \frac{1}{4}\int_\Gamma \left(f(2\ln r - 1)r\cos a - \frac{\partial f}{\partial n}(\ln r - 1)r^2\right)\mathrm{d}s_q + \int_\Gamma \left(\varphi_c\frac{\cos a}{r} - \frac{\partial\varphi_c}{\partial n}\ln r\right)\mathrm{d}s_q \tag{30}$$

The values of the function $\varphi_c(P)$ inside the domain Ω can be established from the integral representation (30) if φ_c were known on the boundary Γ. Thus, using Eq. (12) the integral representation (30) can be written as

$$\varphi_c(P) = \frac{1}{8\pi}\int_\Gamma \left(f(q)(2\ln r - 1)r\cos a - \frac{\partial f(q)}{\partial n}(\ln r - 1)r^2 + 4\varphi_c(q)\frac{\cos a}{r}\right)\mathrm{d}s_q \tag{31}$$

where $r = |P - q|$, $P \in \Omega, q \in \Gamma$. The unknown boundary quantity $\varphi_c(q)$ can be evaluated from the solution of the following singular boundary integral equation, which is derived after substituting Eq. (12) in the integral representation Eq. (30) written for the boundary points of the domain Ω

$$\pi\varphi_c(p) = \frac{1}{4}\int_\Gamma \left(f(q)(2\ln r - 1)r\cos a - \frac{\partial f(q)}{\partial n}(\ln r - 1)r^2 + 4\varphi_c(q)\frac{\cos a}{r}\right)\mathrm{d}s_q \tag{32}$$

where $r = |p - q|$, $p,q \in \Gamma$. Thus, using constant, linear or parabolic boundary elements to approximate the line integrals along the boundary and a collocation technique the following linear system of simultaneous algebraic equations is established

$$[A]\{\Phi\} = \{C\} \tag{33}$$

where

$$\{\Phi\}^T = \{\,(\varphi_c)_1 \quad (\varphi_c)_2 \quad \ldots \quad (\varphi_c)_N\,\} \tag{34}$$

are the values of the boundary quantity of φ_c at the N nodal points of the boundary elements. Moreover, in Eq. (33) $[A]$ and $\{C\}$ are square $N \times N$ and column $N \times 1$ known coefficient matrices, respectively. From the solution of the system of simultaneous algebraic Eq. (33) the values of the warping function φ_c for all boundary nodal points are established. As it was already mentioned the warping function φ_c is determined exactly apart from an arbitrary constant term (Neumann problem). To maintain the pure boundary character of the proposed method, the domain integral of Eq. (14) is converted to a boundary line integral as

$$c = \frac{Q_zI_z - Q_yI_{yz}}{4AG(I_yI_z - I_{yz}^2)}\int_\Gamma y^2z^2n_z\mathrm{d}s + \frac{Q_yI_y - Q_zI_{yz}}{8AG(I_yI_z - I_{yz}^2)}\int_\Gamma y^4n_y\mathrm{d}s - \frac{1}{A}\int_\Gamma \varphi_c y n_y\mathrm{d}s \tag{35}$$

However, the stress components are not influenced by this arbitrary constant, since only the derivatives of the stress function φ_c are required for the evaluation of the aforementioned quantities, as it is easily verified from Eq. (3b,c).

For the calculation of the stress resultants Eq. (3b,c), the derivatives of φ_c with respect to y and z at any interior point of the region Ω are obtained by direct differentiation of the integral representation Eq. (30), after substituting Eq. (12) as

$$\frac{\partial\varphi_c(P)}{\partial y} = \frac{1}{2\pi}\int_\Gamma \left(\varphi_c(q)\frac{\cos(\omega - a)}{r^2}\right)\mathrm{d}s_q - \frac{1}{8\pi}\int_\Gamma \Big[f(q)(2\cos\omega\cos a + (2\ln r - 1)\cos\beta). - \frac{\partial f(q)}{\partial n}(2\ln r - 1)r\cos\omega\Big]\mathrm{d}s_q \tag{36a}$$

$$\frac{\partial\varphi_c(P)}{\partial z} = \frac{1}{2\pi}\int_\Gamma \left(\varphi_c(q)\frac{\sin(\omega - a)}{r^2}\right)\mathrm{d}s_q - \frac{1}{8\pi}\int_\Gamma \Big[f(q)(2\sin\omega\cos a + (2\ln r - 1)\sin\beta). - \frac{\partial f(q)}{\partial n}(2\ln r - 1)r\sin\omega\Big]\mathrm{d}s_q \tag{36b}$$

with $r = |P - q|$, $P \in \Omega$, $q \in \Gamma$ and $\omega = \hat{x}, r$ (as shown in Fig. 1b).

Moreover, since the torsionless bending problem of beams is solved by the BEM, the domain integrals in Eqs. (8a)–(c), (18), (19), (23a)–(23c) have to be converted to boundary line ones in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, for the moments of inertia, the product of inertia and the cross section area we can write the following relations

$$I_{yy} = \int_\Gamma (yz^2\cos\beta)\mathrm{d}s, \quad I_{zz} = \int_\Gamma (zy^2\sin\beta)\mathrm{d}s \tag{37a, b}$$

$$I_{yz} = \frac{1}{2}\int_{\Gamma}(zy^2\cos\beta)\,ds,$$
$$A = \frac{1}{2}\int_{\Gamma}(y\cos\beta + z\sin\beta)\,ds \tag{37c, d}$$

while the $\{y_S, z_S\}$ coordinates of the shear center S are obtained from the calculation of the following boundary line integrals

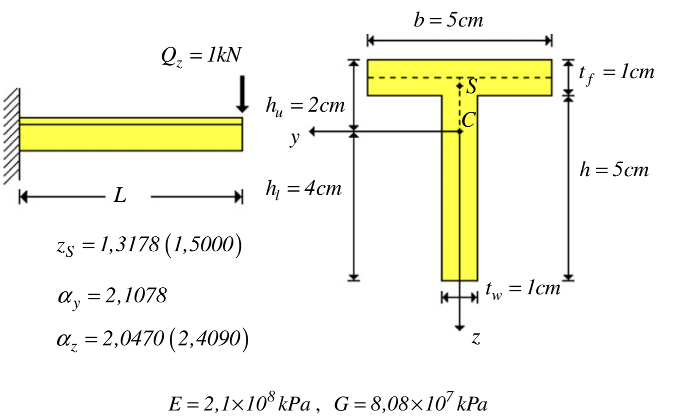


Fig. 2. T-section of the cantilever beam of Example 1 (values in parentheses are obtained from Cowper's definition [18]).

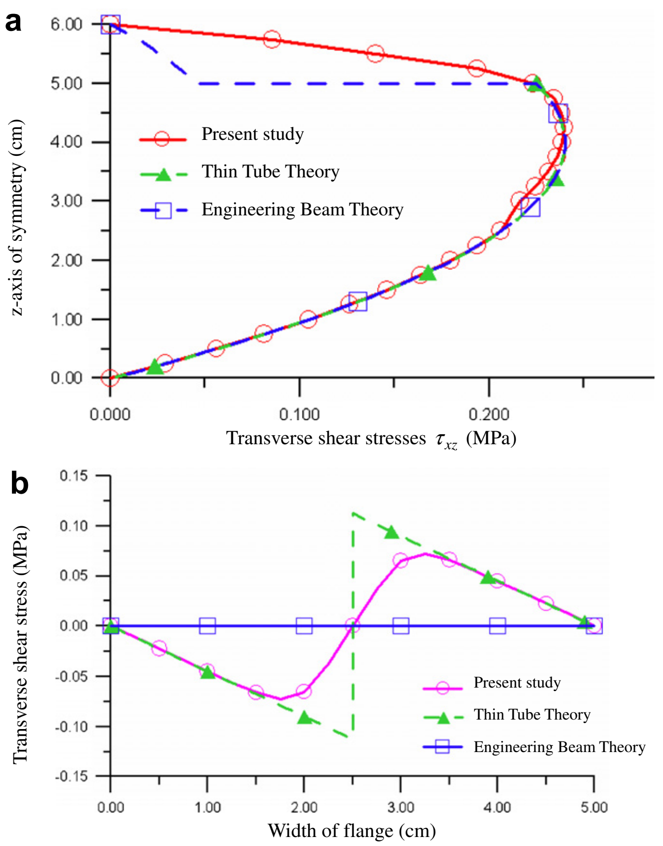


Fig. 3. Shear stresses τ_{xz} along the cross section's axis of symmetry (a) and τ_{xy} along the flange's midline (b) of the cantilever beam of Example 1.

$$y_S = G\int_{\Gamma}[\varphi_{cy}(y\sin\beta - z\cos\beta)]\,ds,$$
$$z_S = G\int_{\Gamma}[\varphi_{cz}(y\sin\beta - z\cos\beta)]\,ds \tag{38a, b}$$

Moreover, the shear deformation coefficients a_y, a_z, $a_{yz} = a_{zy}$ are obtained from the relations

$$a_y = AG^2\left(\frac{I_yI_{yz}}{12D^2}\int_{\Gamma}y^2z^3n_y\,ds - \frac{I_{yz}^2}{30D^2}\int_{\Gamma}z^5n_z\,ds + \frac{I_{yz}}{2D}\int_{\Gamma}\varphi_{cz}z^2n_z\,ds - \frac{I_y^2}{30D^2}\int_{\Gamma}y^5n_y\,ds + \frac{I_yI_{yz}}{12D^2}\int_{\Gamma}y^3z^2n_z\,ds - \frac{I_y}{2D}\int_{\Gamma}\varphi_{cz}y^2n_y\,ds\right) \tag{39a}$$

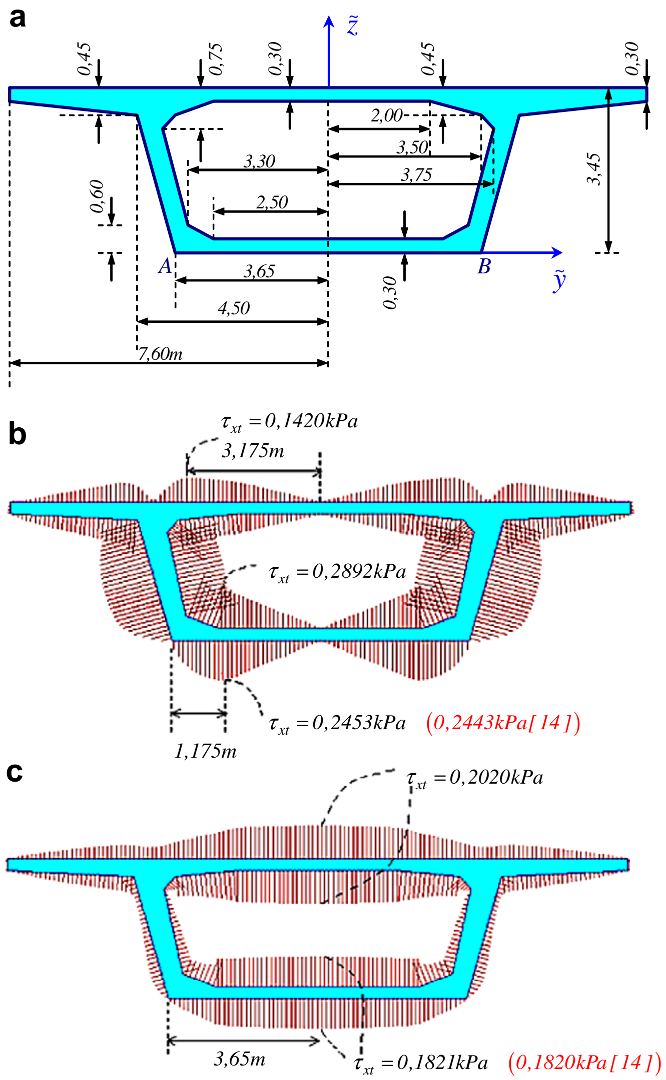


Fig. 4. Box shaped cross section of Example 2 (a) and boundary distributions of shear stress τ_{xt} for (b) $Q_z = -1$ kN and (c) $Q_y = +1$ kN (values in parentheses are obtained from a stress function solution [14]).

$$a_z = AG^2\left(\frac{I_z I_{yz}}{12D^2}\int_\Gamma y^2 z^3 n_y \mathrm{d}s - \frac{I_z^2}{30D^2}\int_\Gamma z^5 n_z \mathrm{d}s + \frac{I_z}{2D}\int_\Gamma \varphi_{cy} z^2 n_z \mathrm{d}s - -\frac{I_{yz}^2}{30D^2}\int_\Gamma y^5 n_y \mathrm{d}s + \frac{I_z I_{yz}}{12D^2}\int_\Gamma y^3 z^2 n_z \mathrm{d}s - \frac{I_{yz}}{2D}\int_\Gamma \varphi_{cy} y^2 n_y \mathrm{d}s\right) \tag{39b}$$

$$a_{yz} = -AG^2\left(\frac{I_y I_z}{12D^2}\int_\Gamma y^2 z^3 n_y \mathrm{d}s - \frac{I_z I_{yz}}{30D^2}\int_\Gamma z^5 n_z \mathrm{d}s. + \frac{I_z}{2D}\int_\Gamma \varphi_{cz} z^2 n_z \mathrm{d}s - -\frac{I_y I_{yz}}{30D^2}\int_\Gamma y^5 n_y \mathrm{d}s + \frac{I_{yz}^2}{12D^2}\int_\Gamma y^3 z^2 n_z \mathrm{d}s - \frac{I_{yz}}{2D}\int_\Gamma \varphi_{cz} y^2 n_y \mathrm{d}s\right) \tag{39c}$$

where D is given as

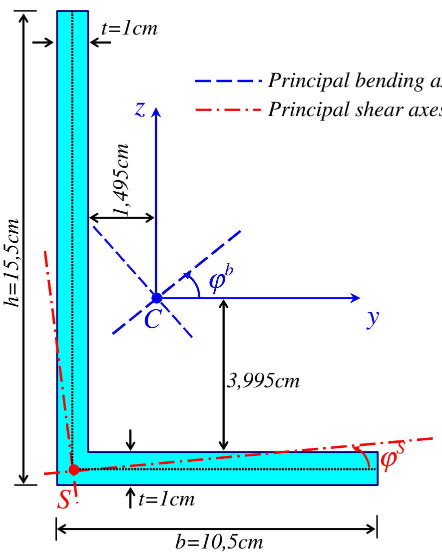
$$D = G(I_y I_z - I_{yz}^2) \tag{40}$$

Finally, the coordinates of the centroid C with respect to the arbitrary coordinate system $O\bar{y}\bar{z}$ are obtained from

$$\bar{y}_c = 2\frac{\int_\Gamma(\bar{y}\bar{z}\sin\beta)\mathrm{d}s}{\int_\Gamma(\bar{y}\cos\beta + \bar{z}\sin\beta)\mathrm{d}s}, \quad \bar{z}_c = 2\frac{\int_\Gamma(\bar{y}\bar{z}\cos\beta)\mathrm{d}s}{\int_\Gamma(\bar{y}\cos\beta + \bar{z}\sin\beta)\mathrm{d}s} \tag{41a, b}$$

Table 1
Shear deformation coefficients a_z, a_y and shear center z_S with respect to the centroid C of the box shaped cross section of Example 2

	Present study	Sapountzakis and Mokos [14]	FEM solutions
z_S	−0.5859	−0.5919	−0.5862 [8]
a_y	1.6691	1.6690	1.6686 [10]
a_z	4.3287	4.3271	4.3253 [10]



$(E = 2{,}1\times10^8\,kPa,\ G = 1{,}05\times10^8\,kPa)$

Fig. 5. L-shaped of unequal legs cross section of Example 3.

4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, the accuracy and

Table 2
Shear deformation coefficients a_y, a_z, $a_{yz} = a_{zy}$, angle φ^S (rad) of the principal shear axes and coordinates of the shear center y_S, z_S with respect to the centroid C of the L-shaped cross section of Example 3

	Present study	Sapountzakis and Mokos [14]	Schramm et al. [20]	TTT
a_y	3.063591	3.063227	3.058207	–
a_z	1.899402	1.899355	1.898375	–
$a_{yz} = a_{zy}$	0.036972	0.037235	0.039510	–
φ^S	0.031715	0.031949	0.034013	–
y_S	−1.9980	−1.998300	−1.997600	−1.995
z_S	−4.4255	−4.425422	−4.423925	−4.495

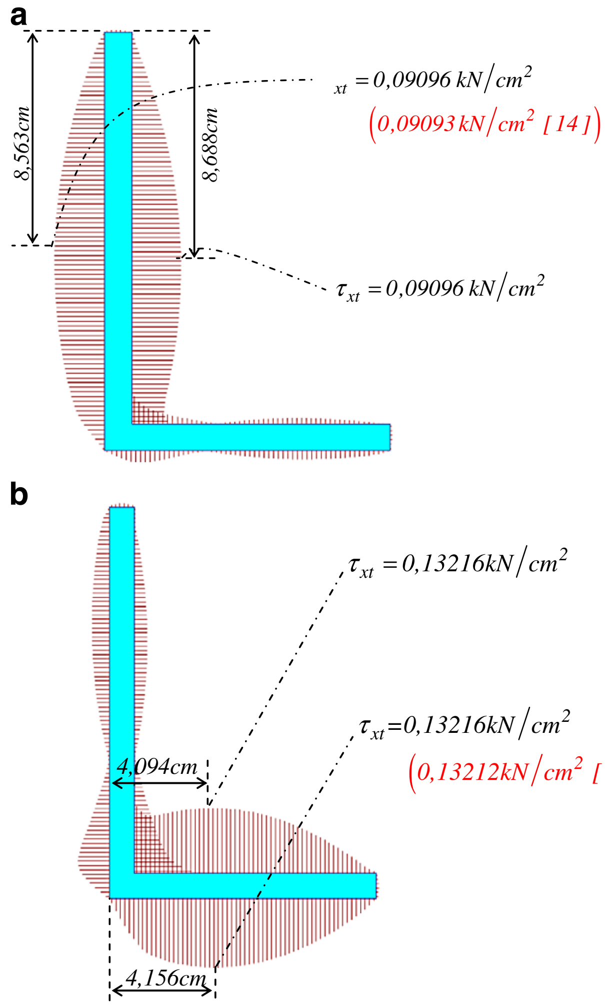


Fig. 6. Boundary distributions of the transverse shear stress τ_{xt} of the L-shaped cross section of Example 3, for (a) $Q_z = 1$ kN and (b) $Q_y = +1$ kN (values in parentheses are obtained from a stress function solution [14]).

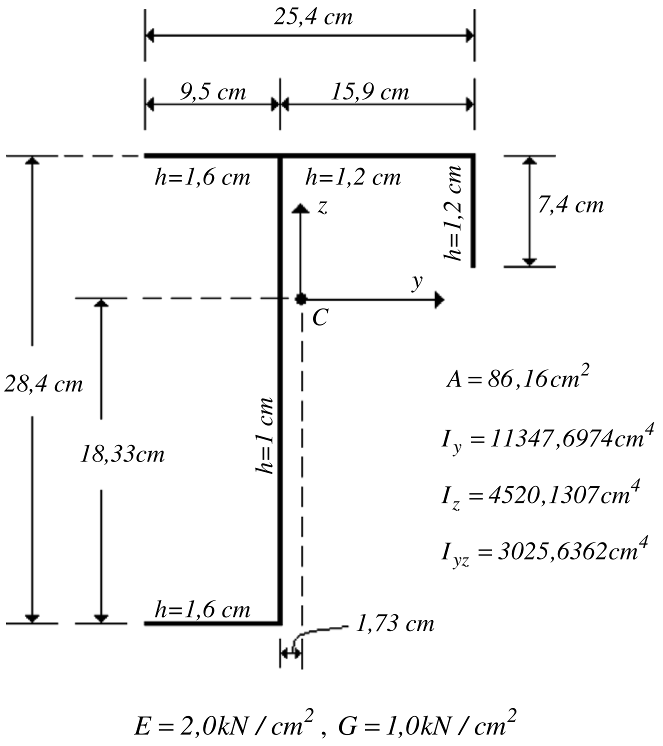


Fig. 7. Unsymmetrical welded profile forming the cross section of Example 4.

the range of applications of the developed method. In all the examples treated the numerical results have been obtained using $N = 300$ constant boundary elements necessary to fulfil the convergence criterion with error less than $\varepsilon = 10^{-3}$.

4.1. Example 1

The T-section cantilever beam of Fig. 2, loaded at its free end by a concentrated force $Q_z = 1$ kN has been studied. In Fig. 3 the distributions of the transverse shear stresses τ_{xz} along the cross section's axis of symmetry and τ_{xy} along the midline of the cross section's flange are presented as compared with those obtained from the thin tube and the engineering beam theories. The discrepancy of the aforementioned theories is remarked.

4.2. Example 2

A box shaped cantilever beam having the cross section shown in Fig. 4a ($E = 2.1 \times 10^8$ kPa, $G = 1.05 \times 10^8$ kPa) has also been analyzed. In Figs. 4b,c the distributions of the boundary shear stress τ_{xt} for two different cases of concentrated loading are presented as compared with those obtained from a stress function solution [14]. Moreover, in Table 1 the shear deformation coefficients a_z, a_y and

Table 3
Shear deformation coefficients a_y, a_z, $a_{yz} = a_{zy}$, angle φ^S (rad) of the principal shear axes and coordinates of the shear center y_S, z_S with respect to the centroid C of the cross section of Example 4

	Present study	Wagner and Gruttmann [16]
a_y	2.92763	–
a_z	3.10697	–
$a_{yz} = a_{zy}$	0.24100	–
φ^S	−0.60729	–
y_S	1.3278	1.3860
z_S	10.0803	10.0580

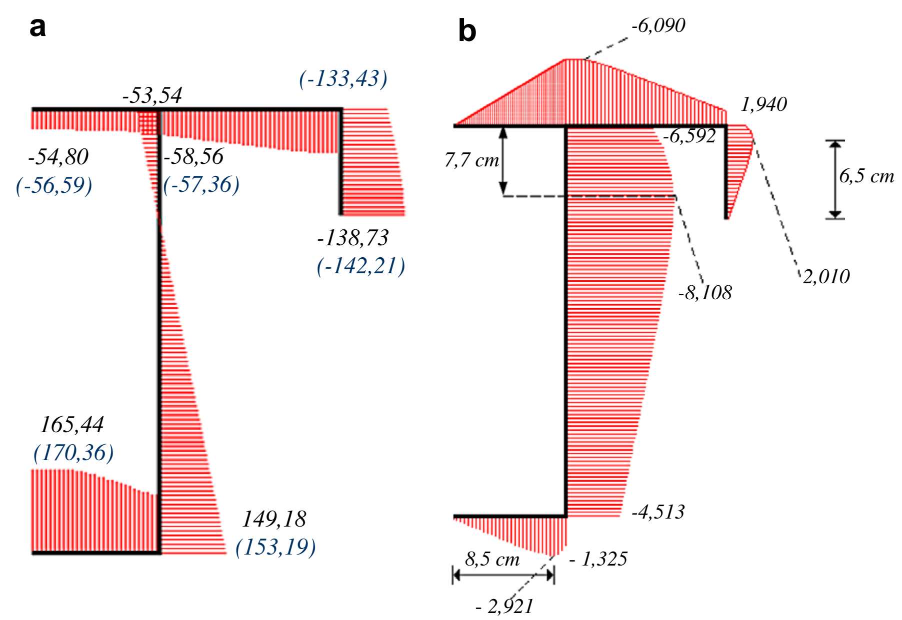


Fig. 8. Warping function φ_c (cm) (a) and transverse shear stress τ_{xt} (kN/cm^2) of the cross section of Example 4 (values in parentheses are obtained from a FEM solution [16]).

the shear center coordinate z_S with respect to the cross section centroid C are presented as compared with those obtained from a stress function solution [14] and FEM solutions [8,10]. The accuracy of the proposed method is verified.

4.3. Example 3

A cantilever beam having an L-shaped of unequal legs cross section shown in Fig. 5 has also been analyzed. In Table 2 the shear deformation coefficients a_y, a_z, $a_{yz} = a_{zy}$, the angle φ^S of the principal shear axes and the shear center coordinates y_S, z_S with respect to the cross section's centroid C are presented as compared with those obtained from a stress function solution [14], a FEM solution [20] and the thin tube theory (TTT). Moreover, in Fig. 6 the distributions of the boundary shear stress τ_{xt} for two different cases of concentrated loading are presented as compared wherever possible with those obtained from a stress function solution [14].

4.4. Example 4

As a final example the unsymmetrical welded profile taken from the text book of Petersen [23], consisting of a U300 (DIN1026) and L160 × 80 × 12 (DIN1029), replaced by a thin-walled cross section as shown in Fig. 7, subjected in the shear forces $Q_y = -120$ kN, $Q_z = -200$ kN has been studied. In Table 3 the shear deformation coefficients a_y, a_z, $a_{yz} = a_{zy}$, the angle φ^S of the principal shear axes and the shear center coordinates y_S, z_S with respect to the cross section's centroid C are presented as compared with those obtained from a FEM solution [16]. Moreover, in Fig. 8 the warping function φ_c and the shear stress τ_{xt} distributions are presented as compared wherever possible with those obtained from a FEM solution [16]. The accuracy, the efficiency and the range of applications of the developed method are easily verified (a small discrepancy in some values may be due to the small discrepancy in geometrical and inertia data).

5. Concluding remarks

The main conclusions that can be drawn from this investigation are

(a) The numerical technique presented in this investigation is well suited for computer aided analysis for prismatic beams of arbitrary simply or multiply connected cross section, while the analysis is performed with respect to an arbitrary system of axes and not necessarily to the principal one.
(b) Accurate results are obtained using a small number of boundary elements.
(c) Engineering beam theory cannot give accurate results especially in cross sections with discontinuous variation of width.
(d) The inaccuracy of the thin tube theory especially near the intersections of the parts of the thin cross section is verified.
(e) The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

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