the ratio of the maximum height of the deformation to the thickness of the dielectric slab. If ϵ is reasonably small, for example $\epsilon < 0.2$, the second-order scattered fields may be neglected without effecting much the accuracy of the previous results.

In conclusion, the present example shows how to apply the formulas in previous sections. It shows how those calculated quantities vary when the shape of the perturbed boundary changes.

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Numerical Aspects on Coupling Between Complementary Boundary Value Problems

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Abstract—**The consequences of nonuniqueness for integral equations used in the numerical resolution of electromagnetic scattering problems are investigated from a practical point of view. The scatterers are closed perfectly conducting cylinders of arbitrary cross section illuminated by a plane wave in both E and H polarizations. It is shown how to detect the frequencies at which nonuniqueness occurs, and how to avoid the resulting errors by the use of the notion of an equivalent problem. This approach is compared to other ones proposed by different authors. A new interpretation of the computed solution, when uniqueness conditions are not satisfied, is given and it is shown how to use such a solution in the computation of the resonant modes of the interior problem, even for degenerate modes.**

I. Introduction

THE INTEGRAL formulation is well suited to the numerical solution of scattering and radiation problems, particularly when the scatterers or radiators are in the resonance domain (i.e., when their dimensions are of the order of the wavelength). This formulation is of increasing interest to engineers.

The integral formulation leads to a great diversity of integral or integro-differential equations which can be reduced, by means of a moment method, to a high-order system of linear equations. As long as existence and uniqueness conditions are satisfied and the limits of storage and speed of actual computers are respected, the solution of the linear system can be obtained without great difficulties.

The discussion concerning existence and uniqueness of a solution would seem to be of only theoretical interest, especially when we deal with a practical problem whose solution exists and is unique. This is true when it is possible to obtain a closed form solution. In this case, nonphysical solutions, which usually appear when the formulation is not accurate, can be eliminated afterwards. This is no longer possible when we try a numerical resolution, and, as we shall see later, the interpretation of the results appearing in an array might be difficult. One can imagine then the practical importance of a discussion about existence and uniqueness conditions.

In fact, this discussion can be found at the base of many theoretical investigations of electromagnetic and acoustic scattering and radiation problems. Though many papers have been written on this subject (for substantial references see [1]–[3]) most of them are arduous for an engineer whose interest is the result of the discussion.

For the equations, ordinarily deduced from the Green's theorem, the existence conditions are always satisfied, but uniqueness is not guaranteed (see for instance [1]). Generally speaking, uniqueness is related to the existence

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of solutions of the associated homogeneous equations. Some of these solutions may be linked to the problem under investigation (i.e., resonant modes of a cavity, eigenmodes of a surface wave structure externally excited $\cdots$), whereas others are parasitic solutions usually related to eigensolutions of other problems. For an exterior scattering or radiation problem in a domain D_e, these solutions correspond to the interior cavity problem in the complementary domain D_i. For these reasons such exterior and interior problems are called complementary.

Our purpose here is to study, from a practical point of view, the consequences of the coupling between complementary problems on numerical resolution of the problem. We shall particularly emphasize two points: first, the detection of the errors resulting from the coupling in the classical formulation, second the use of the notion of an equivalent problem to avoid such errors [6]–[8]. This approach is compared to other ones proposed by different authors [4], [5]. We shall give a new interpretation of the solution computed when uniqueness conditions are not satisfied and show how to use such a solution in the computation of the resonant modes of the interior problem, even for degenerate modes. Finally, a series of numerical results will be given which illustrate the different aspects of the problem.

II. General Considerations

A. Classical Integral Formulation

For the sake of simplicity we shall consider only a two dimensional problem as is the case for a perfectly conducting infinite cylinder receiving an incident wave whose electrical field (E case) or magnetic field (H case) is parallel to its generators. Let C denote the contour of the obstacle in a cross section plane, and D_e and D_i, respectively, the exterior and interior domains with respect to C.

The Green's theorem provides us with the following representation for the total field

$$H_e u(x) = u^0(x) + \int_c \{G(x,x')\partial_{n'}u(x') - \partial_{n'}G(x,x')u(x')\}\, dx' \quad (1)$$

where u and u^0 are the longitudinal components of the total field and the incident field, respectively, H_e the characteristic function of D_e, $\partial_{n'}$ is the symbol of the derivative along the interior normal to C, and G the free-space Green's function. With the time dependence $e^{j\omega t}$, $G(x,x') = (j/4)H_0^{(2)}(k \mid x - x' \mid)$ where k is the propagation constant in D_e and $H_0^{(2)}$ the Hankel function of second type and zero order.

In each case of polarization, one may deduce from (1) a Fredholm's integral equation of second kind

$$\frac{1}{2}\partial_n u - \int_c \partial_n G \partial_{n'} u = \partial_n u^0, \qquad \text{for } E \text{ case} \quad (2)$$

$$\frac{1}{2}u + \int_c \partial_{n'} G\, u = u^0, \qquad \text{for } H \text{ case.} \quad (3)$$

The existence and uniqueness of a solution for these equations can be discussed by means of the Fredholm alternative. (This would not be possible with an integral equation of the first kind). Let $\{k_n^E\}$ and $\{k_n^H\}$ denote the eigenvalues of the interior cavity D_i bounded by perfectly conducting walls, in the E case and the H case, respectively. Equations (2) and (3) then have a unique solution if $k \neq k_n^H$ and $k \neq k_n^E$, respectively. If these conditions are not realized the homogeneous equations associated with (2) and (3) have no vanishing solutions and (2) and (3) have an infinite number of solutions [1].

The existence of these eigensolutions may seem surprising especially if one considers the fact that they do not correspond to any possible physical solution of the scattering problem. It turns out that these parasitic solutions are related to the complementary interior problem for the other polarization and result from the fact that (2) and (3) correspond to the physical boundary conditions of the considered scattering problem only when the total field is zero in D_i. But, this field is precisely nonvanishing for the resonant frequencies of the interior problem. It can be shown that among all the solutions of (2) and (3), only the physical solution of the scattering problem creates a zero field in D_i. Furthermore the parasitic solutions correspond to the resonant modes of the cavity D_i [10], [11].

B. Numerical Problems

The consequences of the aforementioned coupling must not be underestimated in the numerical resolution. By means of the well-known moment method [12], (2) and (3) can be reduced to a system of linear equations of finite dimensions.

We shall briefly recall here the principles of this method. Equations (2) and (3) can be written in the form $Lf = g$, where L is a linear operator, g a known source function and f the unknown function. The solution can be approximated, using a norm, in a Hilbert space $\mathcal{H}$, by a function such that

$$f^{(N)} = \sum_{n=1}^{N} \alpha_n f_n \qquad \lim_{N\to\infty} \| f - f^{(N)} \| = 0 \quad (4)$$

with $\{f_n\}$ a complete set of base functions in $\mathcal{H}$. The unknown coefficients α_n satisfy the matrix equation

$$AX = B \quad (5)$$

with $A_{m,n} = \langle w_m, Lf_n \rangle$, $B_m = \langle w_m, g \rangle$, $X_n = \alpha_n$, for $n = s,2, \cdots,N$; $m = s,2,\cdots,N$; $\{w_m\}$ is another set of base functions in $\mathcal{H}$ and $\langle \cdot \mid \cdot \rangle$ represents the scalar product

$$\langle f_1, f_2 \rangle = \int f_1(x) f_2^*(x)\, dx$$

where the asterisk represents the complex conjugate.

Theoretically, when k is equal to an eigenvalue of the interior complementary problem, and when $N \to \infty$, the system (5) is singular. Practically the problem is more complicated. In fact, when N has a finite value the system is only approximate. Furthermore computations are made with a finite number of significant digits.

These remarks have two consequences: on one hand, the system is never exactly singular and on the other hand, the consequences of the coupling do not only occur exactly at the eigenvalues but rather in the neighborhood of the latter.

Often, the occurrence of an eigenvalue is not sufficient to render useless the algorithm employed for the computation of the solution. Nevertheless, the computed solution appears as an arbitrary linear combination of the researched external physical solution and of parasitic solutions of the internal complementary problem. As we shall see later this kind of error is more hazardous when the orders of magnitude of the solution remain reasonable. It is obvious that any numerical test (double inversion, substitution) is not significant.

C. Numerical Tests

The small value of the determinant can make us presume that an error exists, but only a continuous variation of the determinant with respect to k clearly indicates resonances. As this method is lengthy it is more suitable to use systematic tests.

One test is to substitute the computed solution in the right-hand side of (1), this is done for different points in D_i. If the computed field vanishes at these points, then the computed solution is correct. If not, the frequency used is near an eigenfrequency of the complementary interior problem and the total field computed in D_i determines the resonant mode [15]. This procedure can be easily used in the study of internal problems.

A second test is to check the energy conservation at infinity. In the case of an incident plane wave, with θ_0 the angle of incidence, the power scattering diagram $|F(\theta)|^2$ satisfies the well known relation

$$\int_0^{2\pi} |F(\theta)|^2 \, d\theta = -2\pi \operatorname{Re}\{F(\theta_0)\} \tag{6}$$

where $\operatorname{Re}\{f\}$ denotes the real part of f and $F(\theta_0)$ is evaluated in the forward scattering direction. Thus, the calculation of the quantity ε such that

$$\varepsilon = \left| \int_0^{2\pi} |F(\theta)|^2 \, d\theta + 2\pi \operatorname{Re}\{F(\theta_0)\} \right| \tag{7}$$

indicates how good is the energy conservation. The expression giving ε is somewhat more complicated for an arbitrary distribution of sources. Nevertheless, its computation using the solution of the integral equations remains a suitable check especially because the result can be characterized by a single number which can be easily deduced from $|F(\theta)|^2$, a quantity of practical interest. As we shall see later, it is possible to determine ε_M, the maximum allowed value of ε such that for $\varepsilon > \varepsilon_M$ the result is surely false and for $\varepsilon < \varepsilon_M$ the result can be considered as true with some confidence.

D. The Particular Case of Circular Cylinder

When we examine the coupling between complementary problems, the case of the circular cylinder is particularly interesting, because it is practically the only one for which the external and internal problems possess analytical and easy computable solutions. Thus it is possible to emphasize some important points of the preceding sections.

In the case of a circular cylinder of radius a, Table I gives the analytical expressions of the coefficients $A_{m,n}$ and B_m deduced from (2) [similar consideration occurs for (3)] for different choices of the base functions. When the set of base functions $\{f_n\}$ and $\{W_m\}$ coincides with the eigenfunctions of the kernel of the integral equation, the matrix A is obviously diagonal. Then one can easily see that A is singular if ka is a zero of $J_m'(ka)$ in the E case [or a zero of $J_m(ka)$ in the H case] for $m \leq N$. When an analytical resolution is performed, no difficulty occurs because the corresponding coefficient B_m is also zero. On the other hand, in the case of a numerical resolution one cannot predict the behavior of the system. The $A_{m,n}$ are computed numerically and those which must be zero, have in fact very small values. Thus, matrix A is never exactly singular.

In the case of the second choice of the base functions, matrix A is entirely filled, but the singularities can be easily determined, because in a resonant case all elements in a row are zero. In the third choice it is not obvious that we have a singularity and we do not know for which value of N they will start appearing. Thus one can easily imagine that the extent to which a singularity occurs depends on the choice of the base functions and on the accuracy of computation of the coefficients $A_{m,n}$.

E. Methods of Resolution in a Singular Case

When one of the previously described tests indicates a resonance of the complementary interior problem, (2) and (3) cannot be directly used. One method of retaining only the correct solution consists in formulating, explicitly or not, the condition that the total field vanishes in D_i.

Waterman [4] proposed to use the "generalized boundary condition" in D_i

$$0 = u^0(x) + \int_c \{G(x,x')\partial_{n'}u(x') - \partial_{n'}G(x,x')u(x')\} \, dx', \qquad x \in D_i. \tag{8}$$

Making use of polar coordinates and of the well known expansions of the kernel in series of Bessel and Hankel's functions, we are led to a system of integral equations which allows us to compute the field or its normal derivative on C under the unique condition that the total field vanishes inside a circular domain in D_i. By an analytic continuation of the field in D_i one can show that this necessary condition is also sufficient in order that the field vanishes on the entire domain D_i provided that the contour of the obstacle is not discontinuous [4]. Numerically it becomes less sufficient when the ratio of the largest dimension of the cylinder to the smallest dimension has a large value or in the case of an edge-type contour.

TABLE I

f_n	W_m	A_{mn}	B_m
$e^{jn\theta}$	$e^{jm\theta}$	$-j\frac{\pi ka}{2}J_n'(ka)H_m(ka)\delta_{mn}$	$j^{+m}kJ_m'(ka)$
$e^{jn\theta}$	$\delta(\theta-\theta_m)$	$-j\frac{\pi ka}{2}J_n'(ka)H_n(ka)e^{-jn\theta_m}$	$+j^{e^{+jka\cos\theta_m}}k\cos\theta_m$
$H_n(\theta-\theta_n)$	$\delta(\theta-\theta_m)$	$-j\frac{ka}{4N}\sum_p C_p J_p'(ka)H_p(ka)e^{-jp(\theta_m-\theta_n)}$	$+j^{e^{+jka\cos\theta_m}}k\cos\theta_m$
$H_n=H_n^{(2)}$			$C_p=\frac{\sin\frac{p\pi}{N}}{\frac{p\pi}{N}}$

Schenck [5] resolved (2) and (3) in a similar manner by introducing the additional constraints which force the field to be zero at a finite number of points in D_i. These points must be chosen in such a way that the constraints have a maximal efficiency. The resolution of the resultant overdetermined system, can be effectuated by means of an optimization algorithm.

Another method is based on the idea of an equivalent problem. It allows us to formulate a fictitious problem equivalent to the physical one in D_e only, in such a way that the complementary parasitic problem does not possess resonant frequencies for real values of k. Thus, we may represent the scattered field as a linear combination of single and double layer retarded potentials [6], [7], [13] or more generally as a linear combination of higher order potentials [8]. For instance the field in D_e can be represented by

$$u(x) = u^0(x) + \int_c [\partial_{n'}G(x,x') + jkG(x,x')]r(x')\,dx'. \tag{9}$$

Such a linear combination of potentials can be called hybrid potential. The unknown density $r(x')$, in each case of polarization, is deduced from one of the following equations:

$$\frac{1}{2}r(x) - \int_c [\partial_{n'}G(x,x') + jkG'(x,x')]r(x')\,dx' = u^0(x), \qquad \text{for } E \text{ case} \tag{10}$$

$$-\frac{j}{2}r(x) - j\int_c \partial_n G(x,x')r(x')\,dx' - \text{FP}\int_c \partial_{nn'}{}^2G(x,x')r(x')\,dx' = \partial_n u^0(x), \qquad \text{for } H \text{ case} \tag{11}$$

where FP stands for "finite part in the sense of Hadamard" [14]. The knowledge of the order of singularity of $\partial_{nn'}{}^2G$ allows us to write

$$\text{FP}\int_c \partial_{nn'}{}^2G(x,x')f(x')\,dx' = \lim_{\substack{\gamma\to 0\\ x\in\gamma}}\left\{\int_{c-\gamma}\partial_{nn'}{}^2G(x,x')f(x')\,dx' - \frac{2f(x)}{\pi M(\gamma)}\right\} \tag{12}$$

where $M(\gamma)$ is the length of the arc γ, and x being at the center of γ. The introduction of a finite part avoids the uneasy manipulation of an integro differential equation. The right-hand side of (12) can be easily computed by the usual integration methods.

III. Numerical Results

A. General Considerations

Two approximation processes have been used. In both cases, $\{f_n\}$ are of the rectangular pulse type, but in the second process, Lf_n is computed with a crude approximation consisting in the replacement of the integral by the value of the integrand at the center of the interval of integration. In both cases, the testing base functions W_m are of the Dirac pulse type. The sampling rate on the contour C is about 10 per wavelength. Such a sampling rate provides an accuracy of the order of 10^{-3}, in the absence of interior resonance.

Two different algorithms were used to solve the systems of linear equations. The first one is the well-known Gauss–Jordan algorithm for which it is very difficult to find a genuine criterion to check the singularity of the system. Indeed, computations have shown, that very good results can be obtained even if the determinant of the system is very small. Numerical examples will later illustrate the behavior of this algorithm in quasi-singular cases. The second one is due to Le Foll [16]. For this iterative algorithm, the occurrence of a singularity does not perturb the numerical process: in such a case, or, more generally, when the system is underdetermined, the Le Foll algorithm converges to a core solution of the system.

B. Circular Contour

Let us consider the same problem as in section II-D. The singular character of matrix A clearly appears in (Fig. 1) when the first approximation process is applied to (2) and (3). The variations of the determinant with respect to ka, are very rapid in the neighborhood of a singularity. These variations provide an accurate determination (4 significant digits) of the resonant frequencies of the complementary problem. On the other hand, when (10) and (11) are used, the determinant has monotonic variations (Fig. 1).

The value $ka = 3.8317$ corresponds to a resonance for both E and H cases ($J_0'(ka) = -J_1(ka) = 0$) and thus is a singular value for both (2) and (3). In Figs. 2 and 3 the computed results are compared with the exact solution. The Gauss-Jordan algorithm provides an

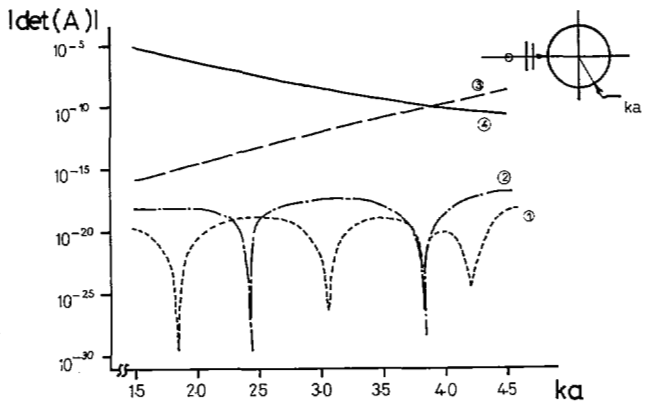

Fig. 1. Determinant variations for linear systems deduced from equations relevant to classical and hybrid potential formulations. Curves ①, ②, ③, ④, correspond, respectively, to (2), (3), (10), (11).

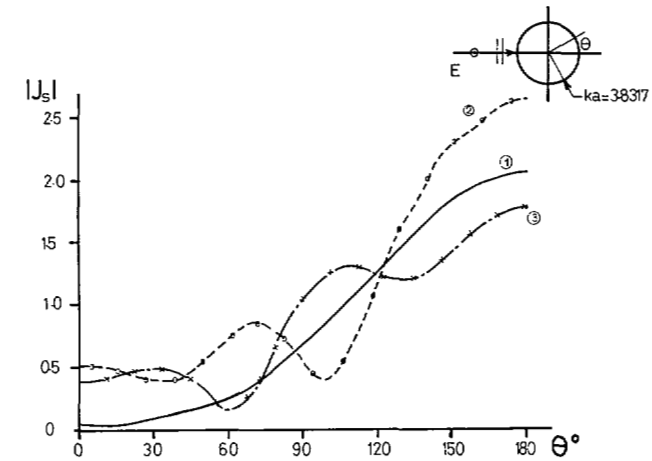

Fig. 2. Normalized current density, E case. Curve ① corresponds to exact solution, curves ② and ③ to computed solutions from (2) with Gauss–Jordan and Le Foll algorithms, respectively.

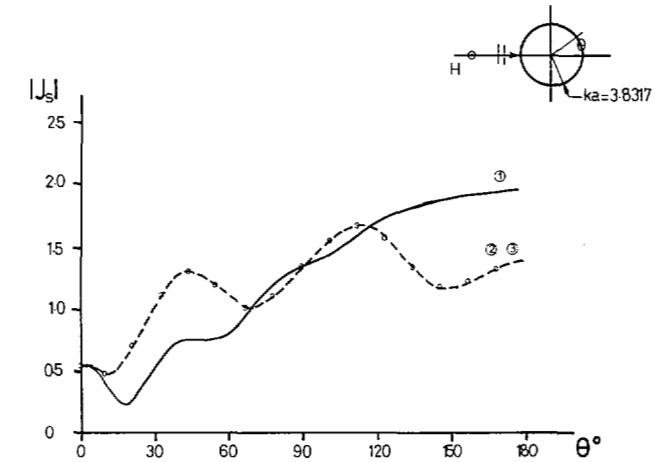

Fig. 3. Normalized current density H case. Curve ① corresponds to exact solution, curves ② and ③ to computed solutions from (2) with the Gauss–Jordan and Le Foll algorithms, respectively.

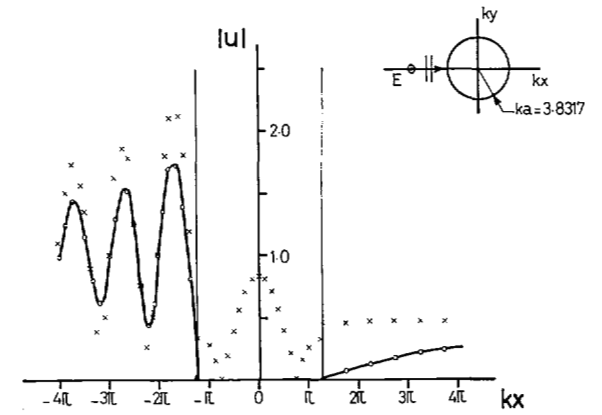

Fig. 4. Near-field distribution computed from classical method (1) (×××) and hybrid potential method (11) (○○○). Continuous line corresponds to exact distribution. For $kx > ka$, u represents external electric field component, for $kx < ka$, u is proportional to magnetic field component of eigenmode.

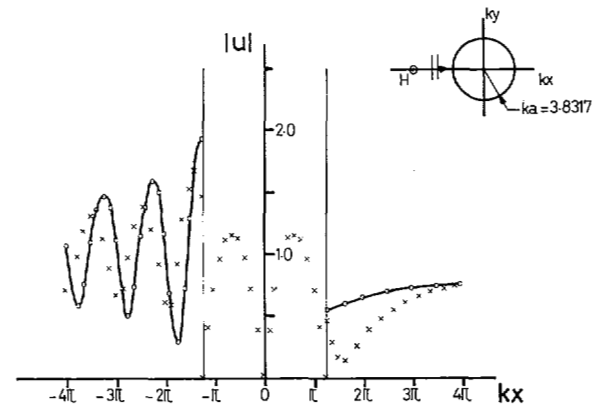

Fig. 5. Near-field distribution computed from classical method (1) (×××) and hybrid potential method (11), (○○○). Continuous line corresponds to exact distribution. For $kx > ka$, u represents external magnetic field component, for $kx < ka$, u is proportional to electric-field component of eigenmode.

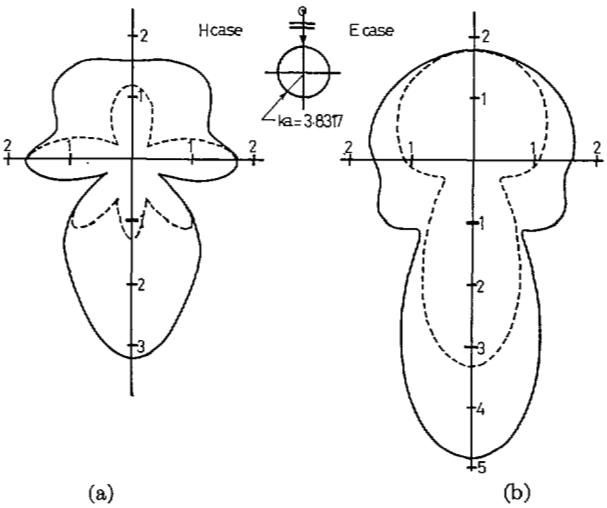

Fig. 6. Far-field distribution computed from classical method (– – –) and hybrid potential method (——), coinciding with exact solution).

arbitrary linear combination of the exact solution and the one related to the complementary interior problem. Le Foll's algorithm gives the core solution. The variations of the computed solutions are very different from those of the exact solution, but it must be remarked that the orders of magnitude are not absurd. Therefore, if these solutions are introduced into (1), one must not be surprised to find erroneous values for the near field (Figs. 4 and 5) and for the scattering diagram (Fig. 6). If the computation concerning the near field have no particular significance in the region exterior to the obstacle, in the region interior to the latter, they provide the configuration of the resonant eigenmode. When the external problem corresponds to an E or H case, the field computed in D_i is relevant to a magnetic or electric eigenmode, respectively. For the problem here considered, the magnetic eigenmode varies like $J_0(kx)$ and the electric one like $J_1(kx)$. The use of hybrid potentials gives satisfactory results in the outside of the scatterer.

C. Rectangular Contour

In the case of a rectangular contour, the second approximation process was applied. The exact solution for the external problem is not analytically known. To check the validity of the solution, the energy test was used. It was experimentally found that the corresponding quantity must be less than a few percent in order that the solution be correct. Fig. 7 shows the correlation between determinant and energy test variations in the case of a quasi-degenerate rectangular contour. Resonant values are not so sharply defined as in the circular contour case because the approximation process here used is less accurate. The curves reported here show how far we can go into the detection of neighboring resonant modes. Fig. 8 shows the continuous deformation of the interior mode between the resonant modes (1,2) and (2,1) When the complementary interior problem is degenerate, the interior field is a linear combination of all degenerate modes. Nevertheless, we can separate them by using different values of the parameter θ_0 (Fig. 9). Thus, in the case of the square contour, modes (1,2) and (2,1) have the same resonant frequency and appear separately for different values of the incident angle θ_0.

D. Elliptic Contour

The case of the elliptic contour enables some comparisons between the methods used to avoid the difficulty resulting from the coupling between complementary problems. We shall consider, for instance, the E case. As a reference we have used two methods deduced from an integral equation of the first kind. This equation can be easily obtained from (1) by direct use of the appropriate boundary condition. It can be shown that the singular values of k for this equation are those of the complementary interior problem for the same boundary condition. A first set of results were obtained by solving directly this equation by the first approximation process. A second

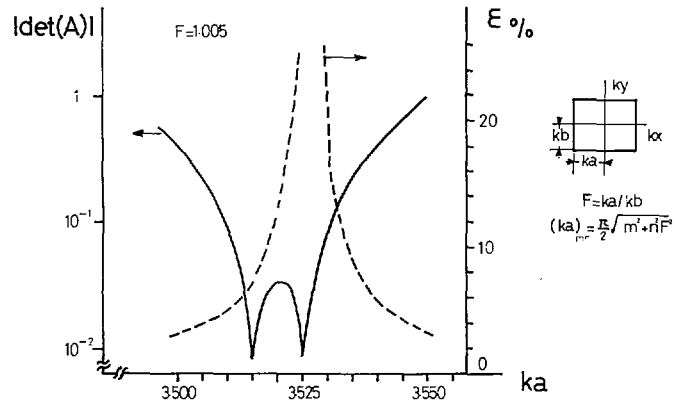

Fig. 7. Correlation between variations of the determinant and energy test for a rectangular cylinder.

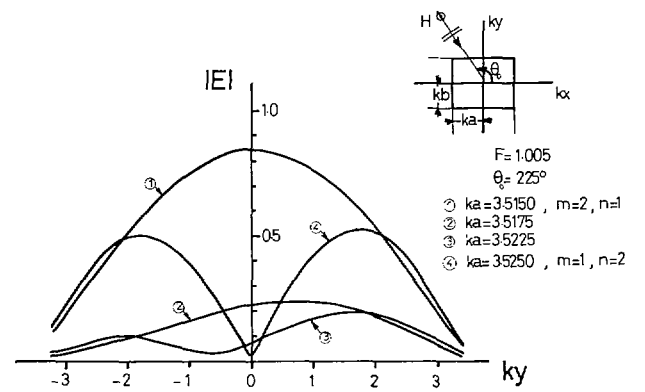

Fig. 8. Computed field distribution inside rectangular cylinder illustrating transition between two neighbored eigenmodes.

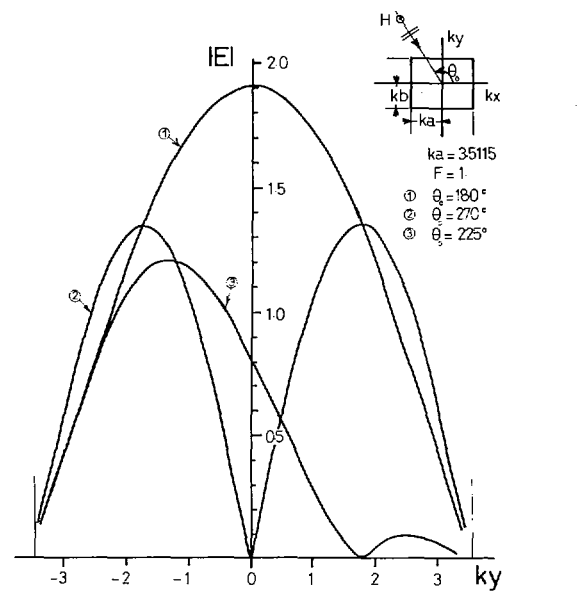

Fig. 9. Computed field distribution inside square cylinder illustrating possibility of separating degenerate eigenmodes by mean of incident angle θ_0.

TABLE II

Method	Classical	Waterman	Classical	Schenk		Hybrid potentials
Type of equations	first kind	System of first kind equations	second kind	second kind + constraints 		second kind.
Algorithm	Gauss Jordan	Gauss Jordan	Gauss Jordan	Le Foll	Le Foll	Gauss Jordan
* $\lvert F_N(\theta)\rvert^2$ θ = 0°	0.946	0.947	0.786	0.785	0.934	0.945
30°	0.047	0.046	0.038	0.039	0.047	0.047
60°	0.058	0.058	0.120	0.119	0.057	0.059
90°	0.071	0.071	0.080	0.081	0.070	0.072
120°	0.099	0.099	0.034	0.034	0.098	0.102
150°	0.144	0.144	0.105	0.104	0.145	0.145
180°	0.169	0.168	0.438	0.432	0.173	0.168
Energy test ε %	0.02	0.03	2.0	2.0	0.6	0.5

* $\lvert F_N(\theta)\rvert^2 = \dfrac{\lvert F(\theta)\rvert^2}{-2\pi \mathrm{Re}(F(\theta_0))}$

TABLE III

Ellipticity ka/kb	ε %
1.0	$< 10^{-4}$
0.8	$< 10^{-4}$
0.6	$< 10^{-3}$
0.4	0.34
0.2	15.0

set was obtained by using the Waterman method. For both sets, the numerical results concerning the scattering diagram are very closely related and the energy test is of the order of 3×10^{-4} (Table II).

Equation (2) was then resolved with and without additional constraints as indicated by Schenk. In the first case, it is interesting to note that in spite of an energy test of 2×10^{-2} there are many differences between the corresponding result and the preceeding ones. Therefore it would seem necessary to have energy test inferior to 1×10^{-2} in order for the solution to be considered as good. In the second case, two different choices for the interior constraint points were made. When the points are spread over the major axis of the ellipse it appears that the constraints are of no use because the major axis probably corresponds to a nodal line of the interior resonant mode (Table II). The corresponding scattering diagram is nearly identical to the one computed without constraints. When on the other hand, a point is taken at the origin and one point is chosen in each quadrant, then the solution is greatly improved. The energy test is equal to 6×10^{-3} and the computed diagram is in good agreement with the reference diagrams. Similarly favorable results are obtained with hybrid potentials, without any of the uncertainty, as in the preceding case, concerning the distribution of the interior points.

At this point, it is interesting to examine the degradation of the energy test as a function of the ellipticity for the Waterman method. Clearly, it appears that the energy test becomes worse and worse when the ellipticity increases in spite of careful renormalization of the systems (Table III).

IV. Conclusion

The consequences of nonuniqueness in the classical formulation have been illustrated by some numerical examples. On the other hand, it appears that representing the diffracted field as a suitable linear combination of single and double layer retarded potentials constitutes a good manner in order to overcome difficulties in the numerical resolution of scattering and radiation problems. This method seems to be very reliable compared to the Waterman and Schenck methods which presents some limitative conditions of applicability.

Whatever the method, the energy test, properly used, constitutes a good check preserving against accidental errors resulting from nonuniqueness. For the sake of simplicity, only scalar problems have been considered but the previous considerations can be extended to vector problems.

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Communications

On Numerical Convergence of Moment Solutions of Moderately Thick Wire Antennas Using Sinusoidal Basis Functions

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Abstract—Wire antennas are solved using a moments solution where the method of subsectional basis is applied with both the expansion and testing functions being sinusoidal distributions. This allows not only a simplification of near-field terms but also the far-field expression of the radiated field from each segment, regardless of the length L. Using sinusoidal basis functions, the terms of the impedance matrix obtained become equivalent to the mutual impedances between the subsectional dipoles. These impedances are the familiar impedances found using the induced EMF method. In the induced EMF method an equivalent radius is usually used in the evaluation of the self-impedance term to reduce computation time. However, it is shown that only for very thin segments that the correct equivalent radius is independent of length. When the radius to length ratio (a/L) is not small, an expansion for the equivalent radius in terms of a/L is given for the self-impedance term. The use of incorrect self-term, obtained by using a constant equivalent radius term, is shown to be responsible for divergence of numerical solutions as the number of sections is increased. This occurrence is related to the ratio of a/L of the subsections and hence becomes a problem for moderately thick wire antennas even for a reasonably small number of segments per wavelength. Examples are given showing the convergence with the correct self-terms and the divergence when only a length independent equivalent radius is used. The converged solutions are also compared to King's second- and third-order solutions for moderately thick dipoles.

I. Introduction

The method of moments is applied to wire antennas as discussed in other papers [1], [2], but carried to a higher order of approximation which allows treating the case where the length to radius ratio is small. The theory will discuss the straight wire antenna but the extension to wires of arbitrary shape is straightforward.

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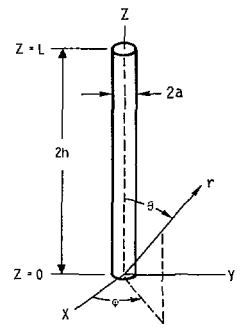

Fig. 1. Straight wire and coordinate system.

Fig. 1 shows a straight section of wire of circular cross section, and defines the coordinate system. The wire extends from $z = 0$ to $z = L$ along the z axis and is of radius a. It is assumed that the radius is small compared to a wavelength but the ratio of a to L need not be small. The only significant component of current on the wire is then the axial component, which can be expressed in terms of the net current $I(z)$ at any point z along the wire. The current distribution will then be modeled as an infinitely thin sheet of current forming a tube of radius a, with the density of current independent of circumferential position on the tube.

An operator equation for the problem is given by

$$\mathfrak{L}I(z) = \frac{j}{4\pi\omega\epsilon}\left(\frac{d^2}{dz^2} + k^2\right)\oint_c \int_0^L \frac{e^{-ikR}}{R} I(z')\, dz'\, dc = E_z^i(z) \quad (1)$$

where $E_z^i(z)$ is the z component of the impressed electric field at the wire surface, $I(z')$ is surface current density, $\oint_c dc$ represents the integration around the circumference, and R is the distance from the source point to the field point. The boundary condition for the current is $I(0) = I(L) = 0$.

II. Theory

The procedure is basically one for which the wire is divided into subsections, and a generalized impedance matrix (Z) obtained to describe the electromagnetic interactions between subsections. The