Journal of Sound and Vibration (1990) **139**(1), 83–97

A STUDY OF THE FAILURE OF NUMERICAL SOLUTIONS IN BOUNDARY ELEMENT ANALYSIS OF ACOUSTIC RADIATION PROBLEMS

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(*Received* 29 *April* 1988, *and in revised form* 14 *August* 1989)

In the numerical implementation of the boundary element method to exterior acoustic problems, the approximating equations give incorrect results for the surface pressure when the frequency is in the neighbourhood of some critical values. In previous papers, this failure has been attributed to ill-conditioning problems which prevail as the singularity is approached, but issues such as whether the coefficient matrix will become singular or whether one will obtain good results when the effects of ill-conditioning can be annihilated have not been discussed. This paper presents a discussion of these questions. Numerical results for two-dimensional sound radiation from a radially vibrating infinite cylinder show that the problem encountered in the vicinity of the critical frequencies is not basically an ill-conditioning problem, and that the coefficient matrix does not become singular for a finite number of boundary elements.

1. INTRODUCTION

The boundary element method has been used extensively in recent years in the numerical solution of acoustic radiation problems. This method essentially consists of discretization of the exterior surface Helmholtz integral equation and solution of the resulting approximating equations, a set of non-homogeneous linear algebraic equations, for the surface pressures at the collocation points. The field pressures are then computed by using the exterior Helmholtz integral formula. A well known drawback of this approach is that the approximating equations give incorrect results for the surface pressures when the forcing frequency is near to a characteristic frequency associated with the interior of the body.

Seybert and Rengarajan [1] have pointed out that this numerical failure experienced in the neighbourhood of the characteristic frequencies is due to the coefficient matrix of the approximating equations becoming nearly singular and ill-conditioned, and have used the matrix condition number to recognize the presence of ill-conditioning. Similar remarks regarding the incidence of singularity and ill-conditioning at or near the characteristic frequencies have been made also by other authors [2–4].

These remarks raise certain questions which have not been discussed in the previous papers.

(1) The coefficient matrix is stated to become nearly singular in the vicinity of the characteristic frequencies, implying that it will become singular for certain frequencies. For these frequencies, as is well known from linear algebra, the approximating equations may have either a non-unique numerical solution or no solution at all. Previous papers imply that the coefficient matrix may be singular but make no comments on the question of existence of a numerical solution when singularity prevails. In the present paper the conditions that are necessary for the approximating equations to possess a non-unique

0022-460X/90/100083+15 $03.00/0

solution at the incidence of singularity are derived, and are shown to be satisfied when one uses constant planar boundary elements of equal area and linear Gaussian quadrature for numerical integration. The problem of two-dimensional sound radiation from a radially vibrating infinite cylinder is solved numerically by using boundary elements of this type. Numerical results indicate that the coefficient matrix will not be singular for a finite number of elements.

(2) The failure of numerical solutions in the neighbourhood of the characteristic frequencies is explained by the effects of ill-conditioning; that is, the round-off errors in the calculation of the solution; but no comment has been made on whether or not one will obtain good solutions if the effects of ill-conditioning are annihilated so that solutions can be computed accurately for all frequencies. It should be noted that here the presence and effects of ill-conditioning are not questioned. The coefficient matrix may become severely ill-conditioned for certain frequencies and round-off errors may take over the computations, producing incorrect results, as reported by the authors cited above. What is questioned is whether or not one will obtain good results when solutions can be computed accurately for all frequencies. In this paper this question is studied numerically, again with reference to the problem of two-dimensional sound radiation from a radially vibrating infinite cylinder. For a uniformly vibrating infinite cylinder, the coefficient matrix comes out in the form of a circulant matrix which can be reduced to a single equation which can be solved accurately for all frequencies: that is, in this case one need not be concerned about the effects of ill-conditioning, if any, of the actual coefficient matrix. However, failure of the numerical solutions in the neighbourhood of the characteristic frequencies still takes place, indicating that this problem is *basically* not an ill-conditioning problem but is probably induced by slow convergence of numerical integration at these frequencies.

2. CONDITIONS FOR THE EXISTENCE OF A NON-UNIQUE NUMERICAL SOLUTION

In this section an analytical study is presented of the question of existence of a numerical solution to the approximating equations when the coefficient matrix becomes singular. First, the conditions that are necessary for the existence of a solution when the coefficient matrix is singular are derived, and then the problem of numerical implementation of these conditions is considered.

2.1. GENERAL CONSIDERATIONS

Let a vibrating body occupy a bounded connected domain with a smooth boundary S and unit outward normal n. The normal velocity amplitude v and the sound pressure amplitude p are related by the classical surface Helmholtz integral equation

$$(\pm)p(\mathbf{x})-\int_S [\partial G(R,f)/\partial n(\mathbf{y})]p(\mathbf{y})\,\mathrm{d}s(\mathbf{y})=-\mathrm{i}2\pi f\rho\int_S G(R,f)v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad (1)$$

where $\mathbf{x}$ and $\mathbf{y}$ denote points on S, $R=|\mathbf{x}-\mathbf{y}|$, $\mathrm{i}=\sqrt{(-1)}$, f denotes the frequency and $\exp(-\mathrm{i}2\pi ft)$ time dependence is assumed. $\mathrm{d}s$ denotes a differential boundary element, $G(R,f)$ is the free-space Green function, i.e., $G=\exp(\mathrm{i}2\pi fR/c)/2\pi R$ in three dimensions, ρ is the density of the medium at rest and c is the speed of sound; the plus sign applies for the exterior problems and the minus sign for the interior problems. If the normal velocity v is prescribed on S (the Neumann boundary condition) then equation (1) takes the form of an inhomogeneous Fredholm integral equation of the second type in p. On the other hand, if the sound pressure p is prescribed on S (the Dirichlet boundary condition), equation (1) takes the form of an inhomogeneous Fredholm integral equation of the first type in v. Alternatively, for the Dirichlet boundary condition, the surface Helmholtz integral equation can be expressed also in the form of a Fredholm integral equation of the second type in v, which is needed here in its homogeneous form: namely,

$$(\pm)v(\mathbf{x})+\int_S [\partial G(R,f)/\partial n(\mathbf{x})]v(\mathbf{y})\,\mathrm{d}s(\mathbf{y})=0. \tag{2}$$

For the exterior problem with the Neumann boundary condition, say $v=u$ on S, discretization of equation (1) by using boundary elements, as shown in section 2.2, yields a set of linear algebraic equations which may be written as

$$[\mathbf{E}-\mathbf{K}(f)]\mathbf{p}=\mathbf{H}(f)\mathbf{u}, \tag{3}$$

where $\mathbf{p}$ and $\mathbf{u}$ are, respectively, the nodal sound pressure and the prescribed nodal normal velocity vectors. Matrices $\mathbf{E}, \mathbf{K}(f)$ and $\mathbf{H}(f)$ are of size $N\times N$, where N is the total number of collocation points (nodes). $\mathbf{K}(f)$ corresponds to the integral operator on the left side of equation (1) and $\mathbf{H}(f)$ to that on its right side. As to the definition of matrix $\mathbf{E}$, a brief digression is in order. Even though the actual boundary is assumed to be smooth, numerical implementation of the boundary element method in general involves some kind of geometric discretization which can introduce edges and corners on the working (approximating) boundary. Then, equation (1) does not apply for a collocation point, if any, that lies on such an edge or corner. The correct equation to use then is the general form of equation (1), where the coefficient of $p(\mathbf{x})$ is given by the outer solid angle divided by 2π at point $\mathbf{x}$ [1]. For this reason, matrix $\mathbf{E}$ in equation (4) in general denotes a diagonal matrix which reduces to a unit matrix if the edges and corners on the working boundary contain no collocation points.

Now, obviously, if matrix $\mathbf{E}-\mathbf{K}(f)$ becomes singular for certain f, say, f_D, then equation (3) may not possess a solution. From linear algebra, a necessary and sufficient condition for equation (3) to have a solution when matrix $\mathbf{E}-\mathbf{K}(f)$ is singular is

$$\mathbf{u}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}(f_D)\mathbf{w}=\mathbf{0}, \tag{4}$$

where the superscript T denotes matrix transpose and $\mathbf{w}$ is the non-trivial solution of the homogeneous equation

$$[\mathbf{E}-\mathbf{K}^{\mathrm{T}}(f_D)]\mathbf{w}=\mathbf{0}. \tag{5}$$

If equation (4) is satisfied, then equation (3) for $f=f_D$ will be a consistent (or, compatible) system of equations, and can be solved to obtain a non-unique solution by assigning n of the unknowns arbitrarily, where n is the nullity of matrix $\mathbf{E}-\mathbf{K}(f_D)$ [5]. To examine the conditions under which equation (3) can possess a non-unique solution for $f=f_D$, it is convenient to consider the interior problem for the same boundary with the homogeneous Dirichlet boundary condition, $p=0$ on S, and proceed by expressing these conditions in the form of assumptions, as follows.

Assumption 1. Numerical implementation of the boundary element method does not involve any surface approximation which introduces edges and/or corners that contain collocation points.

Under this assumption, the normal derivative of G will be uniquely defined at all collocation points and, therefore, equation (2) for the interior problem can be discretized by making exactly the same approximations that have been made in the derivation of equation (3). Hence, equation (2) will yield N equations, which may be written as

$$[\mathbf{E}-\mathbf{L}(f)]\mathbf{v}=\mathbf{0}, \tag{6}$$

where $\mathbf{v}$ is the unknown nodal normal velocity vector. Note that, with assumption 1, matrix $\mathbf{E}$ reduces to a unit matrix.

Assumption 2. This assumption is

$$\mathbf{L}(f)=\mathbf{K}^{\mathrm{T}}(f). \tag{7}$$

With this assumption, equation (6) takes the form of equation (5). Similarly, discretization of equation (1) for the interior problem with the homogeneous Dirichlet boundary condition gives the following set of N equations:

$$\mathbf{H}(f)\mathbf{v}=\mathbf{0}. \tag{8}$$

Assumption 3. This is

$$\mathbf{H}(f)=\mathbf{H}^{\mathrm{T}}(f). \tag{9}$$

Under assumption 2, it is clear that the characteristic values of equation (6) are given by $f=f_D$ and, therefore, a non-trivial solution of equation (6) is also a solution of equation (5). Then, with assumption 3, this solution will satisfy equation (4) if the final assumption is made.

Assumption 4. Equations (6) and (8) have identical characteristic values.

This should be true approximately if the boundary element mesh is sufficiently fine.

In summary, assumptions 1–4 express the conditions required for matrix $\mathbf{E}-\mathbf{K}(f)$ to have a non-unique solution when it becomes singular. The question of how to discretize equation (1) so that these conditions will hold true is considered in the next section.

It should be noted that in the foregoing analysis, it is tacitly assumed that equation (4) is satisfied by virtue of the condition $\mathbf{H}^{\mathrm{T}}(f_D)\mathbf{w}=\mathbf{0}$. However, equation (4) may also be satisfied if $\mathbf{u}$ happens to be such that $\mathbf{H}(f_D)\mathbf{u}=\mathbf{0}$, or if $\mathbf{w}^{\mathrm{T}}\mathbf{H}(f_D)\mathbf{u}=\mathbf{0}$ but $\mathbf{H}^{\mathrm{T}}(f_D)\mathbf{w}\neq\mathbf{0}$ or $\mathbf{H}(f_D)\mathbf{u}\neq\mathbf{0}$. In both cases, equation (3) will have a non-unique solution for $f=f_D$ with no additional conditions and assumptions 1–4 are not required. These possibilities, however, are concerned with some very special forms of $\mathbf{u}$ and are noted here only for the sake of completeness.

2.2. RESTRICTIONS ON NUMERICAL IMPLEMENTATION OF THE BOUNDARY ELEMENT METHOD

Assumptions 1–3 place some severe restrictions on the numerical implementation of the boundary element method. For example, assumption 1 will not be satisfied with isoparametric elements which have nodes on their boundaries unless a special technique is employed for smooth surface approximation. However, as will be shown in this section, the more critical restrictions are imposed by assumptions 2 and 3.

To begin with, assume that assumption 1 is valid and express equation (1) for the exterior problem with the Neumann boundary condition, $v=u$, on S, in the following form which is more convenient for the discussion of its discretization: namely,

$$p(\mathbf{x})-\int_S(-\partial G/\partial R)\cos\gamma(\mathbf{y},\mathbf{x})p(\mathbf{y})\,\mathrm{d}s(\mathbf{y})=-\mathrm{i}2\pi f\rho\int_S G(R,f)u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}). \tag{10}$$

Here $\gamma(\mathbf{y},\mathbf{x})$ denotes the angle between the unit normal vector $\mathbf{n}(\mathbf{y})$ and the vector $\mathbf{x}-\mathbf{y}$. In the numerical implementation of the boundary element method [6], the boundary is divided into a number of elements and the parametric equations of an element are written exactly or, as is more usual, for a working (approximating) boundary which is formulated by a polynomial interpolation. Sound pressure and normal velocity are collocated at a number of points. The collocation points which belong to an element are called the nodes of that element. The distributions of p and u over a boundary element are interpolated in terms of their values at the element nodes by using a polynomial expansion which, in the case of isoparametric elements, is of the same order as that used for geometric discretization. Finally, integration of equation (10) numerically, always by Gaussian quadrature in this paper, gives

$$p_c - \sum_e \sum_g [-G'(R^e_{cg}, f)] \cos [\gamma(\mathbf{y}^e_g, \mathbf{y}_c)] w^e_g J(\mathbf{y}^e_g) \mathbf{q}^e_g \mathbf{p}^e = -\mathrm{i}2\pi f \rho \sum_e \sum_g G(R^e_{cg}, f) w^e_g J(\mathbf{y}^e_g) \mathbf{q}^e_g \mathbf{u}^e, \tag{11}$$

where $e, g = 1, 2, \ldots$ and $c = 1, 2, \ldots, N$; superscript e refers to the elements, subscript g to the integration points (the Gauss points) and the subscript c to the collocation points (the nodes); $\mathbf{y}$ denotes a point on the boundary, $R_{cg} = |\mathbf{y}_c - \mathbf{y}_g|$, J is the Jacobian of the parametric representation of the boundary, $\mathbf{q}$ is a row matrix, the elements of which are the shape functions [6], $\mathbf{p}^e$ and $\mathbf{u}^e$ denote the element nodal pressure and nodal normal velocity vectors, respectively, w denotes the numerical integration weights and $G' = \partial G/\partial R$. In matrix notation, equations (11) for $c = 1, 2, \ldots, N$ can be expressed in the form of equation (3).

Similarly, equation (2) for the interior problem can be expressed as

$$v(\mathbf{x}) - \int_S (-\partial G/\partial R) \cos \gamma(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = 0, \tag{12}$$

where $\gamma(\mathbf{x}, \mathbf{y})$ denotes the angle between the vectors $\mathbf{n}(\mathbf{x})$ and $\mathbf{y} - \mathbf{x}$. Since assumption 1 is supposed to be valid, equation (12) can be discretized by using the same element mesh and the same integration rule to obtain

$$v_c - \sum_e \sum_g [-G'(R^e_{cg}, f)] \cos [\gamma(\mathbf{y}_c, \mathbf{y}^e_g)] w^e_g J(\mathbf{y}^e_g) \mathbf{q}^e_g \mathbf{v}^e = 0. \tag{13}$$

For $c = 1, 2, \ldots, N$ this equation can be expressed in matrix notation as equation (6).

First, consider matrix $\mathbf{H}(f)$, the elements of which are given by the expression on the right side of equation (11). By assumption 3, this matrix is required to be symmetric. Since for an arbitrarily shaped body, the distances R_{cg} may all be distinct and non-related, a necessary condition for matrix $\mathbf{H}(f)$ to be symmetric is the following: *restriction* 1; the collocation points must all be coincident with the integration points.

Now, assume that restriction 1 is valid and let $e_1, e_2, \ldots$ be the collocation points for element e, and $c_1, c_2, \ldots$ those for element c. For simplicity, one may assume, without loss of generality, that elements have no nodes on their boundaries. As can be shown by writing the right side of equation (11) explicitly, an element of matrix $\mathbf{H}(f)$ which is located on the c_ith row and e_jth column, say $H_{c_ie_j}$, is given by a linear combination of $G(R^e_{c_ie_1}, f)$, $G(R^e_{c_ie_2}, f), \ldots$ and the transposal element, $H_{e_jc_i}$, by a linear combination of $G(R^c_{e_jc_1}, f)$, $G(R^c_{e_jc_2}, f), \ldots$. $\mathbf{H}(f)$ will be asymmetric if these two linear combinations are equal. For an arbitrarily shaped body, this can be true if, and only if, restriction 2 holds: *restriction* 2; boundary elements must have only one collocation point(node).

An element with a single node, generally known as a constant element, is characterized by a constant shape function giving a constant distribution of pressure and normal velocity, defined by their values at the element node which, because of restriction 1, shall at the same time be the integration point.

Under restrictions 1 and 2, one can use only a linear integration rule such as linear Gaussian quadrature for numerical integration over the elements. For this reason, assumption 1 shall always be valid if restrictions 1 and 2 are valid because for linear Gaussian quadrature the integration point is located in the "middle" of an element.

With these implications of restrictions 1 and 2, that is, constant boundary elements and linear Gaussian quadrature, equation (11) may be written as

$$p_c - \sum_e [-G'(R_{ce}, f)] \cos[\gamma(\mathbf{y}_e, \mathbf{y}_c)] J(\mathbf{y}_e) w_e p_e = -\mathrm{i}2\pi f \rho \sum_e G(R_{ce}, f) J(\mathbf{y}_e) w_e u_e. \quad (14)$$

In writing this equation, subscript g is replaced by e and the superscript e is dropped when superflous. Comparing equations (3) and (14) shows that the elements of matrices $\mathbf{K}(f)$ and $\mathbf{H}(f)$, say K_{ce} and H_{ce}, respectively, may be written as

$$K_{ce} = -G'(R_{ce}, f) \cos[\gamma(\mathbf{y}_e, \mathbf{y}_c)] J(\mathbf{y}_e) w_e, \qquad H_{ce} = -\mathrm{i}2\pi f \rho G(R_{ce}, f) J(\mathbf{y}_e) w_e, \quad (15, 16)$$

where $c, e = 1, 2, \ldots, N$. Therefore, since $R_{ce} = R_{ec}$, matrix $\mathbf{H}(f)$ will be symmetric if

$$J(\mathbf{y}_e) w_e = \text{constant}. \quad (17)$$

The product $J(\mathbf{y}_e) w_e$ gives the area of an element exactly if the element is planar and approximately if it is non-planar. For this reason, equation (17) will be satisfied if restriction 3 holds: *restriction* 3; boundary elements must be planar and all have equal area.

Under restrictions 1-3, matrix $\mathbf{H}(f)$ will be symmetric.

Now, consider assumption 2 which requires that $\mathbf{L}(f) = \mathbf{K}^{\mathrm{T}}(f)$. With restrictions 1 and 2 assumed to be satisfied, equation (13), after simplification of the indices as described for equation (14), can be expressed as

$$v_c - \sum_e [-G'(R_{ce}, f)] \cos[\gamma(\mathbf{y}_c, \mathbf{y}_e)] J(\mathbf{y}_e) w_e v_e = 0. \quad (18)$$

By comparing this with equation (6), an element which is located on the cth row and eth column of matrix $\mathbf{L}(f)$, say L_{ce}, can be seen to be

$$L_{ce} = -G'(R_{ce}, f) \cos[\gamma(\mathbf{y}_c, \mathbf{y}_e)] J(\mathbf{y}_e) w_e. \quad (19)$$

From equations (15) and (19) it follows that if restriction 3 is valid, too, then $L_{ce} = K_{ec}$, as required by assumption 2.

In summary, it has been shown that assumptions 1–3 shall be valid if one employs constant planar elements of equal area and linear Gaussian quadrature for numerical integration. Here it is important to note that, under restrictions 2 and 3, restriction 1 is not required when integrating over the source element; that is, the element that contains point $\mathbf{y}_c$ in equations (11) or (14). This follows because, with constant elements, the source element contributes only to the diagonal terms of matrices $\mathbf{K}(f)$, $\mathbf{H}(f)$ and $\mathbf{L}(f)$ and, with planar elements, the angle γ for any two points on the same element is 90 degrees. Therefore, the transposal relationships $\mathbf{L}(f) = \mathbf{K}^{\mathrm{T}}(f)$ and $\mathbf{H}(f) = \mathbf{H}^{\mathrm{T}}(f)$ will not be violated if one uses a higher order Gaussian quadrature when integrating over the source element. This is an extremely useful feature because linear Gaussian quadrature gives rise to singular kernels when integrating over the source element. This problem can thus be handled simply by switching over to a suitable higher order Gaussian quadrature.

3. ANALYSIS OF NUMERICAL SOLUTIONS FOR ACOUSTIC RADIATION FROM A RADIALLY VIBRATING INFINITE CYLINDER

The purpose of this section is two fold. First is the examination of the incidence of singularity in the coefficient matrix. In the previous section it has been tacitly assumed that the coefficient matrix will be singular for certain frequencies. In this section an attempt is made to determine these frequencies numerically.

The second purpose of the section is to study the behaviour of the numerical solutions in the vicinity of the characteristic frequencies.

A prerequisite for these studies is the ability to compute the solution to the approximating equation accurately for all frequencies. If the answers are accurate then it is not necessary to be concerned about ill-conditioning. However, it is difficult to tell if a solution is accurate by looking at it. Knowledge of the presence of ill-conditioning can put one on guard against the possible effects of ill-conditioning. The presence of ill-conditioning can be recognized by computing the matrix condition number, but this is not an error-proof test because the matrix condition number may falsely indicate an inaccurate solution. For this reason, in this section, an approach is adopted whereby the effects of ill-conditioning, if any, can be completely annihilated by simplification of the approximating equations. Such an approach is possible in two-dimensional sound radiation from a uniformly radially vibrating infinite cylinder. This problem is solved in this section by using constant planar boundary elements of equal area and linear Gaussian quadrature for numerical integration, so that the conditions derived in the previous section for the existence of a solution to the approximating equation at the incidence of singularity are satisfied.

3.1. DISCRETIZATION IN TWO-DIMENSIONAL PROBLEMS

In two dimensions, the free-space Green function is given by [7]

$$G(R,f)=(\mathrm{i}/2)\mathrm{H}_0^{(1)}(kR), \tag{20}$$

where $k=2\pi f/c$ is the wavenumber and $\mathrm{H}_n^{(1)}$ denotes the Hankel function of the first kind of order n. The boundary can be conceived as a closed curve in a plane and, therefore, planar elements can be represented by two-dimensional lineal elements. The parametric equations of a lineal element e may be written $\mathbf{y}_1=\mathbf{q}(\xi)\mathbf{y}_1^e$ and $\mathbf{y}_2=\mathbf{q}(\xi)\mathbf{y}_2^e$, where ξ denotes the local element co-ordinate $(-1\leqslant\xi\leqslant 1)$, $\mathbf{q}(\xi)=[1-\xi \;\; 1+\xi]/2$, the subscripts 1 and 2 refer to the rectangular co-ordinates of any point on the lineal element; $\mathbf{y}_j^e=\{y_j^1 \;\; y_j^2\}$, $j=1,2$, where the superscripts 1 and 2 refer to the ends of the element. Thus, the co-ordinates of any point on element e may be written as $\mathbf{y}=[y_1 \;\; y_2]=\mathbf{q}(\xi)\mathbf{y}^e$, where $\mathbf{y}^e=[\mathbf{y}_1^e \;\; \mathbf{y}_2^e]$. Then, the co-ordinates of the collocation point on element e will be $\mathbf{y}_c=\mathbf{q}(0)\mathbf{y}^e$ and the angle γ associated with elements e and c can be shown to be given by

$$\cos[\gamma(\mathbf{y}_c,\mathbf{y}_e)]=\mathbf{y}_e'\mathbf{P}\mathbf{y}_{ce}^{\mathrm{T}}/R_{ce}J(\mathbf{y}_e), \tag{21}$$

where $\mathbf{y}_{ce}=\mathbf{y}_c-\mathbf{y}_e$, the prime (′) denotes differentiation with respect to ξ, and

$$\mathbf{P}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

For linear Gaussian quadrature $J(\mathbf{y}_e)=b/2$, where b denotes the length of a lineal element. Hence, the elements of matrices $\mathbf{K}(f)$ and $\mathbf{H}(f)$, equations (15) and (16), respectively, may be written as

$$K_{ce}=\mathrm{i}k\mathrm{H}_1^{(1)}(kR_{ce})\mathbf{y}_e'\mathbf{P}\mathbf{y}_{ce}^{\mathrm{T}}/R_{ce}, \qquad H_{ce}=(2\pi f\rho b/2)\mathrm{H}_0^{(1)}(kR_{ce}). \tag{22, 23}$$

As discussed at the end of section 2.2, in order to circumvent the appearance of singular kernels, equations (22) and (23) should be used for $c\neq e$ and the diagonal elements should be computed by using a higher order Gaussian quadrature. By starting with equation (11) and assuming constant planar elements but a non-linear Gaussian quadrature, the diagonal elements of matrices $\mathbf{K}(f)$ and $\mathbf{H}(f)$ can be shown to be

$$K_{ee}=0, \qquad H_{ee}=(2\pi f\rho b/4)\sum_g \mathrm{H}_0^{(1)}(kR_{eg}^e)w_g^e, \tag{24, 25}$$

where the subscript g refers to the integration points. These expressions are derived by assuming that the integration points $g=1,2,\ldots$ do not coincide with the collocation point on the source element. This condition can be satisfied by using a Gaussian quadrature formula that requires an even number of integration points.

3.2. NUMERICAL RESULTS FOR A RADIALLY VIBRATING INFINITE CYLINDER

Consider the problem of two-dimensional sound radiation from a radially vibrating infinite cylinder. In two dimensions, the boundary will be in the form of a circle and equations (22)–(25) can be applied by approximating the circle by a regular polygon of N equal sides, inscribed inside the circle. Then, as can be deduced from the rotational symmetry of distances R_{ce} and angles $\gamma(\mathbf{y}_c, \mathbf{y}_e)$, matrices $\mathbf{K}(f)$ and $\mathbf{H}(f)$ will come out as symmetrical circulant matrices.

A circulant square matrix $\mathbf{A}$ of size N is of the form

$$\mathbf{A}=\begin{bmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & a_1 & \cdots & a_{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}. \tag{26}$$

The determinant of a circulant matrix, called a circulant, can be computed without recourse to the standard co-factor expansion procedures. Circulant $|\mathbf{A}|$ can be shown to be given by the product $|\mathbf{A}|_1|\mathbf{A}|_2 \cdots |\mathbf{A}|_N$, where [5]

$$|\mathbf{A}|_j = a_1 + a_2 U_j + a_3 U_j^2 + \cdots + a_N U_j^{N-1}. \tag{27}$$

Here, U_j denotes an Nth root of unity. Note that one of the circulant factors will be equal to the sum of the elements in any row or column of the circulant. For ease of reference, this factor will be referred to as the fundamental factor and denoted by $[\mathbf{A}]^+$ for a circulant $|\mathbf{A}|$.

If the cylinder is vibrating with a uniform radial velocity of amplitude u_0, then $u = u_0\{1\ 1\ \cdots\ 1\}$ and the right side of equation (3) becomes

$$\mathbf{H}(f)\mathbf{u} = [\mathbf{H}(f)]^+ u_0\{1\ 1\ \cdots\ 1\}, \tag{28}$$

where $\{\ \}$ denotes a column vector of N elements. Since matrix $\mathbf{E}-\mathbf{K}(f)$ is also a circulant matrix, it follows that the solution to equation (3) can be expressed as $\mathbf{p} = p_0\{1\ 1\ \cdots\ 1\}$, where

$$p_0 = u_0[\mathbf{H}(f)]^+/(1-[\mathbf{K}(f)]^+). \tag{29}$$

Thus, the solution to equation (3) can be computed accurately for all frequencies by using this single simple equation.

The real and imaginary parts of the normalized surface pressure, p_0/p_{ex}, computed by using equation (29) with 100 and 800 elements are shown in Figures 1 and 2, respectively, as functions of the non-dimensional wavenumber ka, where a denotes the radius of the cylinder. The exact surface pressure, p_{ex}, which can be derived from the fundamental equations given in reference [7], can be shown to be

$$p_{ex} = -\mathrm{i}\rho c u_0 \mathrm{H}_0^{(1)}(ka)/\mathrm{H}_1^{(1)}(ka). \tag{30}$$

The accuracy of the surface pressures computed by using equation (29) can be assessed from the real and imaginary parts of p_0/p_{ex}: the closer is the real part to unity and the imaginary part to zero, the more accurate is the computed surface pressure.

As can be seen from Figures 1 and 2, equation (29) gives accurate results in the frequency range considered except in the neighbourhood of $ka = 2{\cdot}4$ and $ka = 5{\cdot}52$, which are approximately the first two zeros of the Bessel function of order zero, which are in turn the first two characteristic wavenumbers for the associated interior problem [7]. As

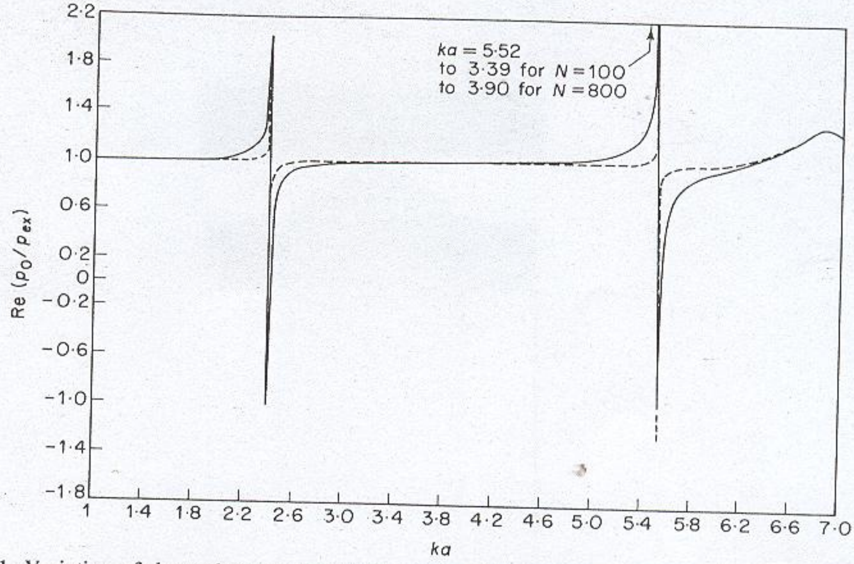

Figure 1. Variation of the real part of the normalized surface pressure, computed by using equation (29), with the dimensionless wavenumber ka; ——, 100 elements; - - -, 800 elements (constant lineal elements with one Gauss point).

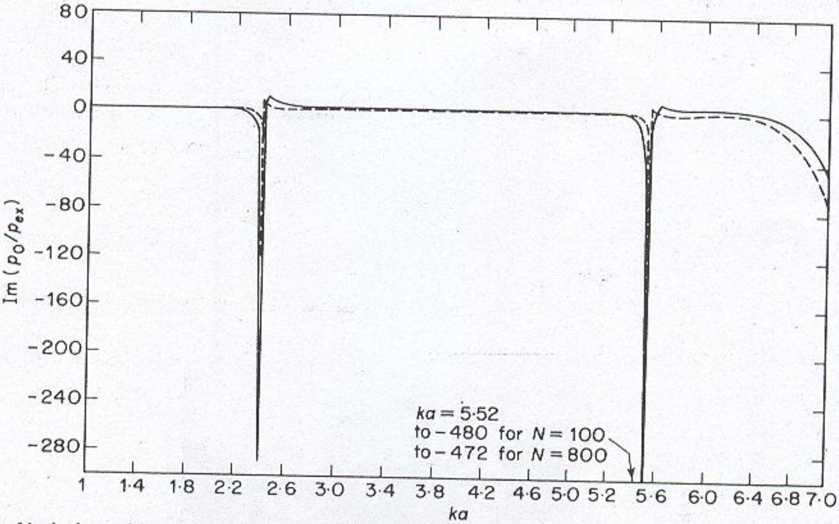

Figure 2. Variation of the imaginary part of the normalized surface pressure, computed by using equation (29), with the dimensionless wavenumber ka. Key as Figure 1.

the number of elements is increased, accurate solutions are obtained in a larger frequency range but the error peaks continue to persist and become sharper. The effect of the number of elements can be seen more clearly in Figure 3, which gives a zoomed view of the first critical region around $ka = 2{\cdot}4$ for the real part of the normalized surface pressure. In this figure, in addition to the results computed by using 100 and 800 elements, some intermediate results for 200 and 400 elements are also shown. A similar behaviour is observed, but not shown here, at the second critical region around $ka = 5{\cdot}52$. Similar numerical phenomena which may be occurring at greater values of ka are not investigated in this paper.

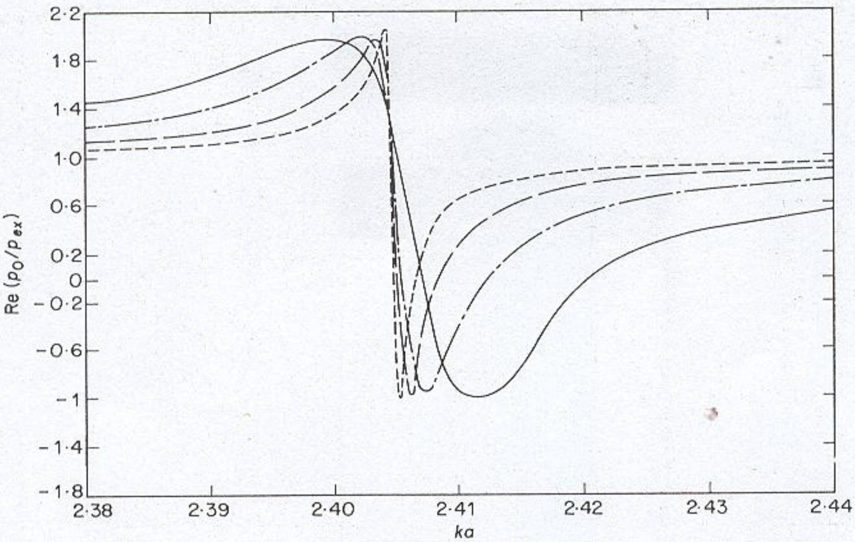

Figure 3. A zoomed view of Figure 1 in the close vicinity of $ka = 2{\cdot}405$; ——, 100 elements; —·—, 200 elements; — —, 400 elements; - - -, 800 elements (constant lineal elements with one Gauss point).

In Figure 4 the zeros of the real and imaginary parts of circulant $|\mathbf{E} - \mathbf{K}(f)|$ that lie in the vicinity of $ka = 2{\cdot}4$ are given as functions of the number of elements. In this range of wavenumbers the real and imaginary parts of this circulant vanish by virtue of the vanishing of its fundamental factor, and the zeros given in Figure 4 were calculated by linear interpolation as the fundamental factor changed sign in a sufficiently narrow frequency interval. When matrix $\mathbf{E} - \mathbf{K}(f)$ becomes singular, both the real and imaginary parts of its fundamental factor must vanish simultaneously. In Figure 4 it is shown that the real and imaginary parts of the fundamental factor have distinct zeros for a finite

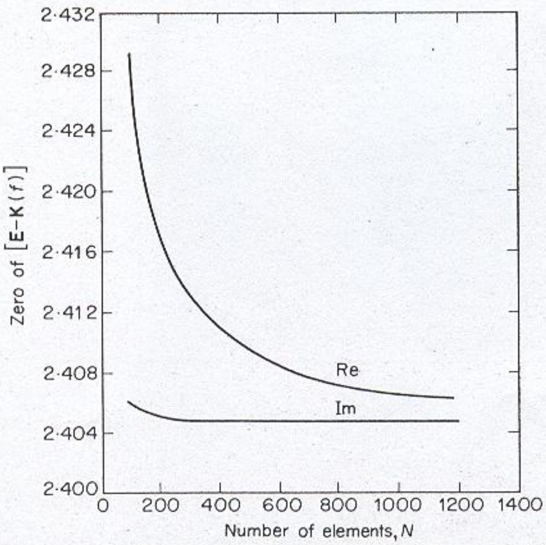

Figure 4. Variation of the zeros of the real and imaginary parts of circulant $|\mathbf{E} - \mathbf{K}(f)|$ with number of elements in the close vicinity of $ka = 2{\cdot}4$ (constant lineal elements with one Gauss point).

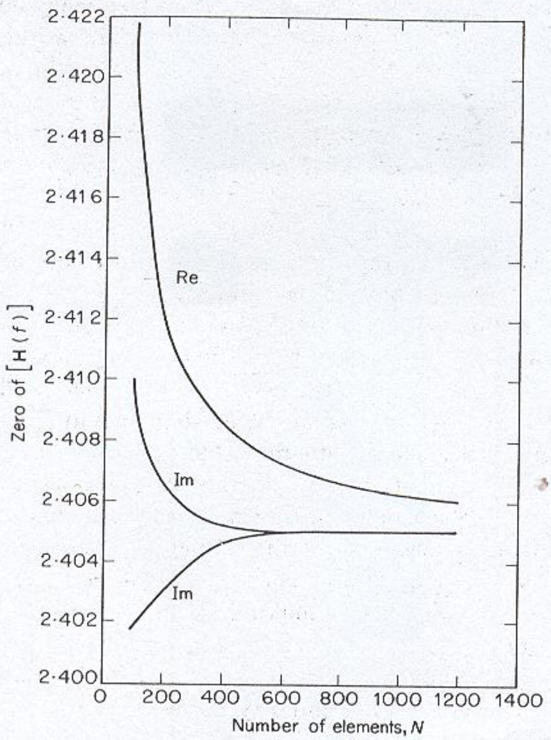

Figure 5. Variation of the zeros of the real and imaginary parts of circulant $|\mathbf{H}(f)|$ with number of elements in the close vicinity of $ka = 2{\cdot}4$ (constant lineal elements with one Gauss point).

number of elements. However, they tend to have a common zero ultimately as the number of elements increases to infinity.

So far, assumption 4 of section 2.1 has been tacitly assumed to be valid throughout the analysis. This assumption requires that when matrix $\mathbf{E}-\mathbf{K}(f)$ is singular, so should be matrix $\mathbf{H}(f)$. The circulant $|\mathbf{H}(f)|$ vanishes also by virtue of the vanishing of its fundamental factor. The zeros of the real and imaginary parts of $[\mathbf{H}(f)]^+$ are shown in Figure 5 as functions of the number of elements, for ka close to 2·4. As can be seen, like matrix $\mathbf{E}-\mathbf{K}(f)$, matrix $\mathbf{H}(f)$, too, tends to be singular ultimately. Note that, near $ka = 2{\cdot}4$, the imaginary part of the fundamental factor has two zeros which converge to the same value as the number of elements is increased. The curves given in Figures 4 and 5 display a tendency to become asymptotic to the value of $ka = 2{\cdot}405$, which is a more accurate value for the smallest exact zero of the Bessel function of zero order than the value of 2·4.

4. DISCUSSION AND CONCLUSIONS

The present analysis has shown that the approximating equations may have a non-unique numerical solution at the incidence of singularity, provided that they are derived by using constant planar elements of equal area and linear Gaussian quadrature for numerical integration. However, it is important to note that this result holds only for a finite number of boundary elements. If the number of elements used is infinitely large, then the approximating equations shall become an exact representation of the exterior surface Helmholtz integral equation irrespectively of the type of boundary elements used in its discretization. Now, it is well known [8] that the exterior surface Helmholtz integral equation possesses a non-unique exact solution when frequency equals a characteristic frequency associated with the interior of the body. Therefore, as the number of elements tends to infinity, the approximating equations, like the parent integral equation, should ultimately possess a non-unique numerical solution at the incidence of singularity, irrespectively of the type of boundary elements used in the discretization.

Then, the following question arises: is it possible that the coefficient matrix may become singular for a finite number of boundary elements? Of course, this question is particularly relevant if the approximating equations are known to possess a solution at the incidence of singularity, as is the case when they are derived by using constant planar elements of equal area and linear Gaussian quadrature. Implementation of this type of boundary element discretization to the problem of two-dimensional sound radiation from a radially vibrating infinite cylinder has shown that the coefficient matrix does not become singular for a finite number of elements. Nevertheless, as the number of elements is increased, the coefficient matrix *converges to singularity*, as depicted in Figure 4, confirming the non-uniqueness feature of the parent integral equation.

This conclusion has been confirmed by further computations carried out by using isoparametric elements in place of the constant lineal elements in the solution of the infinite cylinder problem, as described in the Appendix. As an example, Figure 6 is a counterpart of Figure 4 for the quadratic isoparametric element [6] with two integration points. These results were computed by making use of the properties of block-circulant matrices given in the Appendix and are, therefore, free from the effects of ill-conditioning, if any, of the actual coefficient matrix. As can be seen from Figure 6, with isoparametric quadratic elements, convergence to singularity is faster than that with constant planar elements.

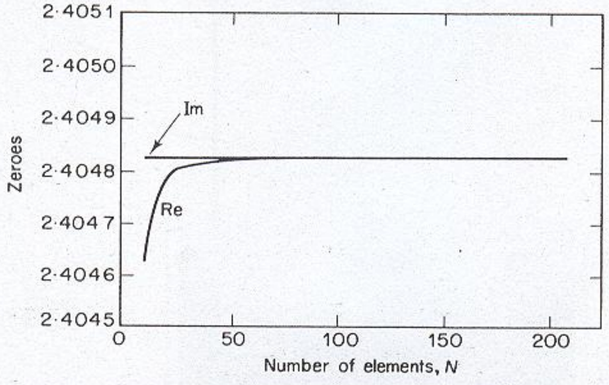

Figure 6. Variation of the zeros of the real and imaginary parts of circulant $|\mathbf{E}-\mathbf{K}(f)|$ with number of elements in the close vicinity of $ka = 2{\cdot}4$ (quadratic isoparametric elements with two Gauss points).

The demonstration of the fact that the coefficient matrix is always non-singular for a finite number of elements is believed to be a contribution of the present paper. An interesting implication of this is that the characteristic frequencies associated with the interior of the body cannot be determined, when using constant planar elements, as non-trivial solutions of equation (6) because these solutions correspond to the states of singularity of the coefficient matrix. However, the characteristic frequencies can be estimated by following the convergence of the zeros of the real and imaginary parts of the coefficient matrix determinant, as has been noted above with reference to Figure 4.

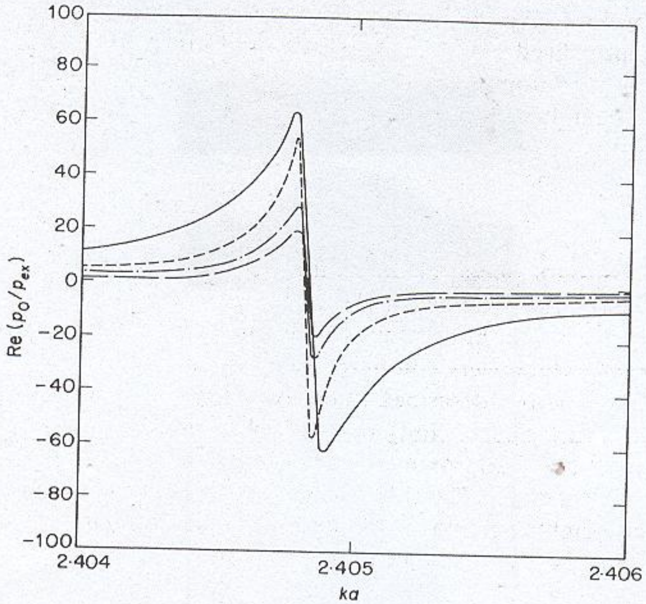

Figure 7. Variation of the real part of the normalized surface pressure, computed by using equation (A2), with the dimensionless wavenumber *ka* in the close vicinity of $ka = 2{\cdot}405$. 60 linear isoparametric elements; ——, 2 Gauss points; - - -, four Gauss points; — · —, eight Gauss points; — —, 16 Gauss points.

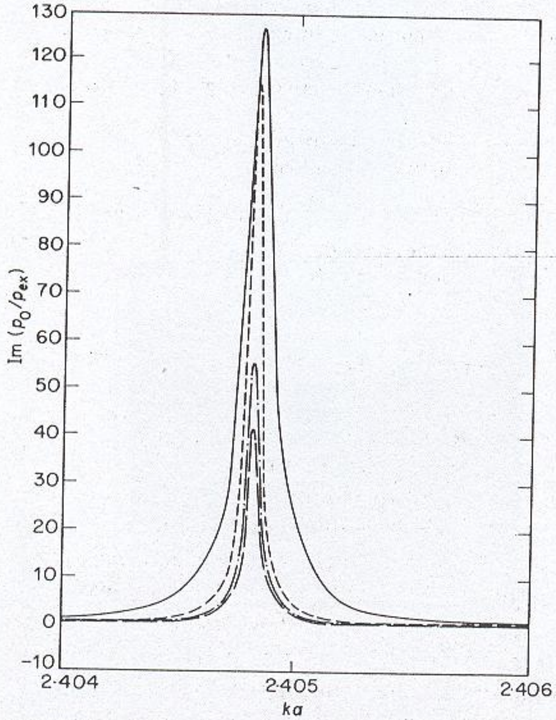

Figure 8. Variation of imaginary part of the normalized surface pressure, computed by using equation (A2), with the dimensionless wavenumber *ka* in the close vicinity of $ka = 2{\cdot}405$. 60 linear isoparametric elements. Gauss point key as Figure 7.

Another contribution of the present analysis has been the demonstration of the fact that the failure experienced in the neighbourhood of the characteristic frequencies is basically not an ill-conditioning problem. Boundary element solutions for the infinite cylinder problem, computed accurately for all frequencies by using equation (29), were found to be grossly in error in the vicinity of the characteristic frequencies. Increasing the number of elements narrows down the range of troublesome frequencies but does not, in general, improve the solution errors in the close vicinity of the characteristic frequencies. Further computations, which were carried out by using isoparametric elements in place of constant planar elements in the infinite cylinder problem have shown that, for a given number of elements, the solution errors in the close vicinity of the characteristic frequencies can be improved by using an improved integration rule. Some representative results, which were computed by using linear isoparametric elements and the block-circulant properties described in the Appendix, are presented in Figures 7 and 8. As can be seen, in the close vicinity of $ka = 2{\cdot}405$, the accuracy of both the real and the imaginary parts of the approximate surface pressure is improved as the order of Gaussian quadrature increases. For this reason, it is believed that the numerical problem experienced in the neighbourhood of the characteristic frequencies is basically more a problem of poor convergence of numerical integration than a problem of ill-conditioning.

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APPENDIX: SOLUTION OF THE INFINITE CYLINDER PROBLEM BY USING ISOPARAMETRIC BOUNDARY ELEMENTS

In two-dimensional acoustic radiation problems, an isoparametric element having M (>1) nodes is obtained by approximation of both the element boundary and the acoustic pressure and the normal velocity distribution over the element boundary by Lagrange interpolation by means of a complete polynomial of order $M-1$. Then, the approximating equations will come out essentially in the form of equation (11) where the free-space Green function is given by equation (20). However, since two of the element nodes have to be located at the ends of the element, it will be necessary to modify equation (11) whenever a collocation point c lies at the junction of two elements, as described in section 2.1. Thus, with an isoparametric element, matrix $\mathbf{E}$ will not normally be a unit matrix.

If the infinite cylinder problem considered in section 3 is discretized by dividing the two-dimensional representation of the boundary into N identical isoparametric elements each having M nodes, matrices $\mathbf{E}-\mathbf{K}(f)$ and $\mathbf{H}(f)$ will come out in the following block-circulant matrix form:

$$\mathbf{A}=\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_N \\ \mathbf{A}_N & \mathbf{A}_1 & \cdots & \mathbf{A}_{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{A}_2 & \mathbf{A}_3 & \cdots & \mathbf{A}_1 \end{bmatrix}. \tag{A1}$$

Here the sub-matrices $\mathbf{A}_i$ are all of size $(M-1)\times(M-1)$. Let $[\mathbf{A}]^+$ denote the sum of the sub-matrices in any one row or column of a block-circulant matrix $\mathbf{A}$.

For a uniformly radially vibrating infinite cylinder, let $\mathbf{u}=\{\mathbf{u}_e \ \mathbf{u}_e \ \cdots \ \mathbf{u}_e\}$, where $\mathbf{u}_e=u_0\{1 \ 1 \ \cdots \ 1\}$ is a vector of size $M-1$. For this case, the solution of equation (3) can be shown to be of the form $\mathbf{p}=\{\mathbf{p}_e \ \mathbf{p}_e \ \cdots \ \mathbf{p}_e\}$, where the vector $\mathbf{p}_e$ is given by

$$[\mathbf{E}-\mathbf{K}(f)]^+\mathbf{p}_e=[\mathbf{H}(f)]^+\mathbf{u}_e. \tag{A2}$$

The elements of the vector $\mathbf{p}_e$ give the acoustic pressure amplitudes at the internal nodes, if there is any, and at one of the end nodes of the element.

The foregoing equation enables the solution to the approximating equations to be obtained by inverting a matrix of size $M-1$. For example, with the well known quadratic isoparametric elements, for which $M=3$, it is necessary only to invert a 2×2 matrix. This N-fold reduction in matrix size enables the computation of the solution to equation (3) accurately for all frequencies.