

A Numerical Method Based on the Boundary Integral Equation and Dual Reciprocity Method for One–dimensional Cahn–Hilliard Equation

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Abstract

This paper describes a numerical method based on the boundary integral equation and dual reciprocity method for solving the one–dimensional Cahn–Hilliard (C–H) equation. A time–stepping method and a predictor–corrector scheme are employed to deal with the time derivative and the nonlinearity respectively. Numerical results are presented for some problems to demonstrate the usefulness and accuracy of this approach. For these test problems the energy functional dissipation and the mass conservation properties are investigated.

Keywords: Boundary integral equation, Dual reciprocity method, Cahn–Hilliard equation, Mass conservation, Energy dissipation.

1 Introduction

The Cahn–Hilliard (C–H) equation [7, 8]

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \Delta \psi'(u) = 0, \quad \gamma > 0, \quad (1.1)$$

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was originally introduced as a phenomenological model of phase separation in a binary systems, such as alloys, glass and polymer mixtures. This equation is very important in materials science (see [7, 8, 25] and the references therein). It is a conservation law, in the sense that the average of the order parameter, which corresponds to a density of atoms, is conserved [33]. Since the pioneering work of Cahn and Hilliard [8], the C–H equation has been extensively applied to the inter-diffusion in binary polymer mixtures by Wang and Shi [41], in interfaces of polymers with dissimilar properties by Jabbari and Peppas [28] and in the formation of interfaces via the inter-diffusion of layers by Puri and Binder [40]. The viscous C–H equation, which arises as a singular limit of the phase field model of phase transition, has been studied by Bai et al [3]. The C–H equation with a logarithmic free energy (ψ contains logarithmic terms) has been considered by Copetti and Elliott [13] and Barrett and Blowery [5], where they also studied finite element approximation of the model equation. Global existence and uniqueness of the solution for (1.1) have been shown by Elliott and Zheng [16] and Yin [48]. Long time behaviour of the solution has been studied by Cart et al [9], Novick-Cohen and Segel [37], and Novick-Cohen [38] using the free energy method. Finite Galerkin approximate solutions have been obtained by Elliott and French [17] and French and Jensen [21] for one–dimensional problems. For multidimensional problems, Elliott and French [18] considered nonconforming finite element method. Elliott and Larsson [19] have obtained error estimates of finite Galerkin solutions for smooth data as well as nonsmooth initial data. Elliott et al [20] have obtained optimal order error bounds using a second order splitting method. Mixed finite element method has been applied to obtain approximate solutions for (1.1) by Dean et al [15]. Furihata et al [22] have examined the boundedness of the solution of a finite difference scheme using discretized Lyapunov functional. Furihata [23] has depicted the dissipation of energy and conservation of mass for this problem. Also this author [24] has proposed a stable and conservative finite difference scheme to solve numerically the C–H equation. Alikakos et al [2] have established rigorously the existence of some extremely slowly evolving solutions. Numerical solutions for the Eq. (1.1) have been considered by Choo and Chung [11] using the Crank–Nicolson type finite difference method for one–dimensional problem and by Choo et al [12] using a nonlinear conservative difference scheme for two–dimensional problem. Another Crank–Nicolson type method has been developed by Khiari et al [31] for (1.1) in one dimension. Ye [45] has employed the Fourier collocation method and Ye and Cheng [47] have developed the Fourier spectral method to solve numerically the C–H equation with periodic boundary conditions. Ye [46] has also used the Legendre collocation method to study this problem with Neumann boundary conditions. Hongjun and Changchun [27] have studied the instability of the traveling waves of the convective–diffusive C–H equation. Mello et al [35] have

presented a stable and fast conservative finite difference scheme to solve (1.1) with two improvements. Danumjaya and Nandakumaran [14] have employed the orthogonal cubic spline collocation method for this equation. Two applications of μ -transform to the C–H equation have been presented by Mitlin [36]. Miranville and Piétrus [33] have obtained a new formulation of the generalizations of the C–H equation based on constitutive equations proposed by Gurtin. A multigrid finite element solver has been presented by Kay and Welford [30], a discontinuous Galerkin finite element method has been developed by Wells *et al* [43] and adaptive finite element methods in two and three dimensions were considered by Bañas and Nürnberg [4] for this problem. He *et al* [26] have analyzed a class of large time-stepping methods for this equation. Kim [32] has considered a conservative nonlinear multigrid method for the C–H equation with a variable mobility. The Adomian decomposition method has been applied for the solution of viscous C–H equation by Momani and Erturk [34]. Xia *et al* [44] have developed the local discontinuous Galerkin (LDG) methods for (1.1) and Abels and Wilke [1] have investigated the asymptotic behavior of the nonlinear C–H equation with a logarithmic free energy and similar singular free energies, etc.

In this research we propose an alternative approach based on the boundary integral equation and dual reciprocity method (DRM) for solving the one–dimensional C–H equation (1.1). The idea behind this approach comes from the DRM introduced by Brebbia and Nardini [6] and Partridge and Brebbia [39] for higher order dimensional problems and it is to expand the inhomogeneous and nonlinear terms in terms of their values at the nodes which lie in the domain of problem. The inhomogeneous and nonlinear terms are approximated by interpolation in terms of some well–known functions $\phi(r)$, called radial basis functions (RBFs), where r is the distance between a source point and the field point. There exist a large class of interpolating RBFs that can be used [29]. These include the linear $1 + r$, the polynomial $P_j(r)$, the thin plate spline (TPS) $r^{2n} \log(r)$, the Gaussian $\exp(-r^2/c^2)$, and the multiquadrics $(c^2 + r^2)^{1/2}$, where c is the constant parameter, etc. In this paper we shall use the linear RBF.

Some authors employed a similar approach for one–dimensional problems. For an example of them see reference [10, 42]. The interested reader can see the Ang’s book [49] for the boundary element method.

This paper is organized as follows: In Section 2, brief discussions for C–H equation are presented. In Section 3 the discretized version of equation is obtained and a predictor–corrector scheme is described to deal with the nonlinearity. In Section 4, numerical results are considered for two test problems and the known properties of equation are tested. Section 5 ends this report with a brief conclusion.

2 Cahn Hilliard equation

Let us consider the following classical Cahn–Hilliard equation (1.1) in one–dimensional

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 \psi'(u)}{\partial x^2} = 0, \quad x \in \Omega, \quad t \geq 0, \quad (2.1)$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \partial\Omega, \quad (2.2)$$

and initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (2.3)$$

where

$$\psi'(u) = \phi(u) = r_2 u^3 + r_1 u^2 + r_0 u, \quad (2.4)$$

and γ is a prescribed positive constant, $\Omega = [a, b]$ and r_0, r_1, r_2 are given constants.

It is known if the initial data $u_0 \in H_E^2(\Omega) = \{f \in H^2(\Omega) : \partial f / \partial x = 0 \text{ on } \partial\Omega\}$ then the problem (2.1)–(2.3) has a unique solution for all times [16].

Let us introduce w , the chemical potential, defined by

$$w = \phi(u) - \gamma u_{xx}, \quad (2.5)$$

then (2.1) may be rewritten as

$$\begin{cases} u_t - w_{xx} = 0, \\ w - \phi(u) + \gamma u_{xx} = 0. \end{cases} \quad (2.6)$$

We have, in view of (2.2) and (2.5), the following boundary condition on w :

$$\frac{\partial w}{\partial x} = 0, \quad x \in \partial\Omega,$$

therefore the boundary conditions

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} = 0, \quad x \in \partial\Omega, \quad (2.7)$$

are equivalent to (2.2).

The initial conditions for u and w are

$$u(x, 0) = u_0(x), \quad w(x, 0) = \phi(u_0) - \gamma \frac{\partial^2 u_0}{\partial x^2}, \quad x \in \Omega. \quad (2.8)$$

The important features of C–H equations are as follows:

(i) For the extended Ginzberg-Landau free energy functional

$$\mathcal{F}(u) = \int_{\Omega} \left[\frac{\gamma}{2} u_x^2 + \psi(u) \right] dx, \quad (2.9)$$

where $\psi(u)$ is a Lyapunov functional (see [16, 17, 19, 45]), $\frac{d}{dt}\mathcal{F}(u) \leq 0$: i.e., the energy is dissipative as time passes.

(ii) The total mass remains constant, i.e:

$$\frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx = \frac{1}{|\Omega|} \int_{\Omega} u(x, 0) dx = \mathcal{M}, \quad t > 0. \quad (2.10)$$

These properties play an important role in analyzing the solution of C–H equation.

3 Mathematical formulation

We use mixed boundary integral equation and dual reciprocity method for the system (2.6)–(2.8) to compute the approximate solutions. In order to derive the integral equation formulations for the problem, we start with the following integral identities of Eq. (2.6):

$$\int_a^b \left[\frac{\partial^2 w}{\partial x^2} - \frac{\partial u}{\partial t} \right] G dx = 0, \quad (3.1)$$

and

$$\int_a^b \left[\gamma \frac{\partial^2 u}{\partial x^2} + w - \phi(u) \right] G dx = 0, \quad (3.2)$$

where the function G is the fundamental solution for the one–dimensional Laplace operator defined by the equation:

$$\frac{\partial^2 G}{\partial x^2}(x, \xi) = \delta(x, \xi), \quad (3.3)$$

where x and ξ are a field point and a source point, respectively and δ is the Dirac delta function. The fundamental solution and its derivative are given as follows:

$$G(x, \xi) = \frac{1}{2}|x - \xi|, \quad (3.4)$$

$$G_{,x}(x, \xi) = G'(x, \xi) = \frac{1}{2}\text{sgn}(x - \xi), \quad (3.5)$$

in which the symbol "sgn" denotes the signum function.

For applying the method, N source points $x_j, j = 1, 2, \dots, N$ are chosen in $[a, b]$ where $a = x_1 < x_2 < \dots < x_{N-1} < x_N = b$ and the following approximations are made:

$$\frac{\partial u}{\partial t} = g(x, t) = \sum_{k=1}^N \mu_k(x) \alpha_k(t), \quad (3.6)$$

$$\frac{1}{\gamma} [\phi(u) - w] = h(x, t) = \sum_{k=1}^N \mu_k(x) \beta_k(t), \quad (3.7)$$

where $\mu_k(x) = 1 + r_k$ is a linear RBF and $r_k = |x - x_k|$ for $k = 1, 2, \dots, N$. From (3.1) and (3.6), we have

$$\int_a^b \frac{\partial^2 w}{\partial x^2} G dx = \sum_{k=1}^N \left[\int_a^b \mu_k(x) G dx \right] \alpha_k(t), \quad (3.8)$$

and from (3.2) and (3.7), we have

$$\int_a^b \frac{\partial^2 u}{\partial x^2} G dx = \sum_{k=1}^N \left[\int_a^b \mu_k(x) G dx \right] \beta_k(t). \quad (3.9)$$

If we represent the value of the function μ_k at source point x_i by μ_{ik} for $i = 1, 2, \dots, N$, and set F as a $N \times N$ matrix that $F(i, k) = \mu_{ik}$ and $E = F^{-1}$, then we have

$$\alpha_k(t) = \sum_{j=1}^N E_{kj} g_j(t), \quad (3.10)$$

and

$$\beta_k(t) = \sum_{j=1}^N E_{kj} h_j(t), \quad (3.11)$$

where $g_j(t) = g(x_j, t)$ and $h_j(t) = h(x_j, t)$. Now consider a set of functions f_k such that

$$\frac{\partial^2 f_k}{\partial x^2} = \mu_k, \quad (3.12)$$

the function f_k is easily determined as:

$$f_k = \frac{1}{2} r_k^2 + \frac{1}{6} r_k^3. \quad (3.13)$$

Substituting (3.12) into (3.8), (3.12) into (3.9) and applying the integration by parts, we can get the following expressions:

$$[G_i(x)p(x, t)]_a^b - [w(x, t)G'_i(x)]_a^b + w_i(t) = \sum_{k=1}^N \left[[G_i f'_k]_a^b - [f_k G'_i]_a^b + f_{ik} \right] \alpha_k(t), \quad (3.14)$$

and

$$[G_i(x)q(x, t)]_a^b - [u(x, t)G'_i(x)]_a^b + u_i(t) = \sum_{k=1}^N \left[[G_i f'_k]_a^b - [f_k G'_i]_a^b + f_{ik} \right] \beta_k(t), \quad (3.15)$$

for $i = 1, 2, \dots, N$, where $p(x, t) = \partial w(x, t)/\partial x$, $q(x, t) = \partial u(x, t)/\partial x$ and $G_i(x) = G(x, \xi)|_{\xi=x_i}$ are the weight functions at the points $i = 1, 2, \dots, N$ which satisfy in Eq. (3.3). $u_i(t) = u(\xi, t)|_{\xi=x_i}$, $w_i(t) = w(\xi, t)|_{\xi=x_i}$, $f_{ik} = f_k(x_i)$ and f'_k is the derivative of f_k and can be expressed as:

$$f'_k = \frac{\partial f_k}{\partial x} = r_k + \frac{1}{2}r_k^2. \quad (3.16)$$

If we set:

$$S_{ik} = [G_i f'_k]_a^b - [f_k G'_i]_a^b + f_{ik}, \quad (3.17)$$

then Eqs. (3.14) and (3.15) take the following forms:

$$[G_i(x)p(x, t)]_a^b - [w(x, t)G'_i(x)]_a^b + w_i(t) = \sum_{k=1}^N S_{ik} \alpha_k(t), \quad (3.18)$$

$$[G_i(x)q(x, t)]_a^b - [u(x, t)G'_i(x)]_a^b + u_i(t) = \sum_{k=1}^N S_{ik} \beta_k(t). \quad (3.19)$$

From (3.10), the right hand side of Eq. (3.18) can be written as:

$$\sum_{k=1}^N S_{ik} \alpha_k(t) = \sum_{k=1}^N S_{ik} \sum_{j=1}^N E_{kj} g_j(t) = \sum_{j=1}^N M_{ij} g_j(t), \quad (3.20)$$

and from (3.11), the right hand side of Eq. (3.19) is:

$$\sum_{k=1}^N S_{ik} \beta_k(t) = \sum_{k=1}^N S_{ik} \sum_{j=1}^N E_{kj} h_j(t) = \sum_{j=1}^N M_{ij} h_j(t), \quad (3.21)$$

where

$$M_{ij} = \sum_{k=1}^N S_{ik} E_{kj}. \quad (3.22)$$

In other hand if:

$$L = \begin{bmatrix} -G_1(a) & G_1(b) \\ -G_2(a) & G_2(b) \\ \vdots & \vdots \\ -G_N(a) & G_N(b) \end{bmatrix}, \quad H = \begin{bmatrix} -G'_1(a) & G'_1(b) \\ -G'_2(a) & G'_2(b) \\ \vdots & \vdots \\ -G'_N(a) & G'_N(b) \end{bmatrix}, \quad (3.23)$$

then Eqs. (3.18), (3.20) and (3.22) yield:

$$[L] \begin{bmatrix} p_1(t) \\ p_N(t) \end{bmatrix} - [H] \begin{bmatrix} w_1(t) \\ w_N(t) \end{bmatrix} + [I] \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_N(t) \end{bmatrix} = [M] \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_N(t) \end{bmatrix}, \quad (3.24)$$

and Eqs. (3.19), (3.21) and (3.22) give:

$$[L] \begin{bmatrix} q_1(t) \\ q_N(t) \end{bmatrix} - [H] \begin{bmatrix} u_1(t) \\ u_N(t) \end{bmatrix} + [I] \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{bmatrix} = [M] \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_N(t) \end{bmatrix}, \quad (3.25)$$

where I is the N by N identity matrix, $p_1(t) = p(a, t)$, $p_N(t) = p(b, t)$, $w_j(t) = w(x_j, t)$, $q_1(t) = q(a, t)$, $q_N(t) = q(b, t)$ and $u_j(t) = u(x_j, t)$. We note that

$$g_j(t) = \frac{\partial u}{\partial t}(x_j, t), \quad (3.26)$$

and

$$h_j(t) = \phi(u(x_j, t)) - w(x_j, t). \quad (3.27)$$

Now, (3.24) and (3.25) constitute a system of $2N$ equations in $2N$ unknown functions of t . Since the Neumann boundary conditions are applied, the unknown variables are given by $u_j(t)$ and $w_j(t)$, for $j = 1, 2, \dots, N$. This system is solved approximately using the iterative scheme which is described in the following.

We make the following approximations:

$$\frac{\partial u}{\partial t} \Big|_j \simeq \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t}, \quad j = 1, 2, \dots, N,$$

$$w_j(t) \simeq \frac{1}{2}[w_j^{(n+1)} + w_j^{(n)}], \quad j = 1, 2, \dots, N, \quad (3.28)$$

$$u_j(t) \simeq \frac{1}{2}[u_j^{(n+1)} + u_j^{(n)}], \quad j = 1, 2, \dots, N,$$

where $u_j^{(n)} = u(x_j, n\Delta t)$ and $w_j^{(n)} = w(x_j, n\Delta t)$. By assuming $\lambda = 1/\Delta t$, $\mathbf{u} = [u_1, u_2, \dots, u_N]^T$ and $\mathbf{w} = [w_1, w_2, \dots, w_N]^T$, (3.24)–(3.28) yield:

$$\begin{aligned} -\frac{1}{2} [H] \left(\begin{bmatrix} w_1 \\ w_N \end{bmatrix}^{(n+1)} + \begin{bmatrix} w_1 \\ w_N \end{bmatrix}^{(n)} \right) + \frac{1}{2} [I] \left([\mathbf{w}]^{(n+1)} + [\mathbf{w}]^{(n)} \right) \\ = \lambda [M] \left([\mathbf{u}]^{(n+1)} - [\mathbf{u}]^{(n)} \right), \end{aligned} \quad (3.29)$$

$$\begin{aligned} -\frac{1}{2} [H] \left(\begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n+1)} + \begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n)} \right) + \frac{1}{2} [I] \left([\mathbf{u}]^{(n+1)} + [\mathbf{u}]^{(n)} \right) \\ = -\frac{1}{2} [M] \left([\mathbf{w}]^{(n+1)} + [\mathbf{w}]^{(n)} \right) + [M] [\Phi], \end{aligned} \quad (3.30)$$

where $\Phi = [\phi(\tilde{u}_1), \phi(\tilde{u}_2), \dots, \phi(\tilde{u}_N)]^T$, \tilde{u}_j is given by the known approximation of $u(x_j, t)$, as described earlier. We note that, due to the zero boundary conditions (2.7) the first terms of left hand sides of Eqs. (3.24) and (3.25) do not appear in (3.29) and (3.30) respectively. At the first time level (when $n = 0$), the initial conditions (2.8) give $u_j^{(0)} = u(x_j, 0)$ and $w_j^{(0)} = w(x_j, 0)$.

In each time level (for example time level $n + 1$) at first we set $\phi(\tilde{u}_j) = \phi(u_j^{(n)})$. Having this, Eqs. (3.29) and (3.30) are solved as a system of linear algebraic equations for unknowns $u_j^{(n+1)}$ and $w_j^{(n+1)}$ for $j = 1, 2, \dots, N$, assuming that $u_j^{(n)}$ and $w_j^{(n)}$, are known from the previous time level. Recompute $\phi(\tilde{u}_j) = \phi(1/2u_j^{(n+1)} + 1/2u_j^{(n)})$, where $u_j^{(n+1)}$ is obtained recently from solving Eqs. (3.29) and (3.30). We iterate between calculating $\phi(\tilde{u}_j)$ and computing the approximate values of the unknowns, until all the unknown quantities converge to within a prescribed number of significant figures, i.e. a predictor–corrector approach is adopted in each time level. Once the prescribed convergence is achieved, we can move on to the following time level. The resulting solutions from the current time level, provide the known values for the next iteration. This process is repeated, until reaching to the desirable time t .

4 Numerical results

To show the efficiency and accuracy of this method, it was applied to various problems. We need to iterate between finding an estimation for $\tilde{\mathbf{u}}$ to recompute Φ and solving the system (3.29)–(3.30) for a new \mathbf{u} that described in previous section. In all the cases considered, the iteration was stopped when the absolute values of all the unknowns from two consecutive iterations differ by less than 10^{-8} . In all cases tested, convergence was achieved after less than 4 iteration.

4.1 Problem 1

In this example, we consider the C–H equation (2.1) in $[0, 2\pi]$ with the initial condition $u_0(x) = 0.5 \cos x$, $\gamma = 0.03$ and $\psi(u) = 0.25u^4 - 0.5u^2$. For numerical solution we set $\Delta t = 0.01$ and $N = 64$ where

$$\{x_i\}_{i=1}^N = \{0, 0.1, 0.2, \dots, 6.1, 6.2, 2\pi\}.$$

In Fig. 1 the evolution from $t = 0$ to $t = 5$ is drawn. In numerical experiments, we observed that the solution $u(x, t)$ for long times remains at patterns that are nearly piecewise constant, and the steady–state solution is very close to a piecewise constant function. In [17] and [45], also the same results were given. In this example, the patterns hardly changed after $t = 3$. Fig. 2 shows the time dependency of energy functional (2.9) of numerical solutions. Due to the energy functional dissipation property, the energy functional of numerical solutions theoretically decreases as time passes. Ye [45] also obtained the same graph for the energy. As described in Section 2 for the C–H equation, the total mass remains constant as time increases. In Table 1 the values of \mathcal{M} are shown for different time levels up to $t = 5$ where the initial mass is zero. Here the evaluation of energy and mass were performed using the composite trapezoidal rule for integration.

Table 1
The values of \mathcal{M} at different time levels

Time t	t=1.0	t=2.0	t=3.0	t=4.0	t=5.0
Mass	1.6777×10^{-6}				

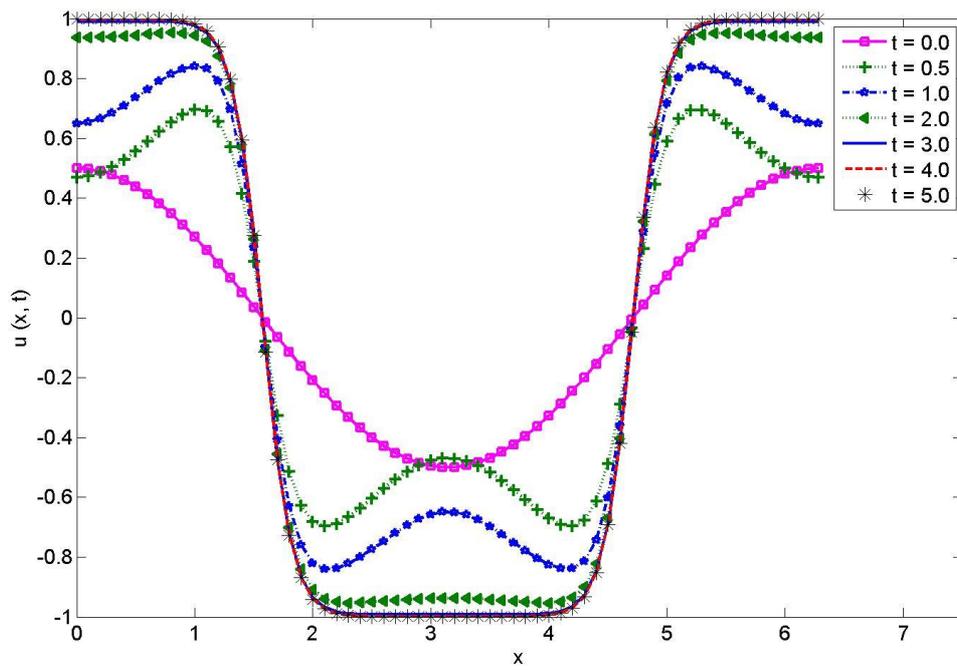


Figure 1: The evolution from $t = 0$ to $t = 5$.

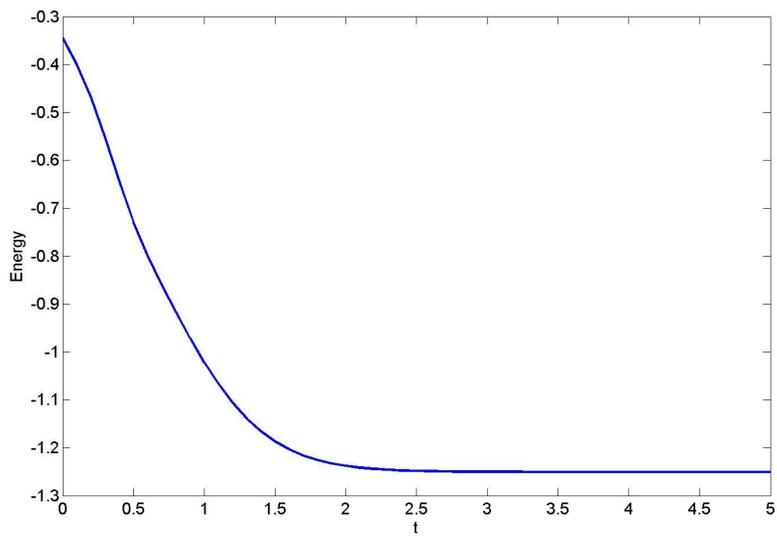


Figure 2: The energy from $t = 0$ to $t = 5$.

4.2 Problem 2

Consider the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \left\{ (-0.31 + 2.31u^2) \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} \right\}, \quad (4.1)$$

in $[0,1]$ with initial condition

$$u(x, 0) = x^4(1 - x)^4. \quad (4.2)$$

For numerical solution $\Delta t = 0.01$ and $N = 51$ are chosen. Fig. 3 shows the numerical solution up to $t = 10$. We observed that the solution $u(x, t)$ for long times remains at patterns that are nearly constant, and the steady–state solution is very close to the constant function $u = 0.00158730158730$ that is the value of initial mass. As before the energy is plotted in Fig. 4 and the values of \mathcal{M} are presented in Table 2 for different time levels to show the energy dissipation and mass conservation respectively. Table 2 depicts that the mass remains constant with absolute error $2.029999866121068e - 012$ at each time level t , .

Table 2
The values of \mathcal{M} at different time levels

Time t	Mass
t= 0.00	0.00158730158730
t= 0.05	0.00158730158933
t= 0.10	0.00158730158933
t= 0.20	0.00158730158933
t= 1.00	0.00158730158933
t= 5.00	0.00158730158933
t= 10.00	0.00158730158933

5 Conclusion

In this paper a numerical method based on the boundary integral equation and dual reciprocity method outlined for the one–dimensional Cahn–Hilliard equation. The DRM was applied to eliminate the domain integrals appearing in the boundary integral equation. Linear RBF was used for DRM and an iterative scheme was employed for the time derivative. A predictor–corrector scheme was used to deal with the nonlinearity.

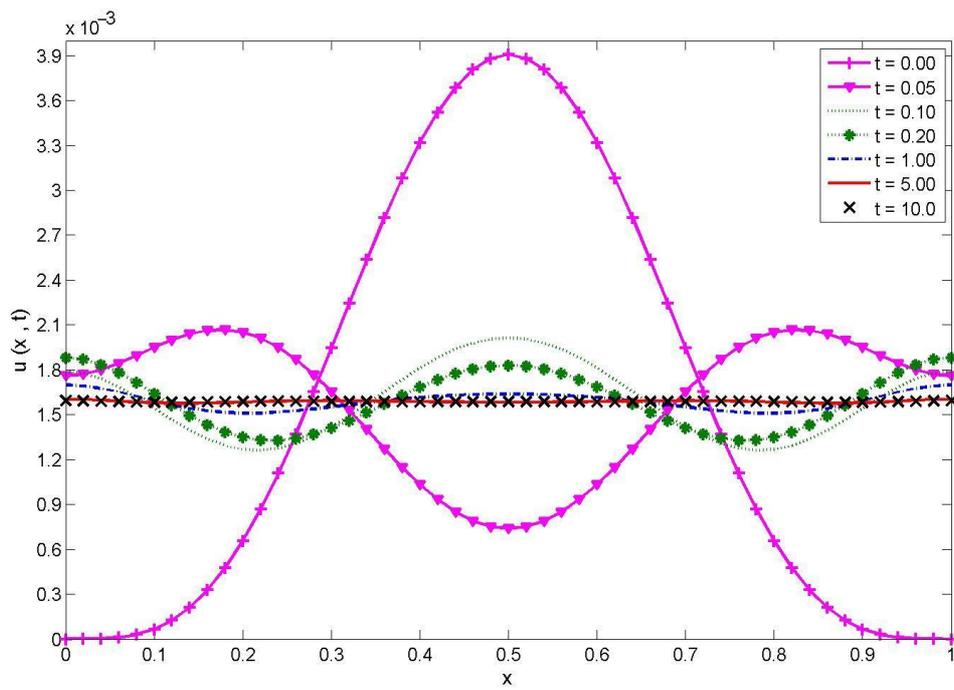


Figure 3: The evolution from $t = 0$ to $t = 10$.

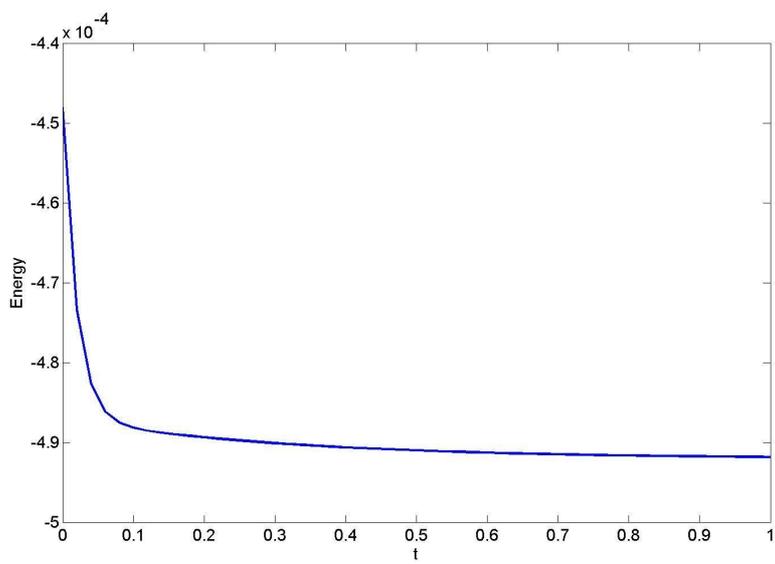


Figure 4: The energy from $t = 0$ to $t = 1$.

Some numerical experiments depicted to demonstrate the efficiency and accuracy of this method. For all examples the energy dissipation and the mass conservation were investigated and the corresponding tables and graphs were presented. These are the known properties of C–H equation.

References

- [1] H. Abels, M. Wilke, Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy, *Nonlinear Analysis* 67(2007) 3176–3193.
- [2] N. Alikakos, P. W. Bates, G. Fusco, Slow motion for the Cahn–Hilliard equation in one space dimension, *J. Diff. Eq.* 90(1991) 81–135.
- [3] F. Bai, C. M. Elliott, A. Gardiner, A. Spence, A. M. Stuart, The viscous Cahn–Hilliard equation: Part I: Computations, *Nonlinearity* 8(1995) 131–160.
- [4] L. Bañas, R. Nürnberg, Adaptive finite element methods for Cahn–Hilliard equations, *J. Comput. Appl. Math.* in press
- [5] J. W. Barrett, J. F. Blowery, An error bound for the finite element approximation of the Cahn–Hilliard equation with logarithmic free energy, *Numer. Math.* 72(1995) 1–20.
- [6] C. A. Brebbia, D. Nardini, Dynamic analysis in solid mechanics by an alternative boundary element procedure, *International Journal of Soil Dynamics and Earthquake Engineering*, 2(1983) 228–233.
- [7] J. W. Cahn, On spinodal decomposition, *Acta Metall.* 9(1961) 795–801.
- [8] J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system, I, interfacial free energy, *J. Chem. Phys.* 28(2)(1958) 258–267.
- [9] J. Carr, M. E. Gurtin, M. Slemrod, Structured phase transitions on a finite interval, *Arch. Rat. Mech. Anal.* 86(1984) 317–351.
- [10] E. Chino, N. Tosaka, Dual reciprocity boundary element analysis of time-independent Burger’s equation, *Engng. Anal. Bound. Elem.* 21(1998) 261–270.
- [11] S. M. Choo, S. K. Chung, Conservative nonlinear difference scheme for the Cahn–Hilliard equation, *Comput. Math. Appl.* 36(7)(1998) 31–39.
- [12] S. M. Choo, S. K. Chung, K. I. Kim, Conservative nonlinear difference scheme for the Cahn–Hilliard equation–II, *Comput. Math. Appl.* 39(2000) 229–243.

- [13] M. I. M. Copetti, C. M. Elliott, Numerical analysis of the Cahn–Hilliard equation with a logarithmic free energy, *Numer. Math.* 63(1992) 39–65.
- [14] P. Danumjaya, A. K. Nandakumaran, Orthogonal cubic spline collocation method for the Cahn–Hilliard equation, *Appl. Math. Comput.* 182(2006) 1316–1329.
- [15] E. J. Dean, R. Glowinski, D. A. Trevas, An approximate factorization least squares solution method for a mixed finite element approximation of the Cahn–Hilliard equation, *Japan J. Indust. Appl. Math.* 13(1996) 495–517.
- [16] C. M. Elliott, S. M. Zheng, On the Cahn–Hilliard equation, *Arch. Rat. Mech. Anal.* 96(1986) 339–357.
- [17] C. M. Elliott, D. A. French, Numerical studies of the Cahn–Hilliard equation for phase separation, *IMA J. Appl. Math.* 38(1987) 97–128.
- [18] C. M. Elliott, D. A. French, A nonconforming finite element method for the two dimensional Cahn–Hilliard equation, *SIAM J. Numer. Anal.* 26(1989) 884–903.
- [19] C. M. Elliott, S. Larsson, Error estimates with smooth and nonsmooth data for a finite element method for the Cahn–Hilliard equation, *Math. Comp.* 58(1992) 603–630.
- [20] C. M. Elliott, D. A. French, F. A. Milner, A second order splitting method of the Cahn–Hilliard equation, *Numer. Math.* 54(1989) 575–590.
- [21] D. A. French, S. Jensen, Long-time behaviour of arbitrary order continuous time Galerkin schemes for some one–dimensional phase transition problems, *IMA J. Numer. Anal.* 14(1994) 421–442.
- [22] D. Furihata, T. Onda, M. Mori, A finite difference scheme for the Cahn–Hilliard equation based on a Lyapunov functional, *GAKUTO Int. Series, Math. Sci. Appl.* 2(1993) 347–358.
- [23] D. Furihata, Finite difference schemes for $\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^\alpha \frac{\delta G}{\delta u}$ that inherit energy conservation or dissipation property, *J. Comput. Phys.* 156(1999) 181–205.
- [24] D. Furihata, A stable and conservative finite difference scheme for the Cahn–Hilliard equation, *Numer. Math.* 87(2001) 675–699.
- [25] M. Gurtin, Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance, *Physica D* 92(1996) 178–192.
- [26] Y. He, Y. Liu, T. Tang, On large time–stepping methods for the Cahn–Hilliard equation, *Appl. Numer. Math.* 57(2007) 616–628.

- [27] G. Hongjun, L. Changchun, Instability of traveling waves of the convective–diffusive Cahn–Hilliard equation, *Chaos, Solitons and Fractals* 20(2004) 253–258.
- [28] E. Jabbari, N. A. Peppas, A model for interdiffusion at interfaces of polymers with dissimilar physical properties, *Polymer* 36(1995) 575–586.
- [29] S. R. Karur, P. A. Ramachandran, Radial basis function approximation in dual reciprocity method, *Math. Comput. Modelling*, 20(7)(1994) 59–70.
- [30] D. Kay, R. Welford, A multigrid finite element solver for the Cahn–Hilliard equation, *J. Comput. Phys.* 212(2006) 288–304.
- [31] N. Khiari, T. Achouri, M. L. Ben Mohamed, K. Omrani, Finite difference approximate solutions for the Cahn-Hilliard equation, *Numer. Methods Partial Differential Eq.* 23(2007) 437–455.
- [32] J. Kim, A numerical method for the Cahn–Hilliard equation with a variable mobility, *Communications in Nonlinear Science and Numerical Simulation* 12(2007) 1560–1571.
- [33] A. Miranville, A. Piétrus, A new formulation of the Cahn–Hilliard equation, *Nonlinear Analysis: Real World Applications* 7(2006) 285–307.
- [34] Sh. Momani, V. S. Erturk, A numerical scheme for the solution of viscous Cahn–Hilliard equation, *Numer. Methods Partial Differential Eq.* in press.
- [35] E. V. L. de Mello, O. T. da S. Filho, Numerical study of the Cahn–Hilliard equation in one, two and three dimensions, *Physica A* 347(2005) 429–443.
- [36] V. Mitlin, On the μ –transform: Applications to the Cahn–Hilliard equation, *Phys. Lett. A* 358(2006) 142–148.
- [37] A. Novick-Cohen, L. A. Segel, Nonlinear aspects of the Cahn–Hilliard equation, *Phys. D* 10(1984) 277–298.
- [38] A. Novick-Cohen, Energy method for the Cahn–Hilliard equation, *Quart. Appl. Math.* XLVI(1988) 681–690.
- [39] P. W. Partridge, C. A. Brebbia, The dual reciprocity boundary element method for the Helmholtz equation. In: C. A. Brebbia, A. Choudouet–Miranda, editors. *Proceedings of the International Boundary Elements Symposium*. Berlin: Computational Mechanics Publications/ Springer, (1990) 543–555.
- [40] S. Puri, K. Binder, Phenomenological theory for the formation of interfaces via the interdiffusion of layem, *Phys. Rev. B* 44(1991) 9735–9738.

- [41] S. Q. Wang, Q. Shi, Interdiffnsion in bilinear polymer mixtures, *Macromolecules* 26(1993) 1091–1096.
- [42] W. T. Ang, A numerical method for the wave equation subject to a non–local conservation condition, *Appl. Numer. Math.* 56(2006) 1054–1060.
- [43] G. N. Wells, E. Kuhl, K. Garikipati, A discontinuous Galerkin method for the Cahn–Hilliard equation, *J. Comput. Phys.* 218(2006) 860–877.
- [44] Y. Xia, Y. Xu, C. W. Shu, Local discontinuous Galerkin methods for the Cahn–Hilliard type equations, *J. Comput. Phys.* in press.
- [45] X. Ye, The Fourier collocation method for the Cahn–Hilliard equation, *Comput. Math. Appl.* 44(2002) 213–229.
- [46] X. Ye, The Legendre collocation method for the Cahn–Hilliard equation, *J. Comput. Appl. Math.* 150(2003) 87–108.
- [47] X. Ye, X. Cheng, The Fourier spectral method for the Cahn–Hilliard equation, *Appl. Math. Comput.* 171(2005) 345–357.
- [48] J. Yin, On the existence of nonnegative continuous solutions of the Cahn–Hilliard equation, *J. Diff. Eqs.* 97(1992) 319–327.
- [49] W. T. Ang, *A Beginner’s Course in Boundary Element Method*, Universal Publisher (2007).