$$U_{ij} = \delta_{ij} \ln r - \frac{y_i y_j}{r^2}$$

$$z_s = \mathrm{Re}^{i\theta} = R(\cos\theta + i\sin\theta)$$

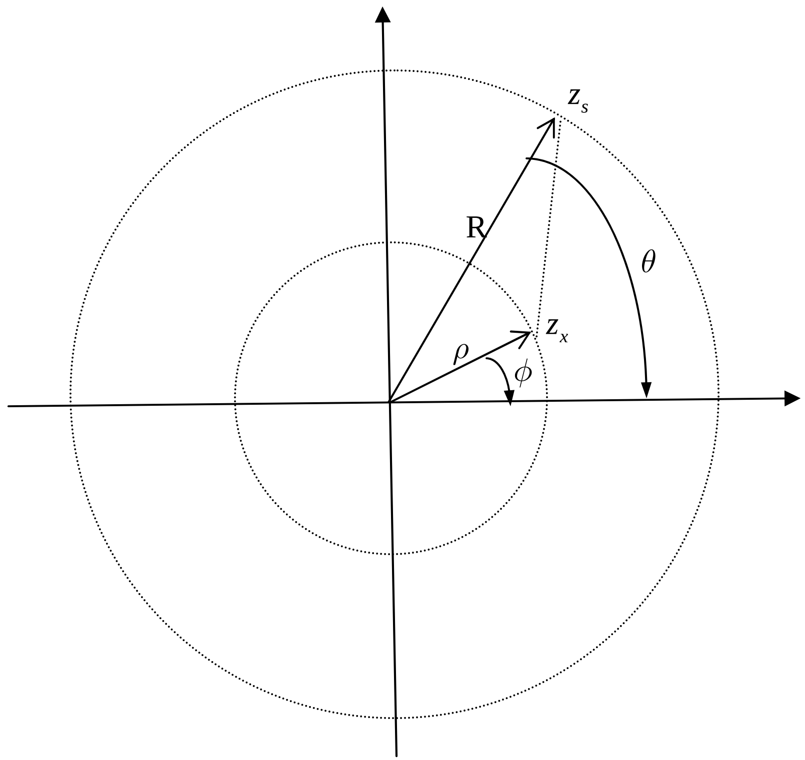
$$z_x = \rho e^{i\phi} = \rho(\cos\phi + i\sin\phi)$$

$$r^2 = \rho^2 + R^2 - 2\rho R\cos\left(\phi - \theta\right)$$

$$y_1 = \rho\cos\phi - R\cos\theta$$

$$y_2 = \rho\sin\phi - R\sin\theta$$

$$\ln r = \begin{cases} \ln R - \sum_{m=1}^{\infty} \frac{1}{m} (\frac{\rho}{R})^m \cos(m(\theta - \phi)), \ R > \rho \\ \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} (\frac{R}{\rho})^m \cos(m(\theta - \phi)), \ R < \rho \end{cases}$$



If $R < \rho$ then

$$\frac{1}{z_x - z_s} = \frac{1}{(\rho\cos\phi + i\rho\sin\phi) - (R\cos\theta + iR\sin\theta)}$$

$$= \frac{1}{(\rho\cos\phi - R\cos\theta) + i(\rho\sin\phi - R\sin\theta)} \frac{(\rho\cos\phi - R\cos\theta) - i(\rho\sin\phi - R\sin\theta)}{(\rho\cos\phi - R\cos\theta) - i(\rho\sin\phi - R\sin\theta)}$$

$$= \frac{(\rho\cos\phi - R\cos\theta) - i(\rho\sin\phi - R\sin\theta)}{\rho^2 + R^2 - 2R\rho\cos(\theta - \phi)}$$

$$= \frac{1}{z_x} \frac{1}{1 - z_s / z_x}$$

$$= \frac{1}{z_x}[1 + \frac{z_s}{z_x} + (\frac{z_s}{z_x})^2 + (\frac{z_s}{z_x})^3 + \cdot\ \cdot\ \cdot + (\frac{z_s}{z_x})^n + \cdot\ \cdot\ \cdot]$$

$$= \frac{1}{\rho} e^{-i\phi}[\sum_{m=0}^{\infty} (\frac{R}{\rho})^m e^{im(\theta-\phi)}]$$

$$= \frac{1}{\rho}(\cos\phi - i\sin\phi)\ \{\sum_{m=0}^{\infty} (\frac{R}{\rho})^m [\cos(m(\theta-\phi)) + i\sin(m(\theta-\phi))]\}$$

$$\frac{y_1}{r^2} = \frac{(\rho\cos\phi - R\cos\theta)}{\rho^2 + R^2 - 2R\rho\cos(\theta-\phi)} = \frac{1}{\rho}\cos\phi\ [\sum_{m=0}^{\infty} (\frac{R}{\rho})^m \cos(m(\theta-\phi))] + \frac{1}{\rho}\sin\phi\ [\sum_{m=0}^{\infty} (\frac{R}{\rho})^m \sin(m(\theta-\phi))] = \frac{1}{\rho}\sum_{m=0}^{\infty} \left(\frac{R}{\rho}\right)^m \cos\left(m\theta - (m+1)\phi\right)$$

$$\frac{y_2}{r^2} = \frac{(\rho\sin\phi - R\sin\theta)}{\rho^2 + R^2 - 2R\rho\cos(\theta-\phi)} = -\frac{1}{\rho}\cos\phi\ [\sum_{m=0}^{\infty} (\frac{R}{\rho})^m \sin(m(\theta-\phi))] + \frac{1}{\rho}\sin\phi\ [\sum_{m=0}^{\infty} (\frac{R}{\rho})^m \cos(m(\theta-\phi))] = \frac{1}{\rho}\sum_{m=0}^{\infty} \left(\frac{R}{\rho}\right)^m \sin\left((m+1)\phi - m\theta\right)$$

$$\frac{y_1^2}{r^2} = \frac{1}{\rho}\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \cos(m\theta-(m+1)\phi)(\rho\cos\phi - R\sin\theta)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \cos(m\theta-(m+1)\phi)(\cos\phi - \frac{R}{\rho}\cos\theta)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos m(\theta-\phi) + \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos(m\theta-(m+2)\phi) - \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m+1)(\theta-\phi)) - \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m-1)\theta-(m+1)\phi)$$

$$\frac{y_2^2}{r^2} = \frac{1}{\rho}\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \sin((m+1)\phi-m\theta)(\rho\sin\phi - R\sin\theta)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \sin((m+1)\phi-m\theta)(\sin\phi - \frac{R}{\rho}\sin\theta)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos m(\theta-\phi) - \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos(m\theta-(m+2)\phi) - \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m+1)(\theta-\phi)) + \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m-1)\theta-(m+1)\phi)$$

$$\frac{y_1 y_2}{r^2} = \frac{1}{\rho}\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \cos(m\theta-(m+1)\phi)(\rho\sin\phi - R\sin\theta)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \cos(m\theta-(m+1)\phi)(\sin\phi - \frac{R}{\rho}\sin\theta)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\sin m(\theta-\phi) - \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\sin(m\theta-(m+2)\phi) - \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\sin((m+1)(\theta-\phi)) + \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\sin((m-1)\theta-(m+1)\phi)$$

If $R > \rho$ then

$$\frac{1}{z_s - z_x} = \frac{1}{(R\cos\theta + iR\sin\theta) - (\rho\cos\phi + i\rho\sin\phi)}$$

$$= \frac{1}{(R\cos\theta - \rho\cos\phi) + i(R\sin\theta - \rho\sin\phi)} \frac{(R\cos\theta - \rho\cos\phi) - i(R\sin\theta - \rho\sin\phi)}{(R\cos\theta - \rho\cos\phi) + i(R\sin\theta - \rho\sin\phi)}$$

$$= \frac{(R\cos\theta - \rho\cos\phi) - i(R\sin\theta - \rho\sin\phi)}{\rho^2 + R^2 - 2R\rho\cos(\theta - \phi)}$$

$$= -\frac{1}{z_s} \frac{1}{1 - z_x / z_s}$$

$$= -\frac{1}{z_s}[1 + \frac{z_x}{z_s} + (\frac{z_x}{z_s})^2 + (\frac{z_x}{z_s})^3 + \cdot \quad \cdot \quad \cdot + (\frac{z_x}{z_s})^n + \cdot \quad \cdot \quad \cdot]$$

$$= \frac{1}{R} e^{-i\theta} [\sum_{m=0}^{\infty} (\frac{\rho}{R})^m e^{im(\phi - \theta)}]$$

$$= \frac{1}{R}(\cos\theta - i\sin\theta) \ \{\sum_{m=0}^{\infty} (\frac{\rho}{R})^m [\cos(m(\phi - \theta)) - i\sin(m(\phi - \theta))]\}$$

$$\frac{y_1}{r^2} = \frac{(R\cos\theta - \rho\cos\phi)}{\rho^2 + R^2 - 2R\rho\cos(\theta - \phi)} = \frac{1}{R}\cos\theta \ [\sum_{m=0}^{\infty} (\frac{\rho}{R})^m \cos(m(\phi - \theta))] + \frac{1}{R}\sin\theta \ [\sum_{m=0}^{\infty} (\frac{\rho}{R})^m \sin(m(\phi - \theta))] = \frac{1}{R} \sum_{m=0}^{\infty} (\frac{\rho}{R})^m \cos\left(m\phi - (m+1)\theta\right)$$

$$\frac{y_2}{r^2} = \frac{(R\sin\theta - \rho\sin\phi)}{\rho^2 + R^2 - 2R\rho\cos(\theta - \phi)} = \frac{1}{R}\sin\theta \ [\sum_{m=0}^{\infty} (\frac{\rho}{R})^m \cos(m(\phi - \theta))] - \frac{1}{R}\cos\theta \ [\sum_{m=0}^{\infty} (\frac{\rho}{R})^m \sin(m(\phi - \theta))] = \frac{1}{R} \sum_{m=0}^{\infty} (\frac{\rho}{R})^m \sin\left((m+1)\theta - m\phi\right)$$

$$\frac{y_1^2}{r^2} = \frac{1}{R}\sum_{m=0}^{\infty}\left(\frac{\rho}{R}\right)^m \cos(m\phi-(m+1)\theta)(R\cos\theta-\rho\cos\phi)$$

$$= \sum_{m=0}^{\infty}\left(\frac{\rho}{R}\right)^m \cos(m\phi-(m+1)\theta)(\cos\theta-\frac{\rho}{R}\cos\phi)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos m(\phi-\theta)+\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos(m\phi-(m+2)\theta)-\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m+1)(\phi-\theta))-\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m-1)\phi-(m+1)\theta)$$

$$\frac{y_2^2}{r^2} = \frac{1}{R}\sum_{m=0}^{\infty}\left(\frac{\rho}{R}\right)^m \sin((m+1)\theta-m\phi)(R\sin\theta-\rho\sin\phi)$$

$$= \sum_{m=0}^{\infty}\left(\frac{\rho}{R}\right)^m \sin((m+1)\theta-m\phi)(\sin\theta-\frac{\rho}{R}\sin\phi)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos m(\phi-\theta)-\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\cos(m\phi-(m+2)\theta)-\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m+1)(\phi-\theta))+\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\cos((m-1)\phi-(m+1)\theta)$$

$$\frac{y_1 y_2}{r^2} = \frac{1}{R}\sum_{m=0}^{\infty}\left(\frac{\rho}{R}\right)^m \cos(m\phi-(m+1)\theta)(R\sin\theta-\rho\sin\phi)$$

$$= \sum_{m=0}^{\infty}\left(\frac{\rho}{R}\right)^m \cos(m\phi-(m+1)\theta)(\sin\theta-\frac{\rho}{R}\sin\phi)$$

$$= \sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\sin m(\phi-\theta)-\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^m \frac{1}{2}\sin(m\phi-(m+2)\theta)+\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\sin((m+1)(\phi-\theta))+\sum_{m=0}^{\infty}\left(\frac{R}{\rho}\right)^{m+1} \frac{1}{2}\sin((m-1)\phi-(m+1)\theta)$$

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