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Hypersingular boundary integral equations have an additional free term

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Abstract In this paper it is shown that hypersingular boundary integral equations may have an additional free term which has been erroneously omitted in former analyses.

1 Introduction

Non-trivial free terms arise in boundary integral equations when the source (collocation) point is located at a non-smooth boundary point (see for example Hartmann (1981), Guiggiani and Gigante (1990), Mantič (1993)).

Free terms for the hypersingular case were considered in Guiggiani et al. (1992), Guiggiani (1992) where, however, it was overlooked that in some particular cases (typically corner points between *curved* boundaries) some additional free terms may arise. The aim of this paper is to correct that error thus providing hypersingular boundary integral equations (HBIE) of full generality. The same results have been obtained independently by Mantič and París (1995).

The missing free terms are zero if the source point is located within one boundary element. The same is true if the collocation point is at a vertex but between flat (straight) elements. Therefore, results obtained so far by means of numerical algorithm employing HBIE's maintain their validity.

2 General form of boundary integral identities

In this section the limiting process for the derivation of boundary integral equations with hypersingular kernels will be outlined. Without loss of generality the analysis will be presented for scalar problems (such as those governed by the Laplace equation).

Let us consider the well-known boundary integral equation for a domain Ω (either 3D or 2D, including the axisymmetric case), bounded by a Kellog's regular surface Γ with unit outward normal $\mathbf{n}(\mathbf{x}) = \{n_i\}$

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} [T(\mathbf{y},\mathbf{x})u(\mathbf{x}) - U(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,d\Gamma_x\right\} = 0, \qquad (1)$$

where u and $q = \partial u/\partial n = u_{,i}n_i$ denote the potential and its normal derivative, respectively, and $\Gamma_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon$ is the boundary of the punctured domain $\Omega_\varepsilon = \Omega - v_\varepsilon$ (Fig. 1).

If $r = [(x_j - y_j)(x_j - y_j)]^{1/2}$ denotes the distance between the source point $\mathbf{y} = \{y_i\}$ and integration point $\mathbf{x} = \{x_i\}$, the fundamental solution U has a weak singularity when $r \to 0$, while the other kernel function $T = \partial U/\partial n(\mathbf{x})$ has a strong singularity.

In Eq. (1) the source point $\mathbf{y}$ lies on the boundary Γ. Since Eq. (1) stems from Green's second identity, it may be only formulated on a domain not including the singular point $\mathbf{y}$. Therefore, a (vanishing) neighbourhood v_ε of $\mathbf{y}$ has been removed from the original domain Ω (Fig. 1).

Equation (1) can be differentiated with respect to any coordinate y_k of the source point. Since all functions of $\mathbf{y}$ in Eq. (1) are regular in the domain Ω_ε (actually, U and $T \in C^\infty$), we can differentiate under the integral sign, to obtain the following boundary integral equation with hypersingular kernel

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} [V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,d\Gamma_x\right\} = 0, \qquad (2)$$

where $W_k = \partial U/\partial y_k$ and $V_k = \partial T/\partial y_k$.

Differentiation rises the order of singularity of the kernel functions. Therefore, W_k have a strong singularity of $O(r^{-2})$ in 3D and of $O(r^{-1})$ in 2D, whereas V_k are *hypersingular* of $O(r^{-3})$ in 3D and of $O(r^{-2})$ in 2D, for $r \to 0$.

It is worth noting that in (2) the integral on Γ_ε is identically zero for any $\varepsilon > 0$. Therefore all singular terms have to cancel out (and without the recourse to any *a-priori* interpretation in the finite-part sense).

Also note that the validity of identities (1) and (2) is by no means restricted to smooth boundary points.

Since the continuity features of u at the point $\mathbf{y}$ never come into play in the limiting process in (1) or in (2), it follows that the density function u does not have to satisfy any continuity requirement at $\mathbf{y}$ for the limits to exist. In other words, the integrals on Γ_ε in Eqs. (1) and (2) are identically zero provided the functions $u(\mathbf{x})$ and $U(\mathbf{y},\mathbf{x})$ are harmonic in the punctured domain $\Omega - v_\varepsilon$, which *never* contains the point $\mathbf{y}$ for any $\varepsilon > 0$. Again, it is relevant the behaviour of $u(\mathbf{x})$ (and of U) in any neighbourhood of $\mathbf{y}$, but not at $\mathbf{y}$.

3 Free terms in hypersingular boundary integral equations

Now, let us assume (as, e.g., Gray et al. (1990), Krishnasamy et al. (1992), Schwab and Wendland (1992), Hildenbrand and Kuhn (1992), Cruse and Suwito (1993)) that the potential u is differentiable at $\mathbf{y}$, with its derivatives satisfying a Hölder condition, that is $u \in C^{1,\alpha}$ at $\mathbf{y}$.

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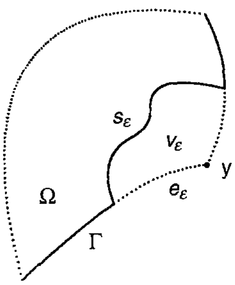

Fig. 1. Exclusion of the singular point y by a vanishing neighbourhood v_ε

As already shown at the end of Section 2, these (quite restrictive) continuity requirements are not strictly necessary for the existence of the limit in (2). However, they allow for the "extraction" of the free terms which seems a necessary step for the subsequent solution by the boundary element method.

The fulfilment of Hölder conditions makes it possible to expand u and q around y

$$u(\mathbf{x}) = u(\mathbf{y}) + u_{,h}(\mathbf{y})(x_h - y_h) + O(r^{1+\alpha}),$$
$$q(\mathbf{x}) = u_{,h}(\mathbf{x})\, n_h(\mathbf{x}) = u_{,h}(\mathbf{y})\, n_h(\mathbf{x}) + O(r^{\alpha}), \qquad (3)$$

where $0 < \alpha \leqq 1$.

By adding and subtracting in (2) the relevant terms of expansions (3), the following form is obtained

$$\lim_{\varepsilon\to 0}\Bigg\{ \int_{(\Gamma - e_\varepsilon)} [V_k(\mathbf{y},\mathbf{x})\, u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})\, q(\mathbf{x})]\, d\Gamma_x$$
$$+ \int_{s_\varepsilon} (V_k[u(\mathbf{x}) - u(\mathbf{y}) - u_{,h}(\mathbf{y})(x_h - y_h)]$$
$$- W_k[q(\mathbf{x}) - u_{,h}(\mathbf{y})\, n_h(\mathbf{x})])\, d\Gamma_x$$
$$+ u_{,h}(\mathbf{y}) \int_{s_\varepsilon} [V_k(x_h - y_h) - W_k n_h(\mathbf{x})]\, d\Gamma_x$$
$$+ u(\mathbf{y}) \int_{s_\varepsilon} V_k\, d\Gamma_x \Bigg\} = 0. \qquad (4)$$

The free terms are given by the limit for $\varepsilon \to 0$ of the integrals on s_ε in (4).

The most convenient shape for v_ε seems to be a ball of radius ε centered at y. As a matter of fact, this simple shape of s_ε allows the analytical evaluation of all integrals on s_ε, for $\varepsilon \to 0$.

Because of the expansions (3) and since on s_ε we have that $d\Gamma_x = O(\varepsilon^2)$ in 3D or $d\Gamma_x = O(\varepsilon)$ in 2D, it follows that

$$\int_{s_\varepsilon} (V_k[u(\mathbf{x}) - u(\mathbf{y}) - u_{,h}(\mathbf{y})(x_h - y_h)]$$
$$- W_k[q(\mathbf{x}) - u_{,h}(\mathbf{y})\, n_h(\mathbf{x})])\, d\Gamma_x = O(\varepsilon), \qquad (5)$$

so that we only have to consider the other integrals.

The second kind of integrals on s_ε gives rise to the following free term coefficients

$$\int_{s_\varepsilon} [V_k(x_h - y_h) - W_k n_h(\mathbf{x})]\, d\Gamma_x = c_{kh}(\mathbf{y}) + O(\varepsilon). \qquad (6)$$

At smooth boundary points these free-term coefficients simply become $c_{kh} = 0.5\,\delta_{kh}$.

So far, the analysis presented in Guiggiani et al. (1992) was correct. The error arose in the evaluation of the last kind of integrals on s_ε. The correct result should be

$$\int_{s_\varepsilon} V_k\, d\Gamma_x = \frac{b_k(\mathbf{y})}{\varepsilon} + a_k(\mathbf{y}) + O(\varepsilon). \qquad (7)$$

The free term coefficients a_k were missing in former analyses. They depend on the *curvature* of the boundary at y. Details on their evaluation are provided in the next section.

According to this analysis, hypersingular boundary integral equations for scalar problems can be written in the following form

$$a_k(\mathbf{y})\, u(\mathbf{y}) + c_{kh}(\mathbf{y})\, u_{,h}(\mathbf{y})$$
$$+ \lim_{\varepsilon\to 0}\Bigg\{ \int_{(\Gamma - e_\varepsilon)} [V_k(\mathbf{y},\mathbf{x})\, u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})\, q(\mathbf{x})]\, d\Gamma_x$$
$$+ \frac{b_k(\mathbf{y})}{\varepsilon}\, u(\mathbf{y}) \Bigg\} = 0. \qquad (8)$$

Note that, at non-smooth boundary points between curved elements it is not possible to obtain free-terms only involving potential derivatives.

Similarly, HBIE's for vector problems (e.g., elasticity) become (cfr. Guiggiani et al. (1991))

$$a_{ikj}(\mathbf{y})\, u_j(\mathbf{y}) + c_{ikjh}(\mathbf{y})\, u_{j,h}(\mathbf{y})$$
$$+ \lim_{\varepsilon\to 0}\Bigg\{ \int_{(\Gamma - e_\varepsilon)} [V_{ikj}(\mathbf{y},\mathbf{x})\, u_j(\mathbf{x}) - W_{ikj}(\mathbf{y},\mathbf{x})\, q_j(\mathbf{x})]\, d\Gamma_x$$
$$+ \frac{b_{ikj}(\mathbf{y})}{\varepsilon}\, u_j(\mathbf{y}) \Bigg\} = 0. \qquad (9)$$

4
Evaluation of the free term coefficients a_k

In this section the free term coefficients $b_k(\mathbf{y})/\varepsilon + a_k(\mathbf{y})$ defined by expression (7) are derived.

For simplicity, let us consider two-dimensional potential problems, whose hypersingular kernel functions are given by

$$V_k(\mathbf{y},\mathbf{x}) = \frac{\partial U(\mathbf{y},\mathbf{x})}{\partial x_i \partial y_k}\, n_i(\mathbf{x}) = -\frac{1}{2\pi r^2}\left[2 r_{,k} \frac{\partial r}{\partial n} - n_k(\mathbf{x}) \right], \qquad (10)$$

where $r = |\mathbf{x} - \mathbf{y}|$, $r_{,i} = \partial r / \partial x_i$ and $U(\mathbf{y},\mathbf{x}) = -(1/2\pi) \ln r$ is the fundamental solution of the Laplace equation.

For convenience, it was assumed s_ε to be an arc of a circle centered at the source point y and of radius ε (Fig. 2). Let (r, φ) be a set of polar coordinates centered at y. Hence the integration point x on s_ε has coordinates (ε, φ), with $\varphi_1(\varepsilon) \leqq \varphi \leqq \varphi_2(\varepsilon)$. Notice that, because of the (possible) curvature of the boundary around y, the interval for φ does depend also on the radius ε, that is on the radial size of s_ε (this is essentially the aspect which was overlooked in Guiggiani et al. (1992), Appendix A).

Owing to the simple shape of s_ε the following relations hold when $\mathbf{x} \in s_\varepsilon$

$$
\begin{aligned}
&r = \varepsilon, && \partial r/\partial n = -1, \\
&r_{,1} = \cos\varphi, && r_{,2} = \sin\varphi, \\
&n_1 = -\cos\varphi, && n_2 = -\sin\varphi, \\
&x_1 - y_1 = \varepsilon\cos\varphi, && x_2 - y_2 = \varepsilon\sin\varphi.
\end{aligned}
\tag{11}
$$

and $d\Gamma_x = \varepsilon d\varphi$.

Accordingly, the hypersingular kernel functions become

$$V_1 = \frac{\cos\varphi}{2\pi\varepsilon^2}, \quad V_2 = \frac{\sin\varphi}{2\pi\varepsilon^2}. \tag{12}$$

In both cases we can set

$$V_k = \frac{f_k(\varphi)}{\varepsilon^2}. \tag{13}$$

The integral in (7) can now be set in the following form (Fig. 2).

$$
\begin{aligned}
I &= \int_{s_\varepsilon} V_k(\mathbf{y}, \mathbf{x})\, d\Gamma_x = \int_{\varphi_1(\varepsilon)}^{\varphi_2(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon d\varphi \\
&= \frac{1}{\varepsilon}[F_k(\varphi_2(\varepsilon)) - F_k(\varphi_1(\varepsilon))],
\end{aligned}
\tag{14}
$$

where $f_k(\varphi) = dF_k/d\varphi$.

Since we are interested in the limit of Eq. (14) for $\varepsilon \to 0$, we can resort in Taylor series expansions. From differential geometry we obtain (e.g., Farin (1992), chapter 11)

$$
\begin{aligned}
\varphi_1(\varepsilon) &= \varphi_1(0) + \frac{\kappa_1(0)}{2}\varepsilon + O(\varepsilon^2), \\
\varphi_2(\varepsilon) &= \varphi_2(0) - \frac{\kappa_2(0)}{2}\varepsilon + O(\varepsilon^2).
\end{aligned}
\tag{15}
$$

where $\kappa_i(0)$ are the *signed* curvatures of the boundary Γ around $\mathbf{y}$ (positive curvature means a convex contour).

In general, given the parametric equations $(x_1(\xi), x_2(\xi))$ of a curve Γ, the signed curvature κ of a planar curve can be computed with the following expression (e.g., Farin (1992), page 386 with sign reversed)

$$\kappa(\xi) = \frac{-x_1'' x_2' + x_2'' x_1'}{[(x_1')^2 + (x_2')^2]^{3/2}}, \tag{16}$$

the prime denoting differentiation with respect to the parameter ξ.

We can also expand $F_k(\varphi(\varepsilon))$ as a function of ε

$$
\begin{aligned}
F_k(\varphi_1(\varepsilon)) &= F_k(\varphi_1(0)) + \left.\frac{dF_k}{d\varphi_1}\frac{d\varphi_1}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \\
&= F_k(\varphi_{10}) + \frac{\kappa_1(0)}{2} f_k(\varphi_{10})\varepsilon + O(\varepsilon^2), \\
F_k(\varphi_2(\varepsilon)) &= F_k(\varphi_2(0)) + \left.\frac{dF_k}{d\varphi_2}\frac{d\varphi_2}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \\
&= F_k(\varphi_{20}) - \frac{\kappa_2(0)}{2} f_k(\varphi_{20})\varepsilon + O(\varepsilon^2).
\end{aligned}
\tag{17}
$$

where expansions (15) of $\varphi_i(\varepsilon)$ were taken into account. We also set $\varphi_i(0) = \varphi_{i0}$.

The integral (14) becomes

$$
\begin{aligned}
I &= \frac{1}{\varepsilon}\left[F_k(\varphi_{20}) - F_k(\varphi_{10}) - \frac{f_k(\varphi_{20})\kappa_2(0) + f_k(\varphi_{10})\kappa_1(0)}{2}\varepsilon + O(\varepsilon^2) \right] \\
&= \frac{F_k(\varphi_{20}) - F_k(\varphi_{10})}{\varepsilon} - \frac{f_k(\varphi_{20})\kappa_2(0) + f_k(\varphi_{10})\kappa_1(0)}{2} + O(\varepsilon) \\
&= \frac{b_k}{\varepsilon} + a_k + O(\varepsilon).
\end{aligned}
\tag{18}
$$

Equation (18) shows that the free-term coefficients a_k depend on the local geometry of Γ at $\mathbf{y}$ but in a more subtle way than the other coefficients. They are affected not only by the inner angle at $\mathbf{y}$ but also by the curvature of Γ.

The same computation can be performed in a slightly different, although equivalent, way. Just observe that

$$I = \int_{\varphi_1(\varepsilon)}^{\varphi_2(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon d\varphi = \int_{\varphi_1(\varepsilon)}^{\varphi_{10}} + \int_{\varphi_{10}}^{\varphi_{20}} + \int_{\varphi_{20}}^{\varphi_2(\varepsilon)} = \int_{\varphi_{10}}^{\varphi_{20}} - \left[\int_{\varphi_{10}}^{\varphi_1(\varepsilon)} + \int_{\varphi_2(\varepsilon)}^{\varphi_{20}} \right]. \tag{19}$$

It is now natural to define

$$\frac{b_k(\mathbf{y})}{\varepsilon} := \int_{\varphi_{10}}^{\varphi_{20}} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon d\varphi = \frac{F_k(\varphi_{20}) - F_k(\varphi_{10})}{\varepsilon}, \tag{20}$$

which coincides with Eqs. (A4) and (A5) in Guiggiani et al.

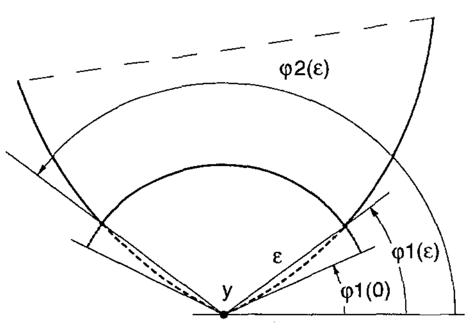

Fig. 2. Local geometry at a non-smooth boundary point

(1992), and

$$a_k(\mathbf{y}) := -\lim_{\varepsilon\to 0}\left[\int_{\varphi_{10}}^{\varphi_1(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon d\varphi + \int_{\varphi_2(\varepsilon)}^{\varphi_{20}} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon d\varphi\right]$$

$$= -\tfrac{1}{2}[f_k(\varphi_{20})\kappa_2(0) + f_k(\varphi_{10})\kappa_1(0)], \qquad (21)$$

which are the so far missing free-term coefficients.

It is worth noting that the coefficients a_k are certainly zero in the following important cases:

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1. $k_1 = k_2 = 0$, that is the contour Γ is piecewise straight around the source point **y**, regardless of **y** being at a corner or not;
2. $k_1 = k_2$ and $\varphi_{20} = \varphi_{10} + \pi$, that is the curvature and the tangent vector are continuous at **y**. This condition can be summarized as Γ having a geometric continuity of order 2 at **y**, as defined, e.g., in Farin (1992), chapter 12 ($\Gamma \in G^2(\mathbf{y})$).

Since all applications of HBIE's presented so far (e.g., Gray et al. (1990), Krishnasamy et al. (1990), Guiggiani et al. (1991), Hildenbrand and Kuhn (1992), Mi and Aliabadi (1992), Huber et al. (1993), Rêgo Silva et al. (1993), Guiggiani (1994), Gallego and Domínguez (submitted), to mention but a few) fall within these two cases, the oversight of a_k's hasn't had any practical relevance.

Notice, however, that a_k may be non-zero at a smooth boundary point with discontinuous curvature.

5 Three-dimensional problems

Analysis for 3D problems is basically the same (Eq. (7)). Of course, it is somehow more involved, but it follows the same steps as for the 2D case.

Since the free-term coefficients c_{kh} and b_k/ε are not affected by the curvature of the boundary, their evaluation can be performed using the procedure described in Hartmann (1981) or in Guiggiani and Gigante (1990) where the solid angle formed by the tangent *planes* at **y** was considered.

For the evaluation of the other set of coefficients a_k it is useful to introduce a system of spherical coordinates (r, φ, ψ) for each boundary element (patch) Γ^e matching at **y** so that the integration point $\mathbf{x}\in s_\varepsilon$ has coordinates $(\varepsilon, \varphi, \psi)$. On each patch Γ^e, it is convenient to refer the spherical coordinates to a cartesian system (z_1, z_2, z_3) with its third axis coinciding with the *inward* normal $\mathbf{m}(\mathbf{y})$ to the tangent plane (hence $\mathbf{m} = -\mathbf{n}$).

The part of s_ε relative to the patch Γ^e that contributes to a_k is therefore delimited by $\psi_1(\varepsilon) \leqq \psi \leqq \psi_2(\varepsilon)$ and $\varphi_e(\varepsilon, \psi) \leqq \varphi \leqq \pi/2$, (or $\pi/2 \leqq \varphi \leqq \varphi_e(\varepsilon, \psi)$, depending on whichever is bigger) with (cf. Eq. (15))

$$\varphi_e(\varepsilon, \psi) = \frac{\pi}{2} - \frac{\kappa(\psi)}{2}\varepsilon + O(\varepsilon^2). \qquad (22)$$

According to Euler's theorem, the *normal curvature* $k(\psi)$ as defined, e.g., in Farin (1992) pp. 372–376, is given by $k(\psi) = k_1\cos^2(\psi) + k_2\sin^2(\psi)$, where k_1 and k_2 are the principal curvatures.

The coefficients a_k can now be defined as an element-by-element summation of integrals

$$a_k + O(\varepsilon) = -\sum_e \int_{\psi_1(\varepsilon)}^{\psi_2(\varepsilon)} \left(\int_{\varphi_e(\varepsilon,\psi)}^{\pi/2} \frac{f_k(\psi, \varphi)}{\varepsilon^3}\varepsilon^2 \sin(\varphi)\, d\varphi \right) d\psi$$

$$= -\sum_e \int_{\psi_1(\varepsilon)}^{\psi_2(\varepsilon)} \left(\frac{\varepsilon\kappa(\psi)}{2}\frac{f_k(\psi, \pi/2)}{\varepsilon} + O(\varepsilon) \right) d\psi$$

$$= -\sum_e \int_{\psi_{10}}^{\psi_{20}} \frac{\kappa(\psi)}{2} f_k(\psi, \pi/2)\, d\psi + O(\varepsilon), \qquad (23)$$

where $V_k = f(\psi, \varphi)/\varepsilon^3$ for $x\in s_\varepsilon$.

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