that are different from those found using pulse functions, but values of the matrix elements are in good agreement for electrically small cells.

The matrix elements found with the new methods do not require a substantial increase in computational effort. For a homogeneous scatterer, if equal size cells are used, the new methods require one-time computation of several additional Bessel functions, but time for calculation of the matrix elements is dominated by recalculation of the zero-order Hankel function which is needed with or without the corrections made in the new methods. If the scatterer is not homogeneous or if different cell sizes are used, then time for calculation of each matrix element is approximately doubled in the new methods. Time spent in calculation of the matrix elements is proportional to N^2, whereas time spent in solving the matrix equation is proportional to N^3, so for large numbers of cells there is still no significant increase in computation time with the new methods.

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Multiple Scattering by Two Conducting Circular Cylinders

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Abstract—Multiple scattering by two conducting circular cylinders is studied by applying three different methods: the method of Zitron and Karp (ZK), the method of Karp and Russek, and an iterative method which gives the exact solution. It is found from the numerical results for the total scattering cross section that the ZK method gives very precise results even for rather large ka and small kd, where a and d are the radius of the cylinder and the separation between the cylinders, respectively.

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I. INTRODUCTION

Asymptotic approximation is a very useful tool in studying the scattering and diffraction of an electromagnetic wave, since it is rather insensitive to the shape of the scatterer, its expression is usually simple, it is easy to evaluate numerically, and it sometimes admits simple physical interpretation. Zitron and Karp [1] proposed a higher order approximation method in multiple scattering by two cylindrical obstacles with an arbitrary cross section (ZK method). It gives solutions which are valid to order $(kd)^{-3/2}$ for two-dimensional scatterers, where d is the separation between the obstacles. Complete asymptotic expansion, which is a convergent series, was given by Twersky [5], [6] and was applied by Young and Bertrand [7] for two circular cylinders. Another method to derive an approximate solution for multiple scattering has been developed by Karp and Russek [2]. They applied the method to diffraction by a wide slit (KR method), which is valid to order $(kd)^{-1}$, although it includes multiple scattering rays bouncing back and forth between the obstacles. It is the purpose of this communication to verify the validity of the ZK method and the KR method numerically when applied to multiple scattering of an electromagnetic plane wave by two conducting infinitely long circular cylinders. To derive a rigorous solution we use an iterative method developed by Germey [3], [4]. It is found from the numerical results for the scattering cross section for E-polarization that the ZK method gives very precise results even for rather large ka and small kd, where a is the radius of the cylinder. The KR method is extremely accurate when applied to a thin scatterer, and can only be significantly improved by the ZK method. The reason may be considered as follows: Some of the terms in the ZK method involve derivatives of plane waves and thus take into account changes in the curvature of the wave front. This is important in the case of a thick scatterer. Scattering patterns for E-polarization are calculated by three methods: the KR method, the ZK method, and the Germey method. The parameters in the scattering patterns for H-polarization are chosen such that the results may be compared with those of Young and Bertrand [7]. The agreement between them is rather good for the backscattering patterns [8].

II. SUMMARY OF THE KR, ZK, AND GERMEY METHODS

A. KR Method

Karp and Russek devised a method to include multiple interactions between the obstacles in calculating the field diffracted by a wide slit. The total field of the slit may be expressed as the sum of a geometric optics term $E_z^{(g)}$ plus a diffracted term due to one edge a, $E_z^{(da)}$ and due to another edge b, $E_z^{(db)}$. The strength of the diffracted fields is determined as follows.

1) $E_z^{(da)}$ is composed of the response to an incident wave and the response to a line source of undetermined strength C_b located at the edge of the opposite half-plane; that is

$$E_z^{(da)} = C(kr_a)[g(\theta_{a0}, \theta_a) + C_b f(2d, \pi, \theta_a)]. \qquad (1)$$

A similar expression for $E_z^{(db)}$ is written by replacing subscript a by b. $C(kr) = \sqrt{\pi/2ikr}e^{-ikr}$ represents a cylindrical wave. $g(\theta_{a0}, \theta_a)$ and $f(2d, \pi, \theta_a)$ are pattern functions of a half-plane due to a plane wave and a cylindrical wave.

2) Since $E_z{}^{da}$ itself may be regarded as representing the cylindrical wave from the edge a in the direction $\theta_a = \pi$, we obtain

$$C_a = g(\theta_{a0}, \pi) + C_b f(2d, \pi, \pi). \tag{2}$$

3) A similar relation to 2) is derived from $E_z{}^{(db)}$, and we can determine the coefficients C_a and C_b.

B. ZK Method

Zitron and Karp derived a formula which expresses the scattered field of two arbitrary-shaped cylinders if the scattered field patterns of an isolated cylinder

$$E_s = \frac{e^{-ikr}}{\sqrt{r}} \sum_n \frac{f_n(\theta, \theta_0)}{r^n} \tag{3}$$

is known, where r is the distance from the axis of the circumscribed circular cylinder, θ is the angle of observation, θ_0 is the angle of incidence, and $f_n(\theta, \theta_0)$ represents the pattern functions of the scattered fields. The total field consists of the incident wave $E_z{}^{(i)}$ and scattered field $E_z{}^{(a)}$ and $E_z{}^{(b)}$ scattered by each cylinder. The procedure for determining $E_z{}^{(a)}$ and $E_z{}^{(b)}$ is as follows.

1) $E_z{}^{(a)}$ and $E_z{}^{(b)}$ include the multiple interactions up to $d^{-3/2}$, where d is the separation between the cylinders.

2) $E_z{}^{(a)}$ may be expressed as $E_z{}^{(a)} = E_z{}^{(a)}(b \to a) + E_z{}^{(a)}(a \to b \to a) + E_z{}^{(a)}(a \to b \to a \to b) + E_z{}^{(a)}(0)$, where $E_z{}^{(a)}(b \to a)$ represents the scattered field by cylinder a due to an equivalent source associated with (3) at cylinder b. $E_z{}^{(a)}(a \to b \to a)$ and $E_z{}^{(a)}(a \to b \to a \to b)$ may be interpreted in a similar way. $E_z{}^{(a)}(0)$ is the scattered field due to plane wave incidence.

3) At a sufficiently large distance from the cylinder, the scattered field resembles a plane wave. Equation (3) for a given cylinder can be represented, in the neighborhood of the other cylinder, as the plane wave in the form

$$E_s = r^{ikd}[Ad^{-1/2} + Bd^{-3/2}], \tag{4}$$

and the scattered field due to the above plane wave is readily derived.

Since the KR method assumes an equivalent line source at each edge, it is considered to be valid to order d^{-1} though it partly picks up any order of multiple interaction.

C. Germey's Iterative Method

Germey presented a method for calculating the field scattered by a conducting elliptic cylinder. The total field consists of an incident wave u^i and the scattered fields $u^{(a)}$ and $u^{(b)}$ due to cylinder a and cylinder b, respectively. Each scattered field may be expanded as

$$u^l = \sum_{n=1}^{\infty} u_n{}^l, \qquad l = a, b \tag{5}$$

where $u_n{}^{(l)}$ satisfies the scalar wave equation and boundary conditions

$$u^i + u_1{}^{(a)} = 0, u_1{}^{(b)} + u_2{}^{(a)} = 0, \cdots, u_{n-1}{}^{(b)} + u_n{}^{(a)} = 0, \quad \text{on } C_a$$

$$u^i + u_1{}^{(b)} = 0, u_1{}^{(a)} + u_2{}^{(a)} = 0, \cdots, u_{n-1}{}^{(a)} + u_n{}^{(b)} = 0, \quad \text{on } C_b. \tag{6}$$

C_a and C_b are contours of surfaces of cylinders a and b, respectively. The process of applying this method to derive the scattered field by two elliptic cylinders is as follows.

1) The incident plane wave u^i and the scattered $u_n{}^{(a)}$ and $u_n{}^{(b)}$ of any orders are expanded in terms of Mathieu functions.

2) Considering that $u_1{}^{(a)}$ and $u_1{}^{(b)}$ are excited by the incident plane wave, we apply the boundary conditions $u^{(i)} + u_1{}^{(a)} = 0$ and $u^{(i)} + u_1{}^{(b)} = 0$ to determine the expansion coefficients.

3) Since $u_n{}^{(a)}$ and $u_n{}^{(b)}$ are considered to be excited by $u_{n-1}{}^{(b)}$ and $u_{n-1}{}^{(a)}$, respectively, the expansion coefficients for $u_n{}^{(a)}$ and $u_n{}^{(b)}$ are determined by applying the boundary condition. To impose the boundary condition, the addition theorem for Mathieu functions is used.

III. STATEMENT OF THE PROBLEM

Consider the two parallel infinitely long conducting circular cylinders shown in Fig. 1. The radii of the cylinders are a and b, and the separation between them is d. An electromagnetic plane wave of E-polarization is incident upon the circular cylinders along a direction which is inclined by an angle ϕ_0 with respect to the x-axis. Since the summary of the analysis is to be found in the previous section, we will give only the final results.

A. Solution Based on ZK Method

1) Incident wave: $E_z{}^i = \exp[-jk(x \cos\phi_0 + y \sin\phi_0)]$.

2) Scattered field:

$$\begin{aligned}
&E_z{}^s C(k\rho)^{-1} \\
&= -f(a, \phi, \phi_0) + C(kd)e^{ij\cos\phi_0}\Big[f(b, \pi, \phi_0)f(a, \phi, \pi) \\
&\quad - \frac{1}{j2kd}\{f(b, \pi, \phi_0)D_{\phi_0}{}^2 f(a, \phi, \pi) + 2D_\phi f(b, \pi, \phi_0) \\
&\quad \cdot D_{\phi_0} f(a, \phi, \pi) + (\tfrac{1}{4} + D_\phi{}^2) f(b, \pi, \phi_0) f(a, \phi, \pi)\}\Big] \\
&\quad - C(kd)^2 f(a, 0, \phi_0) f(b, 0, \pi) f(a, \phi, \pi) \\
&\quad + C(kd)^3 f(b, \pi, \phi_0) f(a, 0, \pi) f(b, \pi, 0) f(a, \phi, \pi) \\
&\quad \cdot e^{jkd\cos\phi_0} - f(b, \phi, \phi_0) e^{jkd(\cos\phi_0 + \cos\phi)} \\
&\quad + C(kd)e^{jkd\cos\phi}\Big[f(a, 0, \phi_0) f(b, \phi, 0) \\
&\quad - \frac{1}{j2kd}\{f(a, 0, \phi_0) D_{\phi_0}{}^2 f(b, \phi, 0) + 2D_\phi f(a, 0, \phi_0) \\
&\quad + (\tfrac{1}{4} + D_\phi{}^2) f(a, 0, \phi_0) f(b, \phi, 0)\}\Big] \\
&\quad - C(kd)^2 f(b, \pi, \phi_0) f(a, 0, \pi) f(b, \phi, 0) \\
&\quad \cdot e^{jkd(\cos\phi_0 + \cos\phi)} + C(kd)^3 f(a, 0, \phi_0) f(b, 0, \pi) \\
&\quad \cdot f(a, 0, \pi) f(b, \phi, 0) e^{jkd\cos\phi}
\end{aligned}$$

(7)

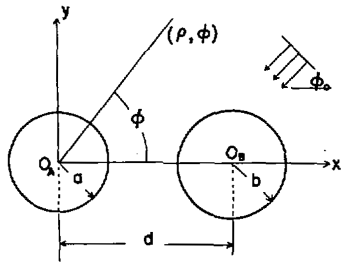

Fig. 1. Geometry of problem. Radius of each cylinder is a and b, and separation between them is d.

where

$$C(x) = \sqrt{\frac{2}{\pi x}}\, e^{-jx + j\frac{\pi}{4}}$$

$$f(x, \phi, \phi_0) = \sum_{n=0}^{\infty} \epsilon_n (-1)^n \frac{J_n(kx)}{H_n^{(2)}(kx)} \cos n(\phi - \phi_0)$$

$$D_{\phi_0} f(x, \phi, 0) = \frac{\partial}{\partial \phi_0} f(x, \phi, \phi_0) \bigg|_{\phi_0 = 0}. \tag{8}$$

B. Solution Based on KR Method

$$\begin{aligned} E_z^s C(k\rho)^{-1} &= f(a, \phi, \phi_0) + C_b C(kd) f(a, \phi, \pi) \\ &\quad + \{f(b, \phi, \phi_0) e^{jkd\cos\phi_0} + C_a C(kd) f(b, \phi, 0)\} \\ &\quad \cdot e^{jkd\cos\phi}, \end{aligned} \tag{9}$$

where

$$C_a = \frac{f(a, 0, \phi_0) + C(kd) f(a, 0, \pi) f(b, \pi, \phi_0) e^{jkd\cos\phi_0}}{1 - C(kd)^2 f(a, 0, \pi) f(b, \pi, 0)}$$

$$C_b = \frac{f(b, \pi, \phi_0) e^{jkd\cos\phi_0} + C(kd) f(a, 0, \phi_0) f(b, \pi, 0)}{1 - C(kd)^2 f(a, 0, \pi) f(b, \pi, 0)}.$$

IV. NUMERICAL RESULTS AND DISCUSSION

For simplicity, we assume that the radii of the cylinders are equal in the actual numerical calculations. In Fig. 2, the scattered field patterns are shown which are calculated based on the ZK method, the RK method, and the iterative method. It is noted that the solution by the iterative method approaches the exact solution as the number of iterations increases. Fig. 2(a) represents the scattering patterns for the case of the thin cylinder $ka = 0.1$, while Fig. 2(b) is for the rather thick cylinder $ka = 1.0$. The separation between the cylinders and the angle of incidence in both cases are chosen to be $kd = 3.0$ and $\phi_0 = 190^\circ$, respectively. From these figures it is seen that the patterns have two lobes. This is expected from the array factor for the present configuration $2\cos[(kd/2)\cos\phi]$.

In Figs. 3 and 4 we present numerical results for the total scattering cross section calculated from the relation

$$\sigma_T = -\frac{4}{k} \operatorname{Re} P(\phi_0),$$

where

$$E_z^s = P(\phi) C(k\rho).$$

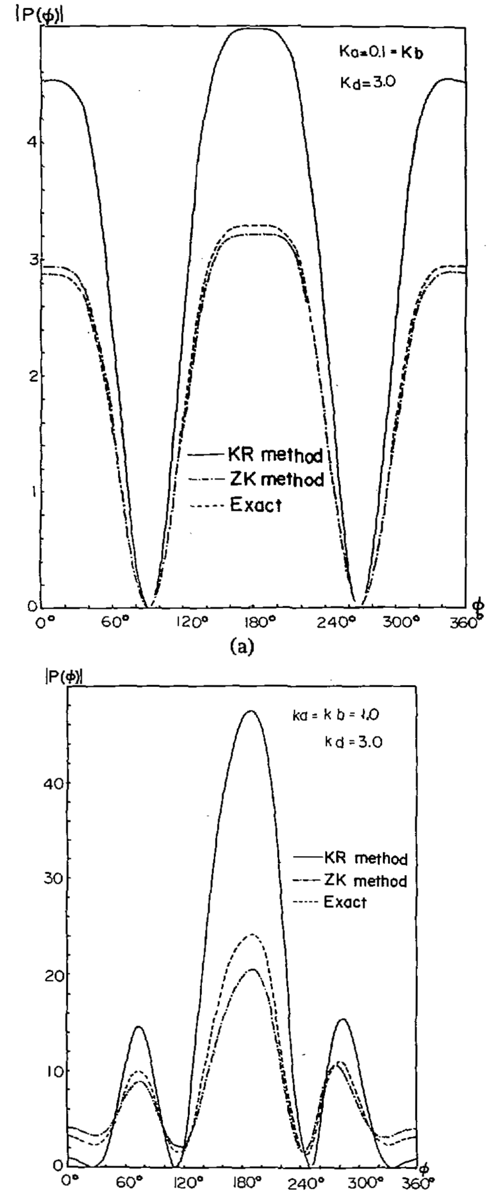

Fig. 2. Radiation pattern of scattered wave due to two parallel conducting cylinders. Separation between cylinders is $kd = 3.0$. (a) $ka = 0.2$. (b) $ka = 1.0$. In each case results based on the ZK method are close to exact results.

Fig. 3 shows the total scattering cross section for $ka = 0.01$, and the corresponding results for $ka = 1.0$ are shown in Fig. 4. It is seen from these figures that the results based on three different methods agree completely for $kd > 2.0$ when $ka = 0.01$. Similarly for $ka = 0.1$, if we choose $kd > 3.0$, each method gives the same results, though it is not shown here. However, for a rather thick cylinder $ka = 1.0$, the result based on the ZK method agrees with the exact solution for $kd > 5$, but the RK's result does not. The reason for this discrepancy may be that the addition terms associated with the derivatives of incident waves play an important role in the case of thick scatterers, since the KR method is extremely accurate when applied to the thin slit. The above numerical results are limited to the case of E-polarization, but the case of H-polarization can be treated similarly. In Fig. 5 we present the numerical results of the scattered patterns for H-polarization when $ka = kb = 5.41$ and $kd = 43.88$. The solid line represents the result based on the ZK method, and the dotted line is the single scattering

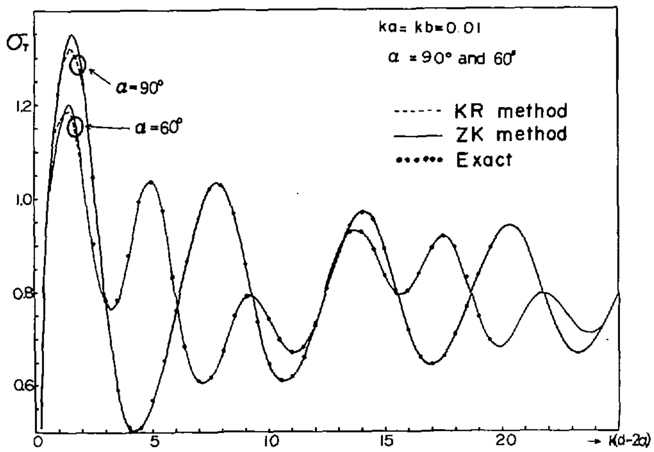

Fig. 3. Variation of total scattering cross section of two parallel conducting cylinders against separation kd for ka = 0.01. Three different methods give same results even for small values of kd.

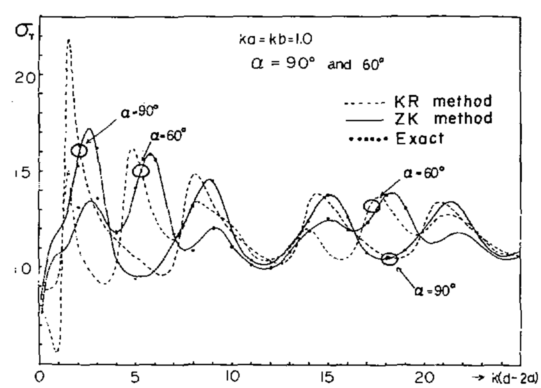

Fig. 4. Same as Fig. 3 except for ka = 1.0 Results of ZK method are closer to exact ones than those of KR method for all ranges of kd.

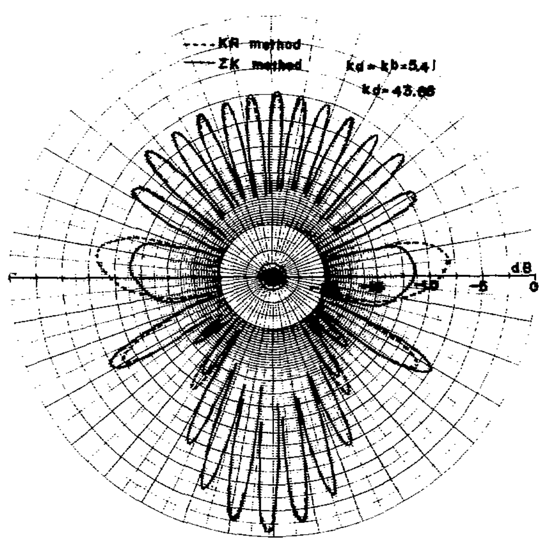

Fig. 5. Radiation pattern of scattered wave by two parallel conducting cylinders for E-polarization when $ka = kb$ = 5.4 and kd = 43.88. Solid line is result based on ZK method and dotted line is single scattering pattern multiplied by array factor.

patterns multiplied by the array factor. The rigorous results based on the iterative method are not given since the separation between the cylinder is too large to apply it. These two results are found to agree well for a wide range of observation angles. The figure shows that the forward scattering is strong compared to the backscattering. This is expected since this property is also valid for the case of a single cylinder. The results are compared with those derived by Young and Bertrand, and the agreement between them is good for the backscattering pattern.[1]

CONCLUSION

The validity of the asymptotic techniques in multiple scattering developed by Zitron and Karp, and by Karp and Russek is investigated by calculating total scattering cross sections numerically. From the results it is found that the ZK method gives accurate results and promises to be a useful tool in treating multiple scattering problems.

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[1] The foward scattering pattern of the results of Young and Bertrand is almost symmetrical to the back-scattering pattern for all cases described in [7]. It is doubtful that this kind of symmetry is valid for the two-cylinder problem. Their experimental method can cover the foward scattering pattern.

A Broadband Compound Waveguide Lens

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Abstract—A broadband compound waveguide lens which is the analog of the optical achromatic doublet is described and analyzed for

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