Low-Frequency Scattering from Two-Dimensional Perfect Conductors

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Abstract—**Exact expressions are obtained for the leading terms in the low-frequency expansions of the far field scattered by an arbitrarily shaped cylinder with finite cross section, an arbitrarily shaped cylindrical bump on a ground plane, and an arbitrarily shaped cylindrical dent in a ground plane. By inserting the low-frequency expansions of the incident plane wave and Green's function into exact integral equations for the surface current, integral equations are obtained for the leading terms in the low-frequency expansions of the surface current. Simple integrations of these leading terms of the current expansion yield the leading terms in the low-frequency expansions of the scattered fields. For the cylinder with finite cross section, the leading term in the low-frequency expansion of the TM scattered far field is explicitly given by an expression that is independent of the shape of the cylinder. The explicit expression for the low-frequency TE scattered far field contains three constants that depend only on the shape of the cylinder. These three constants are found from the solutions to two electrostatic problems. The explicit expressions for the low-frequency diffracted fields of a bump or dent contain one constant that depends only on the shape of the bump or dent. Remarkably, this single constant is the same for both TM and TE polarization and can be found from the solution to either an electrostatic or magnetostatic problem. The general low-frequency expressions are confirmed by comparing them to low-frequency results obtained from exact time-harmonic eigenfunction solutions.**

I. Introduction

The study of low-frequency scattering was initiated by Rayleigh [1] in 1897 and today the term "Rayleigh scattering" is often used instead of "low-frequency scattering." Kleinman [2] gives the following definition of Rayleigh scattering:

> *We are dealing with Rayleigh scattering when the far-zone field may be expanded in a convergent series in positive integral powers of the propagation constant k.*

The starting point in a Rayleigh scattering calculation is the expansion of the unknown fields in powers of k and then the determination, from Maxwell's equations and boundary conditions, of the unknown expansion coefficients, which are functions of the geometry of the scatterer and the angles of incidence and observation. This procedure is used by Stevenson [3] and by Asvestas and Kleinman [4] to determine the low-frequency electromagnetic scattering from perfectly electrically conducting three-dimensional bodies.

However, as noted in [2] this definition of Rayleigh scattering cannot be used for two-dimensional scattering problems since the scattered field in general does not have a convergent expansion in powers of k. The two-dimensional fields cannot be expanded in powers of k because the two-dimensional free-space Green's function is a Hankel function that has a branch point at $k = 0$, whereas the three-dimensional free-space Green's function is analytic in k.

A great deal of work on low-frequency scattering from general three-dimensional bodies has been published [2], but it seems that only van Bladel [5] and MacCamy [6] have done work on low-frequency scattering from general two-dimensional bodies. In [5] the low-frequency scattering from arbitrarily shaped dielectric and conducting cylinders with finite cross sections is considered. For transverse magnetic (TM) polarization [5] uses a time-harmonic integral equation to obtain the low-frequency behavior of the total current from which the low-frequency scattered field is calculated. For transverse electric (TE) polarization an expansion of the current in integral powers of k is assumed. As seen from the previous discussion and from the exact eigenfunction solutions for the circular cylinder [7, Chapter 2] and strip [7, Chapter 4] such an expansion does not exist. However, only the first two terms in this power series expansion are used in [5], and in this paper it will be shown that the final TE results in [5] for perfect conductors are also correct. Reference [6] contains the formal low-frequency expansion of the scattered fields from two-dimensional perfect conductors with finite cross section in cases where the incident field can be expanded in even powers of k. However, no explicit expressions for the expansion coefficients are given.

There are several reasons for being interested in two-dimensional scattering solutions. They can be used directly as approximate solutions to certain three-dimensional scattering problems. For example, the field close to the middle of a long finite rod can often be well approximated by the field from the corresponding infinite rod.

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Also, two-dimensional scattering solutions can be helpful for validating computer codes.

Another important reason for being interested in two-dimensional solutions is that they can be used to determine three-dimensional incremental length diffraction coefficients. These incremental length diffraction coefficients can in turn be used to determine scattering contributions from, for example, curved ridges (bumps) and channels (dents) that have constant cross sections. The three-dimensional incremental diffraction coefficients can be found directly from the corresponding two-dimensional far fields using a direct substitution approach. No integration, differentiation, or specific knowledge of the current on the conductor is needed. This direct substitution procedure was first developed by Shore and Yaghjian [8] for planar surfaces and then extended by Hansen and Yaghjian [9] to general two-dimensional scatterers. To apply this procedure one must, in general, be able to evaluate the two-dimensional far fields for both real and complex angles of observation. Therefore, one needs analytical expressions for the two-dimensional far fields.

Analytical expressions for two-dimensional far fields from single cylindrical bumps and dents can also be used in conjunction with the work of Twersky [10] to calculate the scattering from a random distribution of bumps or dents.

The purpose of this paper is to determine the leading terms in the low-frequency expansions of the scattered electromagnetic far fields for the following three types of two-dimensional perfectly electrically conducting scatterers:

1) An arbitrarily shaped cylinder with finite cross section.
2) An arbitrarily shaped cylindrical bump (protuberance) on a ground plane.
3) An arbitrarily shaped cylindrical dent (indentation) in a ground plane.

The results of this paper apply to bumps and dents that are continuously lined by a conductor and thus they are not generally valid for bumps and dents that contain slits. The scatterers are illuminated by plane waves propagating in directions normal to the axis of the cylinders, and the two polarization cases, TM and TE, are treated separately. Since the scatterers are perfectly conducting, the low-frequency solutions for normal incidence can be immediately generalized to obtain low-frequency solutions for oblique incidence (see, e.g., [11, section 8.15] or [7, chapter 1]).

As mentioned above, the unknown field for two-dimensional scatterers, unlike for three-dimensional scatterers, cannot be expanded in powers of k. Instead we take the following approach: the Green's functions and incident field in the time-harmonic integral equation for the current is expanded for small k to obtain an integral equation for the low-frequency current. From this low-frequency integral equation we find the leading terms in the low-frequency expansion of the current and calculate the low-frequency far field.

The low-frequency expressions for the scattered fields from the cylinders with finite cross section are derived in Section II. For the cylindrical bump and dent, the scattered fields are written as the sum of a known reflected field and a diffracted field. The low-frequency expressions for the diffracted field from the bump and dent are derived in Sections III and IV, respectively. In Section V, the low-frequency expressions are verified in cases where exact time-harmonic eigenfunction solutions exist. A summary of this work was first presented in [12].

II. Cylinder with Finite Cross Section

The cylinder with finite cross section illuminated by the plane-wave field $(\bar{E}^i, \bar{H}^i)$ is situated in a rectangular coordinate system shown in Fig. 1. The scatterer extends uniformly to infinity in the $+z$ and $-z$ directions, and the curve that describes the finite cross section of the scatterer in the $x-y$ plane is denoted by S. The outward normal to the scatterer is $\hat{n}$ and the tangent unit vector to S is $\hat{t} = \hat{z} \times \hat{n}$. A characteristic dimension of the scatterer in the $x-y$ plane is denoted by d. In addition to the rectangular coordinates (x, y, z), circular cylindrical coordinates (r, ϕ, z) given by $x = r\cos\phi$, $y = r\sin\phi$, and $z = z$ will also be used.

The incident plane-wave field propagates in a direction, designated by ϕ^i, normal to the z-axis. As usual, the scattered field $(\bar{E}^s, \bar{H}^s)$ is defined as the total field $(\bar{E}, \bar{H})$ minus the incident field $(\bar{E}^i, \bar{H}^i)$.

Throughout the paper $e^{j\omega t}$ time dependence is suppressed in all the time-harmonic equations.

A. Transverse Magnetic (TM) Polarization

The incident electric field is given by

$$\bar{E}^i(\bar{r}) = \hat{z}e^{jk(x\cos\phi^i + y\sin\phi^i)} \tag{1}$$

where k is the propagation constant. The low-frequency current is determined in Section II-A1), and in Section II-A2) it is integrated to get the low-frequency far field.

1) Low-Frequency Current: The low-frequency TM far field for the cylinder with finite cross section can be determined, as will be shown, from the total current flowing in the z-direction on the cylinder. To find this current begin with the two-dimensional electric field integral equation [13], [14]

$$E_z^i(\bar{r}) = jk\sqrt{\frac{\mu}{\epsilon}} \fint_S G(\bar{r}, \bar{r}')K_z(\bar{r}')\, ds', \qquad \bar{r} \in S \tag{2}$$

where μ and ϵ are the free-space permeability and permittivity, and the bar on the integral sign indicates that the singularity at $\bar{r} = \bar{r}'$ is excluded. Here $G(\bar{r}, \bar{r}')$ is the two-dimensional free-space Green's function

$$G(\bar{r}, \bar{r}') = -\frac{j}{4}H_0^{(2)}(k|\bar{r} - \bar{r}'|) \tag{3}$$

and $\bar{K}(\bar{r}) = \hat{z}K_z(\bar{r})$ is the current on the cylinder.

From the small argument approximation for the Hankel

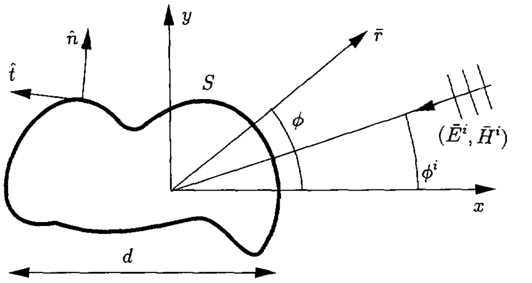

Fig. 1. Cylinder with finite cross section.

function one finds that

$$G(\bar{r}, \bar{r}') = G^0(\bar{r}, \bar{r}') - \frac{1}{2\pi} \ln \frac{kd}{2} - \frac{j}{4} - \frac{\gamma}{2\pi} + O\big((kd)^2 \ln kd\big) \quad (4)$$

where $G^0(\bar{r}, \bar{r}')$ is the static two-dimensional free-space Green's function given by

$$G^0(\bar{r}, \bar{r}') = -\frac{1}{2\pi} \ln \frac{|\bar{r} - \bar{r}'|}{d} \quad (5)$$

γ is Euler's constant, and d is a characteristic dimension of the cylinder. The term of order $(kd)^2 \ln kd$ in (4) is nonsingular at $\bar{r} = \bar{r}'$; in fact, it tends to zero as $\bar{r} \to \bar{r}'$.

Because the incident electric field in (1) for low frequencies is approximately equal to $\hat{z}$ on the circumference S of the scatterer it is assumed that the total current for low frequencies is nonzero, i.e.,

$$\int_S K_z(\bar{r}')\, ds' \neq 0. \quad (6)$$

This assumption is confirmed by the eigenfunction solutions for the circular cylinder [7, chapter 2] and strip [7, chapter 4].

Inserting the expansion (4) for the Green's function into the integral equation (2) and using (6) yield the low-frequency behavior of the total current

$$-\ln kd \frac{jk}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \int_S K_z(\bar{r}')\, ds' \sim 1, \qquad kd \to 0. \quad (7)$$

2) Low-Frequency Scattered Far Field: The low-frequency far field is found by integrating the low-frequency current given in (7). From the asymptotic expansion of the Hankel function it is found that

$$G(\bar{r}, \bar{r}') \sim \frac{e^{-j\pi/4}}{2\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \cdot \big[1 + jk(x' \cos\phi + y' \sin\phi) + O\big((kd)^2\big)\big], \quad r \to +\infty. \quad (8)$$

The expression for the scattered field

$$E_z^s(\bar{r}) = -jk \sqrt{\frac{\mu}{\epsilon}} \int_S G(\bar{r}, \bar{r}') K_z(\bar{r}')\, ds' \quad (9)$$

and the asymptotic formulas (7) and (8), combine to produce the final low-frequency expression for the far field [5]

$$E_z^s(\bar{r}) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-j\pi/4}}{\ln kd} \frac{e^{-jkr}}{\sqrt{kr}}. \quad (10)$$

Note that $\ln kd \sim \ln Akd$ as $kd \to 0$ when A is a positive constant so that d can be any characteristic dimension of the cross section of the scatterer. Further note that the low-frequency scattered field does not depend on the shape of the finite cross section but only on its characteristic dimension.

The expression (10) agrees with the low-frequency results for the circular cylinder [7, chapter 2] and strip [7, chapter 4] obtained from exact eigenfunction solutions.

It should be emphasized that the expression in (10) for the far field is not directly related to the solution to the purely electrostatic or magnetostatic problem in which the scatterer is situated in the static field $\bar{E}^i|_{k=0}$ or $\bar{H}^i|_{k=0}$, respectively.[1] This can be explained as follows.

Since the total electrostatic field has only a z-component, Maxwell's equations imply that it must be a constant and therefore zero because it is zero on the surface of the scatterer. Consequently, the scattered electrostatic field equals the negative of the incident field. The normal component of this electrostatic solution is zero everywhere so that the charge on the conductor is zero. Moreover, this electrostatic solution gives no information about the current on the conductor and thus no information about the scattered time-harmonic fields.

As part of the solution to the magnetostatic equations ($\nabla \times \bar{H} = 0$, $\nabla \cdot \bar{H} = 0$, $\hat{n} \cdot \bar{H} = 0$, and $\hat{n} \times \bar{H} = \bar{K}$ on S) we can have a total current on the conductor. However, the value of this current is not determined by these magnetostatic equations.

B. Transverse Electric (TE) Polarization

In this section the low-frequency far field is determined for TE polarization. The first two terms in the low-frequency expansion of the current are derived in Section II-B1), and these terms are integrated in Section II-B2) to get the far field. For TE polarization, the incident magnetic field is given by the right-hand side of (1).

1) Low-Frequency Current: The low-frequency current is obtained by inserting the low-frequency expansions for the Green's function and the incident field into the two-dimensional magnetic field integral equation [14]

$$H_z^i(\bar{r}) = \fint_S K_t(\bar{r}') \frac{\partial}{\partial n'} G(\bar{r}, \bar{r}')\, ds' - \frac{1}{2} K_t(\bar{r}), \qquad \bar{r} \in S \quad (11)$$

where the bar on the integral sign indicates a Cauchy principal value integration. The expansion (4) of the

[1] Recently, De Smedt has derived a higher order approximation that equals (10) with $\ln kd$ replaced by $\ln(\gamma' kR/2) + j\pi/2$, and with the equivalent radius R determined from a static solution [23].

Green's function shows that

$$\frac{\partial}{\partial n'}G(\bar{r},\bar{r}') = \frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}') + O\big((kd)^2 \ln kd\big) \quad (12)$$

where the term of order $(kd)^2 \ln kd$ is nonsingular at $\bar{r} = \bar{r}'$; in fact, it tends to zero as $\bar{r} \to \bar{r}'$.

Letting $kd \to 0$ in the integral equation (11) and inserting the expansion (12) reveals that the first term K_t^0 in the low-frequency expansion of the current K_t satisfies the integral equation

$$1 = \oint_S K_t^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')\,ds' - \frac{1}{2}K_t^0(\bar{r}), \qquad \bar{r} \in S. \quad (13)$$

The uniqueness of the solution to this integral equation has been proven in [15].

Equation (13) is the integral equation for the static current when the scatterer is situated in the impressed magnetostatic field $\overline{H}^i_{\text{static}} = \overline{H}^i|_{k=0} = \hat{z}$. To prove this, write the magnetostatic field $\overline{H}^0$ as the curl of the vector potential $\overline{A}^0$, i.e., $\overline{H}^0 = \nabla \times \overline{A}^0$, and use the simple relation $\overline{A}^0 = \int_S G^0 \overline{K}^0\,ds'$ along with the boundary condition that $K_t^0 = -H_z^0$ on S.

Since $\nabla \times \overline{H}^0 = 0$ we find that H_z^0 is a constant. Furthermore, the impressed magnetostatic field $\hat{z}$ satisfies the boundary condition $\hat{n} \cdot \overline{H}^0 = 0$ on the conductor, and since the scattered field must vanish far away from the scatterer it follows that the scattered magnetostatic field is zero and therefore $H_z^0 = 1$. Consequently, we have that $K_t^0 = -1$.

To get the second term K_t^1 in the low-frequency expansion of the current K_t, insert the expansion (12) of the Green's function and (13) into the integral equation (11) to obtain

$$jk(x\cos\phi^i + y\sin\phi^i) = \oint_S K_t^1(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')\,ds' - \frac{1}{2}K_t^1(\bar{r}), \qquad \bar{r} \in S \quad (14)$$

which implies that K_t^1 is of order kd.

It will now be shown that K_t^1 can be found from the solution to the electrostatic problem in which the scatterer is situated in the impressed electrostatic field

$$\overline{E}^i_{\text{static}} = \overline{E}^i|_{k=0} = \sqrt{\frac{\mu}{\epsilon}}\,(\hat{x}\sin\phi^i - \hat{y}\cos\phi^i). \quad (15)$$

Let the electrostatic solution be denoted by $\overline{E}^0$. In [16, appendix A] it is proven from $\nabla \times \overline{E}^0 = 0$ and the total charge on the conductor being zero, that the field $-E_y^0\hat{x} + E_x^0\hat{y}$ is conservative. Therefore, it can be written as the gradient of a scalar potential, $-E_y^0\hat{x} + E_x^0\hat{y} = \sqrt{(\mu/\epsilon)}\,\nabla F_z^0$. Defining $\overline{F}^0 = F_z^0\hat{z}$ one finds that $\overline{E}^0 = \sqrt{(\mu/\epsilon)}\,\nabla \times \overline{F}^0$ and that the scalar potential F_z^0 satisfies the same differential equation (Laplace's equation) and the same boundary conditions on the conductor (Neumann type) as the z-component of a magnetostatic field. Thus, F_z^0 satisfies the integral equation (13) with 1 and K_t^0 replaced by F_z^{0i} and $-F_z^0$, respectively. The impressed electrostatic field given in (15) corresponds to the impressed vector potential $F_z^{0i}(\bar{r}) = x\cos\phi^i + y\sin\phi^i + C$ where C is a constant. The integral equation for F_z^0 is then

$$x\cos\phi^i + y\sin\phi^i + C = -\oint_S F_z^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')\,ds' + \frac{1}{2}F_z^0(\bar{r}), \qquad \bar{r} \in S. \quad (16)$$

Using the result from [14, appendix] of the integration near the singularity at $\bar{r}' = \bar{r}$ and the divergence theorem in the region bounded by S one finds that

$$C = -\oint_S \frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')C\,ds' + \frac{1}{2}C, \qquad \bar{r} \in S. \quad (17)$$

Consequently, the constant function C on the left-hand side of the integral equation (16) simply adds the constant C to the solution to (16) with $C = 0$. Because a constant may be added to the potential F_z^0 without changing the field $\overline{E}^0$, C can be set equal to zero in the integral equation (16).

Comparing the integral equations (14) and (16) (with $C = 0$) and recalling that the solutions are unique [15], shows that $K_t^1(\bar{r}) = -jkF_z^0(\bar{r})$, $\bar{r} \in S$; thereby completing the proof that the second term in the low-frequency expansion of the current can be found from the electrostatic solution. The current can thus be written

$$K_t(\bar{r}) = -1 - jkF_z^0(\bar{r}) + K_t^r(\bar{r}) \quad (18)$$

where F_z^0 satisfies the electrostatic integral equation (16) and $(1/kd)K_t^r \to 0$ as $kd \to 0$.

In summary it has been shown that the first and second terms in the TE low-frequency expansion of the current are of order $(kd)^0$ and kd, respectively. Since the second term in the low-frequency expansion (12) of the derivative of the free-space Green's function is of order $(kd)^2 \ln kd$, the integral equation (11) implies that the third term in the low-frequency expansion of the current, in general, will be a function of kd and $\ln kd$. Therefore, as noted in the Introduction, the current cannot be expanded in a power series in kd.

2) Low-Frequency Scattered Far Field: To calculate the far field scattered by the current given in the expansion (18), insert the asymptotic expansion of the derivative of the Green's function

$$\frac{\partial}{\partial n'}G(\bar{r},\bar{r}') \sim k\frac{e^{j\pi/4}}{2\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}} \cdot \Big[1 + jk(x'\cos\phi + y'\sin\phi) + O\big((kd)^2\big)\Big]\hat{n}' \cdot \hat{r} \quad (19)$$

as $r \to +\infty$, and the current expansion (18) into the expression

$$H_z^s(\bar r) = -\int_S K_t(\bar r') \frac{\partial}{\partial n'} G(\bar r, \bar r')\, ds' \tag{20}$$

for the scattered magnetic field to obtain the scattered field to order $(kd)^2$ as given by

$$H_z^s(\bar r) \sim -(kd)^2 \frac{e^{-j\pi/4}}{2\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \frac{1}{d^2} \cdot \left[\hat r \cdot \int_S (x'\cos\phi + y'\sin\phi)\hat n'\, ds' + \hat r \cdot \int_S F_z^0(\bar r')\hat n'\, ds' \right]. \tag{21}$$

From the divergence theorem it is found that the first term in the brackets of (21) equals the area bounded by S, i.e., the area of the cross section of the scatterer. If we introduce the electrostatic potential F_z^{0x} satisfying

$$x = -\oint_S F_z^{0x}(\bar r') \frac{\partial}{\partial n'} G^0(\bar r, \bar r')\, ds' + \frac{1}{2} F_z^{0x}(\bar r), \qquad \bar r \in S \tag{22}$$

and the electrostatic potential F_z^{0y} satisfying (22) with x replaced by y, the total electrostatic potential may be written as $F_z^0 = \cos\phi^i F_z^{0x} + \sin\phi^i F_z^{0y}$. Using this result in (21) along with the dipole moment reciprocity theorem proven in [16, appendix B], one finds that the scattered magnetic far field may be written as

$$H_z^s(\bar r) \sim -(kd)^2 \frac{e^{-j\pi/4}}{2\sqrt{2\pi}} \frac{1}{d^2} \left[A_S + C_1 \sin\phi \sin\phi^i + C_2 \cos\phi\cos\phi^i + C_3 \sin(\phi + \phi^i) \right] \frac{e^{-jkr}}{\sqrt{kr}} \tag{23}$$

where

$$C_1 = -\int_S F_z^{0y}(\bar r) \frac{\partial x}{\partial s}\, ds, \qquad C_2 = \int_S F_z^{0x}(\bar r)\frac{\partial y}{\partial s}\, ds,$$
$$C_3 = -\int_S F_z^{0x}(\bar r) \frac{\partial x}{\partial s}\, ds \tag{24}$$

and A_S is the area of the cross section of the cylinder.

The integral equation (22) for F_z^{0x} and the corresponding one for F_z^{0y} are of the same form as the time-harmonic magnetic field integral equation. Therefore they should be easy to solve numerically when no part of the scatterer is infinitesimally thin [17, p. 168]. When a part of the scatterer is infinitesimally thin they degenerate and cannot be used to determine the constants in (24).

Alternative expressions for these constants can be found that remain valid when part or all of the scatterer is infinitesimally thin. To do this, introduce σ^{0x} and σ^{0y}, which are the electrostatic charges on the cylinder situated in the impressed electrostatic fields $\hat x$ and $\hat y$, respectively. An integral equation for σ^{0x} can be derived from the scalar potential ψ for the electrostatic field. The potential ψ satisfies Laplace's equation and the potential for the impressed electrostatic field is given by $\psi^i = -x + C$, where C is a constant. Noting that the potential for the scattered field is given by $\psi^s = \int_S G^0 \sigma^{0x}/\epsilon\, ds'$ and that the total potential is constant on S, one obtains an integral equation involving an unknown constant. This constant is evaluated at an arbitrary observation point $\bar r_0 = (x_0, y_0) \in S$ so that σ^{0x} satisfies the integral equation

$$\oint_S \left[G^0(\bar r, \bar r') - G^0(\bar r_0, \bar r')\right] \frac{\sigma^{0x}(\bar r')}{\epsilon}\, ds' = x - x_0, \qquad \bar r \in S. \tag{25}$$

In [15] it is shown that the integral equation (25) along with the condition that the total charge on the conductor is zero, i.e., $\int_S \sigma^{0x}\, ds = 0$, determine the electrostatic charge σ^{0x} uniquely. Similarly, σ^{0y} is determined by (25) with x replaced by y.

From the definition of the potentials F_z^{0x} and F_z^{0y} it is easy to show that $(\sigma^{0y}/\epsilon) = -(\partial/\partial s)F_z^{0x}$ and that $(\sigma^{0x}/\epsilon) = (\partial/\partial s)F_z^{0y}$. From these relations and integration by parts in (24), the following alternative formulas are obtained for the three constants C_1, C_2, and C_3

$$C_1 = \int_S \frac{\sigma^{0x}(\bar r)}{\epsilon} x\, ds, \qquad C_2 = \int_S \frac{\sigma^{0y}(\bar r)}{\epsilon} y\, ds,$$
$$C_3 = -\int_S \frac{\sigma^{0y}(\bar r)}{\epsilon} x\, ds. \tag{26}$$

Combining the expression (23) for the scattered field with the expressions (26) for the three constants shows that the scattered field consists of a magnetic dipole in the z-direction (the term with A_S) plus an electric dipole in the $x-y$ plane (the terms with C_1, C_2, and C_3) [11, section 3.8].

We have now given the exact analytical expressions for the first terms in the TM and TE low-frequency far fields scattered from the cylinder with finite cross section. The TM expression is independent of the shape of the cylinder. The TE expression is completely determined by calculating three constants that depend only on the shape of the cross section of the cylinder. These three constants are found from the electrostatic solutions for x and y directed impressed electrostatic fields.

III. Cylindrical Bump on an Infinite Ground Plane

In this section the low-frequency expressions are derived for the field diffracted by a cylindrical bump on a ground plane illuminated by a plane wave. Both the bump and the ground plane are perfectly conducting. The bump extends uniformly to infinity in the $+z$ and $-z$ directions and the curve that describes the cross section of the bump in the $x-y$ plane is denoted by B as shown in Fig. 2. The outward normal to the bump is $\hat n$ and the tangent unit vector to B is $\hat t = \hat z \times \hat n$. The equation for the ground

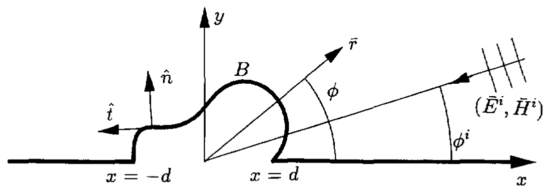

Fig. 2. Cylindrical bump on a ground plane.

plane is $y = 0$ and the intersections between the bump and the ground plane are given by $y = 0$, $x = \pm d$.

The incident field $(\bar{E}^i, \bar{H}^i)$ and the cylindrical coordinates (r, ϕ, z) are the same as in Section II. The field reflected in the ground plane $y = 0$ when there is no bump is denoted by $(\bar{E}^r, \bar{H}^r)$. We write the total field as $(\bar{E}, \bar{H}) = (\bar{E}^i, \bar{H}^i) + (\bar{E}^r, \bar{H}^r) + (\bar{E}^d, \bar{H}^d)$, where $(\bar{E}^d, \bar{H}^d)$ is by definition the diffracted field.

It will be shown that the diffracted field is the scattered field in an equivalent scattering problem where the scatterer is the bump plus its image in the ground plane, and the incident field is $(\bar{E}^i + \bar{E}^r, \bar{H}^i + \bar{H}^r)$ as shown in Fig. 3. The curve that describes the image of the bump in the $y = 0$ plane is denoted by B_i and the cross section of the equivalent scatterer is therefore $B \cup B_i$.

By considering the TM and TE polarizations separately it is easily seen that the scattered field in the equivalent scattering problem satisfies the boundary conditions on the ground plane. Since $(\bar{E}^i + \bar{E}^r, \bar{H}^i + \bar{H}^r)$ also satisfies these boundary conditions it is clear from the uniqueness of the solution that the scattered field in the equivalent scattering problem is the diffracted field in upper half-space of the bump scattering problem. This method of images for constructing an equivalent scattering problem was first used by Rayleigh [18] when he solved for scattering from the semicircular bump.

A. Transverse Magnetic (TM) Polarization

The TM incident electric field is given by (1), and the reflected electric field is also given by (1) with $\hat{z}$ and y replaced by $-\hat{z}$ and $-y$, respectively.

If we use the formula (10) for the scattered far field, the contributions from the incident field $\bar{E}^i$ and the reflected field $\bar{E}^r$ cancel. Consequently, the diffracted field is zero to order $(1/\ln kd)$, and we cannot use the results from Section II-A because we need higher order terms. Therefore we must start a new investigation of the low-frequency current on the equivalent scatterer given by $B \cup B_i$.

1) Low-Frequency Current: The current K_z on the equivalent scatterer satisfies the integral equation (2) with E_z^i replaced by $E_z^i + E_z^r$, and S replaced by $B \cup B_i$. Because $E_z^i + E_z^r$ is an odd function of y and the equivalent scatterer is symmetric about the $y = 0$ plane, it follows that the current K_z is also an odd function of y, i.e., $K_z(x, y) = -K_z(x, -y)$. Consequently, the total current flowing along the equivalent scatterer is zero, i.e., $\int_{B \cup B_i} K_z(\bar{r})\, ds = 0$. Expanding $E_z^i + E_z^r$ in powers of k, and using the expansion (4) of the Green's function in the integral equation for K_z shows that the first term K_z^0 in the low-frequency expansion of K_z satisfies the integral equation

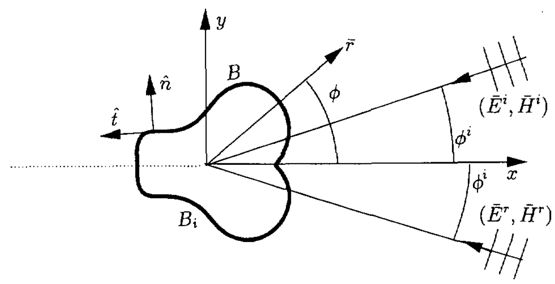

Fig. 3. Equivalent bump scattering problem.

$$2y \sin \phi^i = \sqrt{\frac{\mu}{\epsilon}} \int_{B \cup B_i} G^0(\bar{r}, \bar{r}') K_z^0(\bar{r}')\, ds', \qquad \bar{r} \in B \cup B_i. \quad (27)$$

In [15] it is proven that the integral equation (27) along with the condition that K_z^0 is an odd function of y determine K_z^0 uniquely.

We will show that K_z^0 is the current in the magnetostatic scattering problem with the equivalent scatterer situated in the impressed magnetostatic field

$$\bar{H}^i_{\text{static}} = \bar{H}^i|_{k=0} + \bar{H}^r|_{k=0} = -2\sqrt{\frac{\epsilon}{\mu}} \sin \phi^i \hat{x}. \quad (28)$$

Introduce a vector potential $\bar{A}^0 = A_z^0 \hat{z}$, $A_z^0 = \int_S K_z^0 G^0\, ds'$ and use the boundary condition that a magnetostatic field has zero normal component on $B \cup B_i$ to prove that A_z^0 is constant on $B \cup B_i$. This boundary condition for A_z^0, along with the fact that the potential for the impressed magnetostatic field (28) must be $-2y\sqrt{(\epsilon/\mu)} \sin \phi^i + C$ where C is a constant, now give an integral equation for K_z^0. Since K_z^0 is an odd function of y and $B \cup B_i$ is symmetric about $y = 0$, C is equal to zero and we obtain the integral equation (27).

The leading term in the low-frequency expansion of the current is thus of order $(kd)^0$. From the integral equation for K_z it follows that the next term, in general, is a function of kd and $\ln kd$ so that the current cannot be expanded in powers of kd.

It is convenient to define a current K_z^{0B} equal to the magnetostatic current when the equivalent scatterer is situated in the impressed magnetostatic field $\hat{x}$. From the expression (28) for the impressed magnetostatic field and from the integral equation (27) it follows that the current K_z^{0B} is determined by

$$-y = \int_{B \cup B_i} G^0(\bar{r}, \bar{r}') K_z^{0B}(\bar{r}')\, ds', \qquad \bar{r} \in B \cup B_i \quad (29)$$

and the condition that K_z^{0B} must be an odd function of y.

We then can write the current as

$$K_z = -2\sqrt{\frac{\epsilon}{\mu}} \sin \phi^i K_z^{0B} + K_z^r \qquad (30)$$

where $K_z^r \to 0$ as $kd \to 0$.

2) Low-Frequency Diffracted Far Field: The low-frequency diffracted far field from the bump is the scattered far field in the upper half-space of the equivalent scattering problem of the bump and its image.

The electric field expression (9), the far-field approximation (8) to the Green's function, and the fact that the current is an odd function of y, show that the diffracted far field to order $(kd)^2$ is given by

$$E_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{B_0}{d^2} \sin \phi \sin \phi^i \frac{e^{-jkr}}{\sqrt{kr}} \qquad (31)$$

where

$$B_0 = \int_B y K_z^{0B}(\bar{r})\, ds. \qquad (32)$$

The magnetostatic current K_z^{0B} is an odd function of y and satisfies the static integral equation (29).

It is seen from the formula (31) that the diffracted field is the field of a magnetic dipole in the x-direction [11, section 3.8].

B. Transverse Electric (TE) Polarization

In the TE case the incident magnetic field is given by the right-hand side of (1), and the reflected magnetic field is given by the right-hand side of (1) with y replaced by $-y$. Because the contributions from the incident and reflected fields in the TE case do not cancel, the results of Section II-B apply directly, so we do not have to investigate the current on the equivalent scatterer.

1) Low-Frequency Diffracted Far Field: If we let A_B denote the area of the region bounded by B and the line $y = 0$, $|x| \le d$ (see Fig. 2) then the diffracted far field is given by

$$H_z^d(\bar{r}) = H_z^s(\bar{r})|_{A_S = 2A_B} + H_z^s(\bar{r})|_{A_S = 2A_B,\, \phi^i \to -\phi^i} \qquad (33)$$

where H_z^s is the scattered field given in (23). The first term in (33) is the contribution from $(\bar{E}^i, \bar{H}^i)$ and the second term is the contribution from $(\bar{E}^r, \bar{H}^r)$. Because of the symmetry of the scatterer we find that

$$\int_{B \cup B_i} x\sigma^{0y}(\bar{r})\, ds = 0,$$

$$\int_{B \cup B_i} y\sigma^{0y}(\bar{r})\, ds = 2\int_B y\sigma^{0y}(\bar{r})\, ds \qquad (34)$$

where σ^{0x} and σ^{0y} are the electrostatic charges in the electrostatic problems with the equivalent scatterer situated in the impressed electrostatic fields $\hat{x}$ and $\hat{y}$, respectively.

With substitution of (34) into the field expression (23), (33) becomes

$$H_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{e^{-jkr}}{\sqrt{kr}}\frac{1}{d^2} \cdot \left[A_B + \cos\phi \cos\phi^i \int_B \frac{\sigma^{0y}(\bar{r}')}{\epsilon} y'\, ds'\right]. \qquad (35)$$

The diffracted field therefore consists of the contribution from a magnetic dipole in the z-direction and an electric dipole in the y-direction [11, section 3.8].

We will now show that, remarkably, the integral in the expression (35) for the TE diffracted field equals the negative of the constant B_0 in (32) occurring in the expression (31) for the TM diffracted field. To do this, introduce a scalar potential for the electrostatic field and use the fact that the impressed scalar potential must be $C - y$ where C is a constant. Furthermore, the boundary condition of zero tangential electric field requires that the total scalar potential be constant on the equivalent scatterer, and the charge σ^{0y} therefore must satisfy the integral equation

$$y - C = \int_{B \cup B_i} G^0(\bar{r}, \bar{r}')\frac{\sigma^{0y}(\bar{r}')}{\epsilon}\, ds', \qquad \bar{r} \in B \cup B_i \qquad (36)$$

where the constant C is undetermined at this point of the derivation.

Because the impressed electrostatic field is symmetric about the ground plane $y = 0$, the electrostatic charge σ^{0y} must be an odd function of y. Consequently, the integral in (36) must be an odd function of y and the constant C must be zero.

Comparing the integral equation (36) (with $C = 0$) to the integral equation (29) for the current K_z^{0B}, and noting that the solutions are unique [15], one finds that $\epsilon K_z^{0B} = -\sigma^{0y}$ and therefore the TE diffracted field is given by

$$H_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{1}{d^2} \cdot \left[A_B - B_0 \cos\phi \cos\phi^i\right]\frac{e^{-jkr}}{\sqrt{kr}} \qquad (37)$$

where B_0 is given in (32).

We have now derived the exact expressions for the leading terms in the low-frequency expansions of the diffracted far fields from arbitrarily shaped bumps. Our general analysis has proven the rather remarkable result that the constant in the TM magnetic dipole term equals the negative of the constant in the TE electric dipole term for arbitrarily shaped bumps. In the following section we will show that this result also holds for diffraction from a two-dimensional dent in a ground plane.

IV. Cylindrical Dent in a Ground Plane

In this section we derive the low-frequency expressions for the fields diffracted by a cylindrical dent in a ground plane illuminated by a plane wave. Both the dent and the ground plane are perfectly conducting. The dent extends uniformly to infinity in the $+z$ and $-z$ directions and the curve that describes the cross section of the dent in the $x-y$ plane is denoted by D as shown in Fig. 4. The ground plane is given by $y = 0$ and the dent intersects the ground plane at $y = 0$, $x = \pm d$. The line segment given by $y = 0$, $|x| \leq d$ that caps the dent is called the aperture and is denoted by A. The image of D with respect to the ground plane $y = 0$ is denoted by D_i, and the part of the upper half plane ($y > 0$) outside the image D_i is called F. The normal to D is $\hat{n}$ and the tangent unit vector to D is $\hat{t} = \hat{z} \times \hat{n}$. Normal and tangent unit vectors to A are $\hat{y}$ and $-\hat{x}$, respectively. The incident field $(\bar{E}^i, \bar{H}^i)$, the reflected field $(\bar{E}^r, \bar{H}^r)$, the diffracted field $(\bar{E}^d, \bar{H}^d)$, and the cylindrical coordinates (r, ϕ, z) are defined as in the previous section.

Although the dent problem is very similar to the bump problem, the analysis of the dent is much more complicated and thus warrants a separate analysis. The main reason for the complication is that the dent problem, unlike the bump problem, cannot be reduced using image theory to an equivalent problem that involves only a cylinder of finite cross section without a ground plane. One therefore has to deal with integral equations more complicated than the usual two-dimensional electric and magnetic field integral equations. It is essential to the derivation that the integral equation involves only fields and currents in a region whose maximum dimension tends to zero as the characteristic dimension of the dent tends to zero. We found that the most convenient integral equations for this purpose were coupled integral equations first derived by Asvestas and Kleinman [19]. Here we give an alternative derivation of these coupled integral equations directly from the two-dimensional Stratton-Chu formulas [20, p. 466]:

$$\bar{E}(\bar{r}) - \bar{E}^i(\bar{r}) = \int_\Omega [j\omega\mu(\hat{n}' \times \bar{H}(\bar{r}'))G(\bar{r},\bar{r}') - (\hat{n}' \times \bar{E}(\bar{r}')) \times \nabla' G(\bar{r},\bar{r}') - (\hat{n}' \cdot \bar{E}(\bar{r}'))\nabla' G(\bar{r},\bar{r}')]\, ds' \quad (38)$$

and

$$\bar{H}(\bar{r}) - \bar{H}^i(\bar{r}) = -\int_\Omega [j\omega\epsilon(\hat{n}' \times \bar{E}(\bar{r}'))G(\bar{r},\bar{r}') + (\hat{n}' \times \bar{H}(\bar{r}')) \times \nabla' G(\bar{r},\bar{r}') + (\hat{n}' \cdot \bar{H}(\bar{r}'))\nabla' G(1,\bar{r}')]\, ds' \quad (39)$$

where Ω is a curve enclosing free space and the sources of the incident field, $\hat{n}$ is the outward normal to Ω, and $\bar{r}$ is a point inside Ω. If Ω does not enclose the sources of the incident field (only free space) the right-hand sides of (38) and (39) equal simply the total fields.

A. Transverse Magnetic (TM) Polarization

The incident electric field is given by (1), and the reflected electric field is also given by (1) with $\hat{z}$ and y replaced by $-\hat{z}$ and $-y$, respectively.

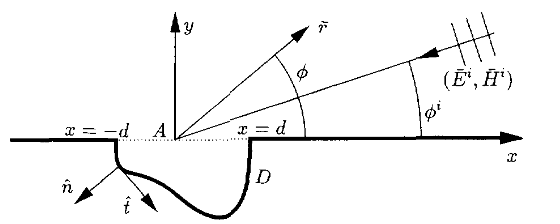

Fig. 4. Cylindrical dent in a ground plane.

1) Low-Frequency Current: To get the first integral equation let Ω consist of the surface of the perfect conductor and the semicircle of infinite radius above the ground plane. (The radiation condition on the fields can be used to prove that the integration over the semicircle of infinite radius is zero.) Inserting the boundary conditions $\hat{n} \times \bar{E} = 0$, $\hat{n} \times \bar{H} = -\bar{K}$, and $\hat{n} \cdot \bar{H} = 0$ on the conductor into (39), placing the observation point in the aperture A, and using the fact that $(\partial/\partial y')G(\bar{r},\bar{r}') = 0$ when $y = y' = 0$, $x \neq x'$, yields

$$\int_D K_z(\bar{r}') \frac{\partial}{\partial y'} G(\bar{r},\bar{r}')\, ds' = H_x^i(\bar{r}) - H_x(\bar{r}), \quad \bar{r} \in A \quad (40)$$

which is a relation between the current in the dent and the magnetic field in the aperture.

To determine the current in the dent we need one more relation of this kind. Let $\Omega = A \cup D$, so that the formula (38) without the incident field (since Ω here does not enclose the sources of the incident field) and with the free-space Green's function replaced by the Neumann Green's function G_N

$$G_N(\bar{r},\bar{r}') = G(\bar{r},\bar{r}') + G(\bar{r},\bar{r}_i'), \qquad \bar{r}_i = x\hat{x} - y\hat{y} \quad (41)$$

produces a relation between the current in D, the magnetic field in A, and the electric field in the interior of $A \cup D$. With the observation point on D and the boundary condition that $\bar{E} = E_z\hat{z} = 0$ on D, this relation becomes

$$\int_D G_N(\bar{r},\bar{r}')K_z(\bar{r}')\, ds' + 2\int_A G(\bar{r},\bar{r}')H_x(\bar{r}')\, ds' = 0, \quad \bar{r} \in D. \quad (42)$$

The relations (40) and (42) constitute a pair of coupled integral equations that determine the current K_z in D and the magnetic field H_x in A. These coupled integral equations will now be used to determine the first term in the low-frequency expansion of the current in the dent.

With the low-frequency expansion (12) of the derivative of the Green's function, the integral equation (40) becomes

$$\int_D K_z^0(\bar{r}') \frac{\partial}{\partial y'} G^0(\bar{r},\bar{r}')\, ds' = -\sqrt{\frac{\epsilon}{\mu}} \sin\phi^i - H_x^0(\bar{r}), \quad \bar{r} \in A \quad (43)$$

where K_z^0 and H_x^0 are the first terms in the low-frequency expansions of the current in the dent and the magnetic field in the aperture, respectively. G^0 is the static two-dimensional free-space Green's function given in (5). Inserting the low-frequency expansion of the free-space Green's function (4) and the corresponding expansion of the Neumann Green's function in the integral equation (42) leads to

$$\int_D K_z^0(\bar{r}')\,ds' + \int_A H_x^0(\bar{r}')\,ds' = 0 \tag{44}$$

and

$$\int_D G_N^0(\bar{r},\bar{r}')K_z^0(\bar{r}')\,ds' + 2\int_A G^0(\bar{r},\bar{r}')H_x^0(\bar{r}')\,ds' = 0, \qquad \bar{r} \in D. \tag{45}$$

Equations (43)–(45) constitute a set of coupled integral equations that determine the low-frequency current on D. In [15] it is proven that these three coupled integral equations have a unique solution.

We will now show that they are the integral equations for the magnetostatic current when the scatterer is situated in the impressed magnetostatic field

$$\bar{H}^i_{\text{static}} = -\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i\hat{x}. \tag{46}$$

For the dent problem we consider the magnetostatic problem in which the total magnetostatic field $\bar{H}^0$ satisfies $\hat{n}\cdot\bar{H}^0 = 0$ and $\hat{n}\times\bar{H}^0 = -\bar{K}^0$ on the conductor. Furthermore, it can be proven that these boundary conditions make the total static field zero below the ground plane and dent.

It is easy to show that a magnetostatic field $\bar{H}^0$ satisfies the Stratton-Chu formula (39) with $\bar{H}$ replaced by $\bar{H}^0$, G replaced by G^0, and $\omega = 0$. One may now repeat the derivation that led to (40) and show that (43) is indeed satisfied by the magnetostatic field and current when the impressed field is given by (46). To get (44) and (45) introduce a vector potential $\bar{A}^0 = A_z^0\hat{z}$ for the magnetostatic field $\bar{H}^0$ so that $\bar{H}^0 = \nabla\times\bar{A}^0$, and use the boundary condition $\hat{n}\cdot\bar{H}^0 = 0$ on D to find that A_z^0 is constant on D. Choosing $A_z^0 = 0$ on D and using Green's second identity one finds that (44) and (45) are satisfied by every magnetostatic field and current. Since the solution to (43)–(45) is unique [15], $\bar{H}^0$ and $\bar{K}^0$ are the magnetostatic field and current in the static problem with the dent situated in the impressed field (46).

If we let $\bar{K}^{0D}$ and $\bar{H}^{0D}$ be the magnetostatic current and field in the magnetostatic field problem when the scatterer is situated in the impressed magnetostatic field $(1/2)\hat{x}$, we obtain from (46) that the current in the dent can be written as

$$K_z = -2\sqrt{\frac{\epsilon}{\mu}}\sin\phi^i K_z^{0D} + K_z^r \tag{47}$$

where $K_z^r \to 0$ as $kd \to 0$. From the integral equations (43)–(45) it follows that $\bar{K}^{0D}$ and $\bar{H}^{0D}$ satisfy the static integral equations

$$\int_D K_z^{0D}(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')\,ds' = \frac{1}{2} - H_x^{0D}(\bar{r}), \qquad \bar{r}\in A \tag{48}$$

$$\int_D K_z^{0D}(\bar{r}')\,ds' + \int_A H_x^{0D}(\bar{r}')\,ds' = 0 \tag{49}$$

and

$$\int_D G_N^0(\bar{r},\bar{r}')K_z^{0D}(\bar{r}')\,ds' + 2\int_A G^0(\bar{r},\bar{r}')H_x^{0D}(\bar{r}')\,ds' = 0, \qquad \bar{r}\in D. \tag{50}$$

In the next section the low-frequency current is integrated to get the low-frequency diffracted far field.

2) Low-Frequency Diffracted Far Field: To determine an expression for the low-frequency diffracted far field we will first derive a relation between the current in the dent and the diffracted field in F, the region in the upper half plane ($y > 0$) outside the image of the dent. When $\bar{r}\in F$ and $\bar{r}' \in \text{int}(A\cup D)$ the Dirichlet Green's function

$$G_D(\bar{r},\bar{r}') = G(\bar{r},\bar{r}') - G(\bar{r},\bar{r}_i'), \qquad \bar{r}_i = x\hat{x} - y\hat{y} \tag{51}$$

satisfies the homogeneous Helmholtz equation. Then Green's second identity along with the boundary conditions $E_z = 0$ on D and $G_D = 0$ on A give

$$\int_D G_D(\bar{r},\bar{r}')\frac{\partial}{\partial n'}E_z(\bar{r}')\,ds' = \int_A E_z(\bar{r}')\frac{\partial}{\partial y'}G_D(\bar{r},\bar{r}')\,ds', \qquad \bar{r}\in F. \tag{52}$$

Because the right-hand side of (52) is the standard expression for the diffracted field, and K_z equals $(j/k)\sqrt{(\epsilon/\mu)}(\partial/\partial n)E_z$ one finds

$$E_z^d(\bar{r}) = -jk\sqrt{\frac{\mu}{\epsilon}}\int_D G_D(\bar{r},\bar{r}')K_z(\bar{r}')\,ds', \qquad \bar{r}\in F \tag{53}$$

which is the relation between the current in the dent and the diffracted field in F. This expression was first derived by Asvestas and Kleinman [19].

From the asymptotic expansion of the Hankel function one finds that

$$G_D(\bar{r},\bar{r}') \sim \frac{e^{j\pi/4}}{\sqrt{2\pi}}\frac{e^{-jkr}}{\sqrt{kr}}k\sin\phi[y' + O(kd)], \qquad r\to\infty. \tag{54}$$

The expansion (54) and the current expansion (47) inserted into the field expression (53) yield the diffracted far field to order $(kd)^2$

$$E_z^d(\bar{r}) \sim -(kd)^2 e^{-j\pi/4}\sqrt{\frac{2}{\pi}}\frac{D_0}{d^2}\sin\phi\sin\phi^i\frac{e^{-jkr}}{\sqrt{kr}} \tag{55}$$

with

$$D_0 = \int_D K_z^{0D}(\bar{r}')y'\,ds' \tag{56}$$

where K_z^{0D} is the solution to the coupled static integral equations (48)–(50). Equation (55) is the final expression for the TM low-frequency diffracted far field of the dent D in the ground plane. We see that this TM low-frequency far field is that of a magnetic dipole in the x-direction [11, section 3.8].

B. Transverse Electric (TE) Polarization

For TE polarization, the incident magnetic field is given by the right-hand side of (1), and the reflected magnetic field is given by the right-hand side of (1) with y replaced by $-y$.

1) Low-Frequency Current: We will start by giving a short alternative derivation of Asvestas and Kleinman's [19] coupled TE integral equations for the current in the dent and the magnetic field in the aperture.

Using exactly the same procedure that led to (40) we obtain

$$\int_D K_t(\bar{r}')\frac{\partial}{\partial n'}G(\bar{r},\bar{r}')\,ds' = -H_z(\bar{r}) + H_z^i(\bar{r}), \qquad \bar{r}\in A \tag{57}$$

which is a relation between the current in the dent and the magnetic field in the aperture. To determine the current in the dent we need one more relation of this kind. Applying the Stratton-Chu formula (39) with $\Omega = A \cup D$ and the Dirichlet Green's function (51) replacing the free-space Green's function gives a relation between the magnetic field and the current in D. Letting the observation point in this relation approach D with $K_t = H_z$ on D produces

$$\frac{1}{2}K_t(\bar{r}) = -\fint_D K_t(\bar{r}')\frac{\partial}{\partial n'}G_D(\bar{r},\bar{r}')\,ds' - 2\int_A H_z(\bar{r}')\frac{\partial}{\partial y'}G(\bar{r},\bar{r}')\,ds', \qquad \bar{r}\in D \tag{58}$$

which is the second relation between the current in the dent and the magnetic field in the aperture. The relations (57) and (58) constitute a pair of coupled integral equations for the current in the dent.

We can use these coupled integral equations to determine the first two terms in the low-frequency expansion of the current in the dent. Inserting the low-frequency expansion (12) for $(\partial/\partial n')G$ and the similar expansion for $(\partial/\partial n')G_D$ into the integral equations (57) and (58) yields

$$\int_D K_t^0(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')\,ds' = -H_z^0(\bar{r}) + 1, \qquad \bar{r}\in A \tag{59}$$

and

$$\frac{1}{2}K_t^0(\bar{r}) = -\fint_D K_t^0(\bar{r}')\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')\,ds' - 2\int_A H_z^0(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')\,ds', \qquad \bar{r}\in D \tag{60}$$

where K_t^0 and H_z^0 are the first terms in the low-frequency expansions of the current K_t and magnetic field H_z, respectively. In [15] it is proven that these two coupled integral equations have a unique solution.

By a derivation similar to the one explained after (46) one finds that a static magnetic field H_z^0 and current K_t^0 satisfy (57) and (58) with G replaced by G^0, G_D replaced by G_D^0, and H_z^i replaced by $H_{\text{static},z}^i$. Here $H_{\text{static},z}^i$ is the z-component of the impressed magnetostatic field. Therefore, (59) and (60) show that H_z^0 and K_t^0 are the magnetostatic field and current in the case where the scatterer is situated in the impressed magnetostatic field $\hat{z}$.

Applying the divergence theorem to ∇G^0, and converting the integration near the singularity to a principal value [14, Appendix] result in the relations

$$\int_D C\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')\,ds' = -\frac{1}{2}C, \qquad \bar{r}\in A \tag{61}$$

and

$$\frac{1}{2}C = -\fint_D C\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')\,ds' - 2\int_A C\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')\,ds', \qquad \bar{r}\in D \tag{62}$$

where C is an arbitrary constant. Comparing (61) and (62) with $C = 2$ to (59) and (60), and recalling the uniqueness of the solution [15], reveals that the first term in the low-frequency expansion of the current in the dent is $K_t^0 = H_z^0 = 2$.

The second term in the low-frequency expansion of the current in the dent can be determined as follows. From the static integral equations (59) and (60), as well as the low-frequency expansions of the Green's functions in the integral equations (57) and (58), one obtains

$$\int_D K_t^1(\bar{r}')\frac{\partial}{\partial n'}G^0(\bar{r},\bar{r}')\,ds' = -H_z^1(\bar{r}) + jkx\cos\phi^i, \qquad \bar{r}\in A \tag{63}$$

and

$$\frac{1}{2}K_t^1(\bar{r}) = -\fint_D K_t^1(\bar{r}')\frac{\partial}{\partial n'}G_D^0(\bar{r},\bar{r}')\,ds' - 2\int_A H_z^1(\bar{r}')\frac{\partial}{\partial y'}G^0(\bar{r},\bar{r}')\,ds', \qquad \bar{r}\in D \tag{64}$$

where K_t^1 and H_z^1 are the second terms in the low-frequency expansion of the current in the dent and the magnetic field in the aperture.

We have shown that the first two terms in the low-frequency expansion of the current in the dent are of order $(kd)^0$ and kd. Because the second term in the expansion of the derivative of the Green's function [see (12)] is of order $(kd)^2 \ln kd$, the third term in the expansion of the current will in general be a function of both kd and $\ln kd$. Consequently, the current cannot be expanded in a power series in k, as mentioned in the Introduction.

The current K_t^1 can be found from the solution to the electrostatic problem with the scatterer situated in the impressed electrostatic field

$$\bar{E}^i_{\text{static}} = -\sqrt{\frac{\mu}{\epsilon}} \cos\phi^i \hat{y}. \tag{65}$$

To prove this, introduce a vector potential $\bar{F}^0 = F_z^0 \hat{z}$ for the electrostatic solution $\bar{E}^0$ so that $\bar{E}^0 = \sqrt{(\mu/\epsilon)} \nabla \times \bar{F}^0$. The scalar F_z^0 satisfies Laplace's equation and the Neumann boundary condition on the conductor like the z component of a magnetostatic field. Thus F_z^0 satisfies (59) and (60) with K_t^0, H_z^0, and 1 replaced by F_z^0, F_z^0, and F_z^{0i}, respectively. Here F_z^{0i} is the potential for the impressed electrostatic field (65). This potential is given by $F_z^{0i} = \cos\phi^i x + C$ where C is a constant. Therefore, the coupled integral equations for $\bar{F}^0$ are

$$\int_D F_z^0(\bar{r}') \frac{\partial}{\partial n'} G^0(\bar{r}, \bar{r}')\, ds' = -F_z^0(\bar{r}) + x\cos\phi^i + C, \qquad \bar{r} \in A \tag{66}$$

and

$$\frac{1}{2} F_z^0(\bar{r}) = -\int_D F_z^0(\bar{r}') \frac{\partial}{\partial n'} G_D^0(\bar{r}, \bar{r}')\, ds' - 2\int_A F_z^0(\bar{r}') \frac{\partial}{\partial y'} G^0(\bar{r}, \bar{r}')\, ds', \qquad \bar{r} \in D. \tag{67}$$

From (61) and (62) one finds that the constant C on the right-hand side of the integral equation (66) adds only the constant C to the solution to (66) and (67) with $C = 0$. Because an arbitrary constant may be added to the potential F_z^0 without changing the electrostatic solution $\bar{E}^0$, it is permissible to let $C = 0$ in (66). Therefore, comparing the integral equations (63) and (64) to (66) and (67) (with $C = 0$) and recalling that the solution is unique [15], shows that $K_t^1(\bar{r}) = jkF_z^0(\bar{r})$ when $\bar{r} \in D$. This completes the proof that the second term in the low-frequency expansion of the current can be found from the corresponding electrostatic solution.

It is convenient to introduce a potential F_z^{0D} as the solution to the equations

$$\int_D F_z^{0D}(\bar{r}') \frac{\partial}{\partial n'} G^0(\bar{r}, \bar{r}')\, ds' = -F_z^{0D}(\bar{r}) + \frac{1}{2}x, \qquad \bar{r} \in A \tag{68}$$

and

$$\frac{1}{2} F_z^{0D}(\bar{r}) = -\int_D F_z^{0D}(\bar{r}') \frac{\partial}{\partial n'} G_D^0(\bar{r}, \bar{r}')\, ds' - 2\int_A F_z^{0D}(\bar{r}') \frac{\partial}{\partial y'} G^0(\bar{r}, \bar{r}')\, ds', \qquad \bar{r} \in D \tag{69}$$

since the current can thereby be written as

$$K_t = 2 + 2jk\cos\phi^i F_z^{0D} + K_t^r \tag{70}$$

where $(1/kd)K_t^r \to 0$ as $kd \to 0$.

2) Low-Frequency Diffracted Far Field: Having derived the first two terms of the low-frequency expansion of the current in the dent, we will integrate these terms to get the low-frequency diffracted far field. We start by deriving a formula that expresses the TE diffracted field in the region F in terms of the current in the dent.

Since the Neumann Green's function $G_N(\bar{r}, \bar{r}')$ in (41) satisfies the homogeneous Helmholtz equation when $\bar{r} \in F$ and $\bar{r}' \in \operatorname{int}(A \cup D)$, Green's second identity along with the facts that G_N and H_z satisfy the Neumann boundary condition on A and D, respectively, give

$$\int_D H_z(\bar{r}') \frac{\partial}{\partial n'} G_N(\bar{r}, \bar{r}')\, ds' = 2\int_A G(\bar{r}, \bar{r}') \frac{\partial}{\partial y'} H_z(\bar{r}')\, ds', \qquad \bar{r} \in F. \tag{71}$$

Noting that the right-hand side of this equation is the negative of the standard expression for the diffracted field and that $H_z = K_t$ on D, we get the relation

$$H_z^d(\bar{r}) = -\int_D K_t(\bar{r}') \frac{\partial}{\partial n'} G_N(\bar{r}, \bar{r}')\, ds', \qquad \bar{r} \in F \tag{72}$$

between the diffracted field and the current in the dent first derived in [19].

From the asymptotic expansion of the Hankel function it is found that

$$\frac{\partial}{\partial n'} G_N(\bar{r}, \bar{r}') \sim k \frac{e^{j\pi/4}}{\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \hat{n}' \cdot \Big[\hat{x}\cos\phi + \hat{x}jkx'\cos^2\phi + \hat{y}jky'\sin^2\phi + O\big((kd)^2\big)\Big], \qquad r \to \infty. \tag{73}$$

Substituting the expansion (70) of the current and the expansion (73) of the Green's function into the field expression (72) shows that to order $(kd)^2$ the diffracted far field is given by

$$H_z^d(\bar{r}) \sim -(kd)^2 \frac{e^{j\pi/4}}{\sqrt{2\pi}} \frac{e^{-jkr}}{\sqrt{kr}} \frac{2j}{d^2} \cdot \Bigg[\int_D (\hat{x}x'\cos^2\phi + \hat{y}y'\sin^2\phi) \cdot \hat{n}'\, ds' + \cos\phi\cos\phi^i \int_D F_z^{0D}(\bar{r}') \frac{\partial y'}{\partial s'}\, ds'\Bigg]. \tag{74}$$

The divergence theorem can be used to prove that the first integral in this equation simply equals the area of the region bounded by $A \cup D$, i.e., the area of the cross section of the dent. The second integral in (74) may be written as

$$\int_D F_z^{0D}(\bar{r}') \frac{\partial y'}{\partial s'} \, ds' = \int_D y' \hat{n}' \cdot \bar{E}^{0D}(\bar{r}') \, ds' \tag{75}$$

where we have integrated by parts once and have introduced the electrostatic solution $\bar{E}^{0D}$ to the problem in which the scatterer is situated in the impressed electrostatic field $(1/2)\hat{y}$.

Before writing down the final expression for the diffracted far field let us prove the relationship

$$\hat{n} \cdot \bar{E}^{0D}(\bar{r}) = K_z^{0D}(\bar{r}), \qquad \bar{r} \in D \tag{76}$$

between the TE electrostatic solution in (75) and the TM magnetostatic solution in (56).

Consider the scalar potential ψ such that $\bar{E}^{0D} = -\nabla\psi$ with ψ satisfying Laplace's equation and the Dirichlet boundary condition on the conductor. Following the procedure explained after (46) shows that ψ satisfies the coupled integral equations

$$\int_D \frac{\partial}{\partial n'} \psi(\bar{r}') \frac{\partial}{\partial y'} G^0(\bar{r}, \bar{r}') \, ds' = -\frac{1}{2} - \frac{\partial}{\partial y'} \psi(\bar{r}), \qquad \bar{r} \in A \tag{77}$$

$$\int_{A \cup D} \frac{\partial}{\partial n'} \psi(\bar{r}') \, ds' = 0 \tag{78}$$

and

$$\oint_D G_N^0(\bar{r}, \bar{r}') \frac{\partial}{\partial n'} \psi(\bar{r}') \, ds' + 2\int_A G^0(\bar{r}, \bar{r}') \frac{\partial}{\partial y'} \psi(\bar{r}') \, ds' = 0, \qquad \bar{r} \in D. \tag{79}$$

Making use of the relation $\hat{n} \cdot \bar{E}^{0D}(\bar{r}) = -(\partial/\partial n)\psi(\bar{r})$, $\bar{r} \in D$, and comparing (77)–(79) to (48)–(50), respectively, shows that (76) is indeed a valid identity.

According to the relations (75) and (76) the diffracted far field in (74) may be written as

$$H_z^d(\bar{r}) \sim (kd)^2 e^{-j\pi/4} \sqrt{\frac{2}{\pi}} \frac{1}{d^2} \cdot \left[A_D + D_0 \cos\phi \cos\phi^i \right] \frac{e^{-jkr}}{\sqrt{kr}} \tag{80}$$

where A_D is the area of the cross section of the dent and

$$D_0 = \int_D K_z^{0D}(\bar{r}') y' \, ds' = \int_D F_z^{0D}(\bar{r}') \frac{\partial y'}{ds'} \, ds'. \tag{81}$$

The magnetostatic current $\bar{K}_z^{0D}$ is found from the coupled integral equations (48)–(50) and the electrostatic vector potential $\bar{F}_z^{0D}$ is found from the coupled integral equations (68) and (69).

We see that the diffracted far field in (80) consists of the contribution from a magnetic dipole in the z-direction (the term with A_D) and an electric dipole in the y-direction (the cosine term) [11, section 3.8]. Furthermore, the constant in the TE electric dipole term in (80) is the same as the constant in the TM magnetic dipole term (55).

Remarkably, the TM and TE low-frequency scattering from a two-dimensional dent in a ground plane, like that for the bump, reduces to the evaluation of a single constant D_0. This constant is determined from a static solution and depends only on the shape of the cross section of the dent.

Throughout the derivation that led to (80) it was assumed that the dent was continuously lined by a conductor so that the results in this paper do not apply *per se* to the slit in a ground plane. However, since the slit is the compliment of the strip it is seen from (10) that the low-frequency fields diffracted by a slit for TE polarization must be of order $(1/\ln kd)$ where d is the width of the slit [7, ch. 4], [8]. Thus, the low-frequency fields diffracted by slits and dents are of different order and, in particular, the slit scatters more strongly than the dent at low frequencies. The physical reason for this larger scattering by the slit is that the slit completely "stops" the currents on the ground plane while the dent only "diverts" them.

V. Verification of Low-Frequency Expressions

We will verify our general low-frequency expressions by comparing them to low-frequency results obtained from exact time-harmonic eigenfunction solutions.

Begin by considering the circular cylinder with radius d. The low-frequency results obtained from the exact time-harmonic eigenfunction solution [7, sections 2.2.1.2 and 2.2.2.2] are in agreement with our general results (10) and (23) provided that

$$\frac{C_1}{d^2} = \frac{C_2}{d^2} = 2\pi, \qquad C_3 = 0. \tag{82}$$

Solving the problem of the circular cylinder in a static electric field by the method of separation of variables, it is easily shown that $\sigma^{0x} = 2\epsilon \cos\phi$ and $\sigma^{0y} = 2\epsilon \sin\phi$. Inserting these expressions for the charge density into (26) one recovers (82).

We also solved numerically the static integral equation (25) for σ^{0x} and the corresponding equation for σ^{0y} and inserted the numerical solutions into (26). Again the results agreed with (82). Furthermore, the static integral equation (22) for F_z^{0x} and the corresponding equation for F_z^{0y} were solved numerically for the circular cylinder. The numerical results were inserted into (24) and agreement with (82) was obtained.

Next we consider the semicircular bump of radius d. Rayleigh [18] used the exact eigenfunction solution for the circular cylinder to find the low-frequency expressions for the diffracted field. Rayleigh's expressions agree with our general low-frequency expressions (31) and (37) provided that

$$\frac{B_0}{d^2} = -\pi. \tag{83}$$

Using the method of separation of variables to solve the equivalent statics problem, one finds that $K_z^{0B} = -2\sin\phi$. Substitution of this current into (32) recovers the value of B_0 given in (83). We also solved numerically the simple integral equation (29) for the static current K_z^{0B}, which upon integration in (32) again produced (83).

Finally consider the semicircular dent with radius d. The dual-series eigenfunction solution [21] agrees with our general results (55) and (80) provided

$$\frac{D_0}{d^2} = 0.58. \tag{84}$$

We solved numerically the coupled integral equations (48)–(50) for the magnetostatic current K_z^{0D} used in (56) to calculate D_0. The result agreed with (84). We also solved numerically the coupled integral equations (68) and (69) for the electrostatic vector potential $\hat{z}F_z^{0D_z}$ used in (81) to calculate D_0. Again the result agreed with (84).

VI. Conclusion

We have evaluated the low-frequency electromagnetic scattering from the perfectly conducting cylinder with finite cross section and the perfectly conducting cylindrical bump and dent in a ground plane.

For the cylinder with finite cross section the low-frequency scattered far field for TM polarization is independent of the shape of the cross section of the cylinder and is of order $(1/\ln kd)$ where d is a characteristic dimension of the cylinder. This low-frequency result is not related to a corresponding static field problem; see Footnote 1.

For TE polarization the scattered field is of order $(kd)^2$ and it consists of a contribution from a magnetic dipole along the axis of the cylinder and an electric dipole in a direction normal to the axis of the cylinder. The magnetic dipole moment is found directly from the area of the cross section of the cylinder. The electric dipole moment is found by solving an electrostatic problem, i.e., a two-dimensional potential problem, for two impressed fields and integrating these two electrostatic solutions around the cylinder. These electrostatic solutions are determined from simple static integral equations and depend only on the shape of the cylinder.

For both the cylindrical bump on a ground plane and dent in a ground plane the low-frequency diffracted field for TM and TE polarization is of order $(kd)^2$, where d is a characteristic dimension of the bump or dent. The low-frequency TM diffracted far field is that of a magnetic dipole normal to the axial direction and parallel to the ground plane.

The low-frequency TE diffracted far field for both the cylindrical bump and dent consists of a contribution from a magnetic dipole in the axial direction and an electric dipole normal to both the axial direction and the ground plane. The TE magnetic dipole moment is found directly from the area of the cross section of the bump or dent.

Both the TM magnetic dipole field and the TE electric dipole field can be written as a constant times a known simple function. It is proven that, remarkably, this constant, which depends only on the shape of the bump or dent, is the same for both the TM and TE polarizations. This constant can be found by solving either a magnetostatic or electrostatic field problem, i.e., a two-dimensional potential problem, and performing an integration of these solutions over the bump or dent. This means that both the TM and TE low-frequency diffracted far fields for an arbitrarily shaped bump or dent are completely determined by calculating a single constant for the bump or dent.

The low-frequency expressions were confirmed from exact time-harmonic eigenfunction solutions to the circular cylinder, the semicircular bump, and the semicircular dent. As mentioned in the Introduction, the low-frequency expressions derived in this paper are given in closed form and can therefore be used directly to determine incremental length diffraction coefficients for calculating the scattered fields from curved narrow ridges and channels in conductors [9]. These incremental length diffraction coefficients determined from the low-frequency expressions derived in this paper have been applied recently to calculate the effects of ridges and channels on the fields of a reflector antenna [22].

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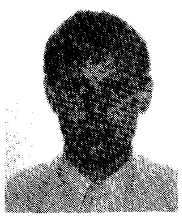

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Arthur D. Yaghjian (S'68–M'69–SM'84–F'92), for a photograph and biography please see p. 312 of the March 1992 issue of this TRANSACTIONS.