J Elasticity (2007) 86:19–39
DOI 10.1007/s10659-006-9078-9

Stress Distribution Around a Crack in Plane Micropolar Elasticity

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Received: 29 April 2006 / Accepted: 10 August 2006 /
Published online: 12 December 2006

Abstract In this paper we formulate the boundary value problem of plane micropolar elasticity for a domain containing a crack in Sobolev spaces and prove the existence and continuous dependence on the data of the corresponding weak solutions. We consider the cases of both finite and infinite domain and find the solutions in terms of modified single layer and modified double layer potentials with distributional densities.

Key words Cosserat elasticity · cracks · boundary integral equation method

Mathematics Subject Classifications (2000) 74B99 · 74R99 · 74E99 · 45E05

1 Introduction

The micropolar theory of elasticity (also known as *Cosserat* or *asymmetric* theory of elasticity) was introduced by Eringen in [1] (see [2] for a review of works in this area and an extensive bibliography) to eliminate discrepancies between classical theory of elasticity and experiments in cases when material microstructure was known to have a significant effect on the body's overall deformation, for example, materials with granular microstructure such as polymers or human bones (see [3–5]). In the last 30 years various problems of Cosserat elasticity have been investigated by a variety of methods. For example, three dimensional problems of Cosserat elasticity have been formulated in a rigorous setting and solved by means of potential theory methods by Kupradze in [6]. In [7–10], the corresponding boundary value problems for plane deformations of a micropolar homogeneous, linearly elastic solid were shown to be

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well-posed and subsequently solved in a rigorous setting using the boundary integral equation method.

In the case when a domain is weakened by a crack the nature of the boundary conditions across the crack region presents formidable difficulties in the boundary integral analysis in a classical setting. Several studies of a crack problem have been undertaken in the classical elastic setting under assumptions of a simplified theory of plane Cosserat elasticity [5, 11, 12] and recently there has been some activity in the area of crack analysis in three-dimensional Cosserat elasticity [13–16]. The rigorous analysis of the corresponding crack problems in plane Cosserat elasticity in the general case still remains absent from the literature.

Recently, Chudinovich and Constanda [17] have used the boundary integral equation method in a weak (Sobolev) space setting to obtain the solution for several crack problems in a theory of bending of classical elastic plates. In spite of the fact that the methods used are extremely complicated (mathematically) they seem to be very effective and give very good results for applications. We continue to study the effectiveness of these methods with a view to the analysis and solution of the plane problems of interest here.

In this paper we formulate boundary value problems for both finite and infinite domains which contain cracks in the case of plane micropolar elasticity when displacements and microrotations or stress and couple stress are prescribed along the two sides of the crack in Sobolev spaces and find the corresponding weak solutions in terms of integral potentials with distributional densities.

2 Preliminaries

In what follows Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, the convention of summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$- matrices, E_n is the identity element in $\mathcal{M}_{n\times n}$, the columns of a (3×3)-matrix P are denoted by $P^{(i)}$, a superscript T indicates matrix transposition, the generic symbol c denotes various strictly positive constants, and $(...)_{,\alpha} \equiv \partial(...)/\partial x_\alpha$. Also, if X is a space of scalar functions and v is a matrix, $v \in X$ means that every component of v belongs to X.

Let S be a domain in $\mathbb{R}^2$ occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants $\lambda, \mu, \alpha, \gamma$ and ε. We use the notations $\|\cdot\|_{0;S}$ and $\langle\cdot,\cdot\rangle_{0;S}$ for the norm and inner product in $L^2(S) \cap \mathcal{M}_{m\times 1}$ for any $m \in \mathbb{N}$. When $S = \mathbb{R}^2$, we write $\|\cdot\|_0$ and $\langle\cdot,\cdot\rangle_0$.

The state of plane micropolar strain is characterized by a displacement field $u\left(x'\right)=\left(u_1\left(x'\right), u_2\left(x'\right), u_3\left(x'\right)\right)^T$ and a microrotation field $\phi\left(x'\right)=\left(\phi_1\left(x'\right),\ \phi_2\left(x'\right),\ \phi_3\left(x'\right)\right)^T$ of the form

$$u_\alpha\left(x'\right) = u_\alpha\left(x\right), \qquad u_3(x') = 0,$$
$$\phi_\alpha\left(x'\right) = 0, \qquad \phi_3(x') = \phi_3\left(x\right), \tag{2.1}$$

where $x' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. The equilibrium equations of plane micropolar strain written in terms of displacements and microrotations are given by [7, 8]

$$L(\partial_x)u(x) + q(x) = 0, \qquad x \in S, \tag{2.2}$$

in which now, denoting ϕ_3 by u_3,we have $u(x) = (\overline{u}, u_3)^T$, where $\overline{u} = (u_1, u_2)^T$, the matrix partial differential operator $L(\partial_x) = L(\partial/\partial x_\alpha)$ is defined by

$$L(\xi) = L(\xi_\alpha) = \begin{pmatrix} (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_1^2 & (\lambda+\mu-\alpha)\xi_1\xi_2 & 2\alpha\xi_2 \\ (\lambda+\mu-\alpha)\xi_1\xi_2 & (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_2^2 & -2\alpha\xi_1 \\ -2\alpha\xi_2 & 2\alpha\xi_1 & (\gamma+\varepsilon)\Delta - 4\alpha \end{pmatrix},$$

where $\Delta = \xi_\alpha\xi_\alpha$, and vector $q = (q_1, q_2, q_3)^T$ represents body forces and body couples.

Together with L we consider the boundary stress operator $T(\partial_x) = T(\partial/\partial x_\alpha)$ defined by

$$T(\xi) = T(\xi_\alpha) = \begin{pmatrix} (\lambda+2\mu)\,\xi_1 n_1 + (\mu+\alpha)\,\xi_2 n_2 & (\mu-\alpha)\xi_1 n_2 + \lambda\xi_2 n_1 & 2\alpha n_2 \\ (\mu-\alpha)\xi_2 n_1 + \lambda\xi_1 n_2 & (\lambda+2\mu)\,\xi_2 n_2 + (\mu+\alpha)\,\xi_1 n_1 & -2\alpha n_1 \\ 0 & 0 & (\gamma+\varepsilon)\xi_\alpha n_\alpha \end{pmatrix},$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S. To guarantee the ellipticity of system (2.2), in what follows we assume that

$$\lambda + \mu > 0, \quad \mu > 0, \quad \gamma + \varepsilon > 0, \quad \alpha > 0.$$

The internal energy density is given by

$$\begin{aligned} 2E(u, v) &= 2E_0(u, v) \\ &\quad +\mu(u_{1,2} + u_{2,1})(v_{1,2} + v_{2,1}) \\ &\quad +\alpha(u_{1,2} - u_{2,1} + 2u_3)(v_{1,2} - v_{2,1} + 2v_3) \\ &\quad +(\gamma+\varepsilon)(u_{3,1}v_{3,1} + u_{3,2}v_{3,2}), \\ 2E_0(u, v) &= (\lambda+2\mu)\left(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}\right) \\ &\quad +\lambda(u_{1,1}v_{2,2} + u_{2,2}v_{1,1}). \end{aligned}$$

Clearly, $E(u, u)$ is a positive quadratic form.

The space of rigid displacements and microrotations $\mathcal{F}$ is spanned by the columns of the matrix

$$\mathbb{F} = \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$$

from which it can be seen that $L\mathbb{F} = 0$ in $\mathbb{R}^2$, $T\mathbb{F} = 0$ on ∂S and a general rigid displacement can be written as $\mathbb{F}k$, where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary.

A Galerkin representation for the solution of Eq. (2.2) when $q(x) = -\delta(|x - y|)$, where δ is the Dirac delta distribution, yields the matrix of fundamental solutions [8]

$$D(x, y) = L^*(\partial x)t(x, y), \tag{2.3}$$

where L^* is the adjoint of L,

$$t(x, y) = \frac{a}{8\pi k^4}\left\{[k^2\,|x-y|^2+4]\ln|x-y| + 4K_0(k|x-y|)\right\}, \tag{2.4}$$

K_0 is the modified Bessel function of order zero and the constants a, k^2 are defined by

$$a^{-1} = (\gamma+\varepsilon)(\lambda+2\mu)(\mu+\alpha), \quad k^2 = \frac{4\mu\alpha}{(\gamma+\varepsilon)(\mu+\alpha)}.$$

In view of Eqs. (2.3) and (2.4)

$$D(x, y) = D^T(x, y) = D(y, x).$$

Along with matrix $D(x, y)$ we consider the matrix of singular solutions

$$P(x, y) = (T(\partial y)D(y, x))^T. \tag{2.5}$$

It is easy to verify that $D^{(i)}(x, y)$ and $P^{(i)}(x, y)$ satisfy Eq. (2.2) with $q(x) = 0$ at all $x \in \mathbb{R}^2, x \neq y$.

First, we consider an infinite domain with a crack modelled by an open arc Γ_0. We assume that Γ_0 is a part of a simple closed C^2-curve Γ that divides $\mathbb{R}^2$ into interior and exterior domains Ω^+ and Ω^-. In what follows we denote by the superscripts $+$ and $-$ the limiting values of functions as $x \to \Gamma$ from within Ω^+ or Ω^-. We define $\Omega = \mathbb{R}^2\backslash\overline{\Gamma_0}$ and $\Gamma_1 = \Gamma\backslash\overline{\Gamma_0}$. Regarding the definition of Ω, we can also use $\|\cdot\|_0$ and $\langle\cdot,\cdot\rangle_0$ for the norm and inner product in $L^2(\Omega)$.

We say that $u \in C^k(\overline{\Omega})$ if $u \in C^k(\overline{\Omega}^+)$, $u \in C^k(\overline{\Omega}^-)$ and $(\partial^\alpha u)^+(x) = (\partial^\alpha u)^-(x)$, $x \in \Gamma_1$, for all two-component multi-indices α such that $|\alpha| \leqslant k$.

For any $m \in \mathbb{R}$, let $H_m(\mathbb{R}^2)$ be the standard real Sobolev space of three-component distributions, equipped with the norm

$$\| u \|_m^2 = \int_{\mathbb{R}^2} (1+|\xi|^2)^m |\widetilde{u}(\xi)|^2 \mathrm{d}\xi,$$

where $\widetilde{u}$ is the Fourier transform of u. In what follows we do not distinguish between equivalent norms and denote them by the same symbol; thus, the norm in $H_1(\mathbb{R}^2)$ can be defined by

$$\| u \|_1^2 = \| u \|_0^2 + \sum_{i=1}^{3} \| \nabla u_i \|_0^2 .$$

The spaces $H_m(\mathbb{R}^2)$ and $H_{-m}(\mathbb{R}^2)$ are dual with respect to the duality induced by $\langle\cdot,\cdot\rangle_0$.

We introduce the space $L^2_\omega(\mathbb{R}^2)$ of (3×1)-vector functions $u = (\overline{u}, u_3)^T$, where $\overline{u} = (u_{1,} u_2)^T$, such that

$$\| u \|_{0,\omega}^2 = \int_{\mathbb{R}^2} \frac{|\overline{u}(x)|^2}{(1+|x|)^2(1+\ln|x|)^2} dx + \int_{\mathbb{R}^2} \frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln|x|)^2} dx < \infty..$$

Also, we consider the bilinear form $b(u, v) = 2\int_{\mathbb{R}^2} E(u, v)\, dx$. Let $H_{1,\omega}(\mathbb{R}^2)$ be the space of three-component distributions on $\mathbb{R}^2$ for which

$$\| u \|^2_{1,\omega} = \| u \|^2_{0,\omega} + b(u, u) < \infty,$$

$H_{-1,\omega}(\mathbb{R}^2)$ is dual to $H_{1,\omega}(\mathbb{R}^2)$ with respect to the duality generated by $\langle \cdot, \cdot \rangle_0$. The norm in $H_{-1,\omega}(\mathbb{R}^2)$ is denoted by $\| \cdot \|_{-1,\omega}$.

Let $\overset{\circ}{H}_m(\Omega^+)$ be the subspace of $H_m(\mathbb{R}^2)$ consisting of all u which have a compact support in Ω^+. $H_m(\Omega^+)$ is the space of the restrictions to $\Omega^+$of all $u \in H_m(\mathbb{R}^2)$. Denoting by $\pi^\pm$ the operators of restrictions from $\mathbb{R}^2$ to $\Omega^\pm$, respectively, we introduce the norm of $u \in H_m(\Omega^+)$ by $\| u \|_{m;\Omega^+} = \inf_{v \in H_m(\mathbb{R}^2):\pi^+ v = u} \| v \|_m$. If $m = 1$, then the norms of $u \in \overset{\circ}{H}_1(\Omega^+)$ and $u \in H_1(\Omega^+)$ are equivalent to

$$\left\{ \| u \|^2_{0;\Omega^+} + \sum_{i=1}^{3} \int_{\Omega^+} |\nabla u_i(x)|^2 dx \right\}^{1/2}.$$

The spaces $\overset{\circ}{H}_m(\Omega^+)$ and $H_{-m}(\Omega^+)$ are dual with respect to the duality induced by $\langle \cdot, \cdot \rangle_{0;\Omega^+}$.

Let $\overset{\circ}{H}_{1,\omega}(\Omega^-)$ be the subspace of $H_{1,\omega}(\mathbb{R}^2)$ consisting of all u which have a compact support in Ω^-. $H_{1,\omega}(\Omega^-)$ is the space of the restrictions to $\Omega^-$of all $u \in H_{1,\omega}(\mathbb{R}^2)$. The norm in $H_{1,\omega}(\Omega^-)$ is defined by $\| u \|_{1,\omega;\Omega^-} = \inf_{v \in H_{1,\omega}(\mathbb{R}^2):\pi^- v = u} \| v \|_{1,\omega}$. From the definition it follows that $H_{1,\omega}(\Omega^-)$ is isometric to $H_{1,\omega}(\mathbb{R}^2) \backslash \overset{\circ}{H}_1(\Omega^+)$. It can be shown that the norm of $u \in H_{1,\omega}(\Omega^-)$ is equivalent to

$$\left\{ \| u \|^2_{0,\omega;\Omega^-} + b_-(u, u) \right\}^{1/2},$$

where

$$\| u \|^2_{0,\omega;\Omega^-} = \int_{\Omega^-} \frac{|\overline{u}(x)|^2}{(1 + |x|)^2 (1 + \ln |x|)^2} dx + \int_{\Omega^-} \frac{|u_3(x)|^2}{(1 + |x|)^4 (1 + \ln |x|)^2} dx$$

and $b_-(u, v) = 2\int_{\Omega^-} E(u, v)\, dx$. This norm is compatible with asymptotic class $\mathcal{A}$ introduced in [8].

The dual of $\overset{\circ}{H}_{1,\omega}(\Omega^-)$ with respect to the duality generated by $\langle \cdot, \cdot \rangle_{0;\Omega^-}$ is the space $H_{-1,\omega}(\Omega^-)$, with norm $\| \cdot \|_{-1,\omega;\Omega^-}$; the dual of $H_{1,\omega}(\Omega^-)$ is $\overset{\circ}{H}_{-1,\omega}(\Omega^-)$, which can be identified with a subspace of $H_{-1,\omega}(\mathbb{R}^2)$.

Let $H_m(\Gamma)$ be the standard Sobolev space of distributions on Γ, with norm $\| \cdot \|_{m;\Gamma}$. $H_m(\Gamma)$ and $H_{-m}(\Gamma)$ are dual with respect to the duality generated by the inner product $\langle \cdot, \cdot \rangle_{0;\Gamma}$ in $L^2(\Gamma)$. We denote by $\overset{\circ}{H}_m(\Gamma_0)$ the subspace of all $f \in H_m(\Gamma)$ with compact support on Γ_0, and by $H_m(\Gamma_0)$ the space of the restrictions to Γ_0 of all $f \in H_m(\Gamma)$. Let π_0 and π_1 be the operators of restriction from Γ to Γ_0 and Γ_1. The norm of $f \in H_m(\Gamma_0)$ is defined by $\| f \|_{m;\Gamma_0} = \inf_{v \in H_m(\Gamma):\pi_0 v = f} \| v \|_{m;\Gamma}$. For any $m \in \mathbb{R}$, $\overset{\circ}{H}_m(\Gamma_0)$ and $H_{-m}(\Gamma_0)$ are dual with respect to the duality generated by the inner product $\langle \cdot, \cdot \rangle_{0;\Gamma_0}$ in $L^2(\Gamma_0)$.

Let γ^+ and γ^- be the continuous trace operators from $H_1(\Omega^+)$ and $H_{1,\omega}(\Omega^-)$ to $H_{1/2}(\Gamma)$. Also, let $\gamma_i^\pm = \pi_i\gamma^\pm$, $i = 0, 1$. For any u defined in Ω (or $\mathbb{R}^2$) we write $u = \{u_+, u_-\}$, where $u_\pm = \pi^\pm u$.

Let $H_{1,\omega}(\Omega)$ be the space of all $u = \{u_+, u_-\}$ such that $u_+ \in H_1(\Omega^+)$, $u_- \in H_{1,\omega}(\Omega^-)$ and $\gamma_1^+ u_+ = \gamma_1^- u_-$. The norm in $H_{1,\omega}(\Omega)$ is defined by

$$\| u \|^2_{1,\omega;\Omega} = \| u_+ \|^2_{1;\Omega^+} + \| u_- \|^2_{1,\omega;\Omega^-} .$$

$\overset{\circ}{H}_{1,\omega}(\Omega)$ is the subspace of $H_{1,\omega}(\Omega)$ consisting of all u such that $\gamma_0^+ u_+ = \gamma_0^- u_- = 0$; therefore, $\overset{\circ}{H}_{1,\omega}(\Omega)$ can be identified with a subspace of $H_{1,\omega}(\mathbb{R}^2)$.

We denote by $H_{-1,\omega}(\Omega)$ and $\overset{\circ}{H}_{-1,\omega}(\Omega)$ the duals of $\overset{\circ}{H}_{1,\omega}(\Omega)$ and $H_{1,\omega}(\Omega)$ with respect to the duality induced by $\langle\cdot,\cdot\rangle_0$. The norms in $H_{-1,\omega}(\Omega)$ and $\overset{\circ}{H}_{-1,\omega}(\Omega)$ are denoted by $\|\cdot\|_{-1,\omega;\Omega}$ and $\|\cdot\|_{-1,\omega}$.

Further, we introduce the corresponding area, single layer, and double layer potentials given, respectively, by

$$(U\varphi)(x) = \int_{\mathbb{R}^2} D(x, y)\varphi(y)\, dy,$$

$$(V\varphi)(x) = \int_{\Gamma_0} D(x, y)\varphi(y)\, ds(y),$$

$$(W\varphi)(x) = \int_{\Gamma_0} P(x, y)\varphi(y)\, ds(y),$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is an unknown density matrix.

It is not difficult to check that $L(Uq) = q$ in $\mathbb{R}^2$.

The properties of single and double layer integral potentials are well known and may be formulated in the following theorem, which can be proved using technique described in [8–10].

Theorem 1

(a) *If* $\varphi \in C(\partial S)$, *then* $V\varphi$, $W\varphi$ *are analytic and satisfy* $L(V\varphi) = L(W\varphi) = 0$ *in* $S^+ \cup S^-$.

(b) *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, *then the direct values* $V_0\varphi$, $W_0\varphi$ *of* $V\varphi$, $W\varphi$ *on* ∂S *exist (the latter as the principal value), the functions* $V^+(\varphi) = (V\varphi)|_{\overline{S}^+}$, $V^-(\varphi) = (V\varphi)|_{\overline{S}^-}$ *are of class* $C^{1,\alpha}(\overline{S}^+)$ *and* $C^{1,\alpha}(\overline{S}^-)$, *respectively, and* $TV^+(\varphi) = (W_0^* + \frac{1}{2}I)\varphi$, $TV^-(\varphi) = (W_0^* - \frac{1}{2}I)\varphi$ *on* ∂S, *where* W_0^* *is the adjoint of* W_0 *and* I *- the identity operator.*

(c) *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$, *then the functions*

$$W^+(\varphi) = \begin{cases} (W\varphi)|_{S^+}, & in\ S^+, \\ (W_0 - \frac{1}{2}I)\varphi, & on\ \partial S, \end{cases} \qquad W^-(\varphi) = \begin{cases} (W\varphi)|_{S^-}, & in\ S^+, \\ (W_0 + \frac{1}{2}I)\varphi, & on\ \partial S, \end{cases}$$

are of class $C^{1,\alpha}(\overline{S}^+)$ *and* $C^{1,\alpha}(\overline{S}^-)$, *respectively, and* $TW^+(\varphi) = TW^-(\varphi)$ *on* ∂S.

3 Boundary Value Problems

We consider two types of boundary values problems: Dirichlet and Neumann boundary value problems. The first one consists of finding $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $u_- \in \mathcal{A}^*$ such that

$$Lu(x) + q(x) = 0, \quad x \in \Omega,$$
$$u^+(x) = f^+(x), \quad u^-(x) = f^-(x), \quad x \in \Gamma_0, \tag{D}$$

where f^+ and f^- are prescribed on Γ_0.

The second problem consists of finding $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u_- \in \mathcal{A}$ such that

$$Lu(x) + q(x) = 0, \quad x \in \Omega,$$
$$(Tu)^+(x) = g^+(x), \quad (Tu)^-(x) = g^-(x), \quad x \in \Gamma_0, \tag{N}$$

where g^+ and g^- are prescribed on Γ_0. Asymptotic classes $\mathcal{A}^*$ and $\mathcal{A}$ were introduced in [8].

The variational formulations are based on the Betti formulae [18] and [19]. The variational formulation of (D) is as follows. We seek $u \in H_{1,\omega}(\Omega)$ such that

$$b\,(u, v) = \langle q, v\rangle_0 \quad \forall v \in \overset{\circ}{H}_{1,\omega}(\Omega),$$
$$\gamma_0^+ u_+ = f^+, \quad \gamma_0^- u_- = f^-, \tag{3.1}$$

where $q \in H_{-1,\omega}(\Omega)$ and f^+, $f^- \in H_{1/2}(\Gamma_0)$ are given.

The variational formulation of (N) is as follows. We seek $u \in H_{1,\omega}(\Omega)$ such that

$$b\,(u, v) = \langle q, v\rangle_0 + \left\langle g^+, \gamma_0^+ v_+\right\rangle_{0;\Gamma_0} - \left\langle g^-, \gamma_0^- v_-\right\rangle_{0;\Gamma_0} \quad \forall v \in H_{1,\omega}(\Omega), \tag{3.2}$$

where $q \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ and $g^+, g^- \in H_{1/2}(\Gamma_0)$ are given.

In what follows we write $\delta f = f^+ - f^-$ and $\delta g = g^+ - g^-$ for the jump of these quantities across the crack.

Theorem 2 *Problem* (3.1) *has a unique solution* $u \in H_{1,\omega}(\Omega)$ *for any* $q \in H_{-1,\omega}(\Omega)$ *and any* f^+, $f^- \in H_{1/2}(\Gamma_0)$ *such that* $\delta f \in \overset{\circ}{H}_{1/2}(\Gamma_0)$*, and this solution satisfies the estimate*

$$\| u \|_{1,\omega;\Omega} \leqslant c\left(\| q \|_{-1,\omega;\Omega} + \| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}\right). \tag{3.3}$$

Proof Assume first that $f^+ = f^- = 0$. To prove this assertion it is sufficient to verify that $b\,(u, v)$ is coercive on $\overset{\circ}{H}_{1,\omega}(\Omega)$. In [18] and [19] it was shown that any $u = \{u_+, u_-\} \in \overset{\circ}{H}_{1,\omega}(\Omega)$ satisfies $\| u_+ \|^2_{1;\Omega^+} \leqslant cb_+(u_+, u_+)$ and $\| u_- \|^2_{1,\omega;\Omega^-} \leqslant cb_-(u_-, u_-)$, where $b_\pm(u, v) = 2\int_{\Omega^\pm} E(u, v)dx$; consequently,

$$\| u \|^2_{1,\omega;\Omega} = \| u_+ \|^2_{1;\Omega^+} + \| u_- \|^2_{1,\omega;\Omega^-} \leqslant c\left[b_+(u_+, u_+) + b_-(u_-, u_-)\right] = cb\,(u, u).$$

By the Lax–Milgram lemma, (D) with $f^+ = f^- = 0$ has a unique solution $u \in \overset{\circ}{H}_{1,\omega}(\Omega)$ and

$$\| u \|_{1,\omega} \leqslant c \| q \|_{-1,\omega;\Omega} \,. \tag{3.4}$$

In the full problem (D), we consider an operator l_0 of the extension from Γ_0 to Γ, which maps $H_{1/2}(\Gamma_0)$ continuously to $H_{1/2}(\Gamma)$. Let $F^+ = l_0 f^+$ and let F^- be the extension of f^- to Γ such that $\pi_1 F^+ = \pi_1 F^-$. We denote by $l_\pm$ operators of the extension from Γ to $\Omega^\pm$, which map $H_{1/2}(\Gamma)$ continuously to $H_1(\Omega^+)$ and $H_{1,\omega}(\Omega^-)$, respectively. Let $w_+ = l_+ F^+ \in H_1(\Omega^+)$ and $w_- = l_- F^- \in H_{1,\omega}(\Omega^-)$. Clearly, $w = \{w_+, w_-\} \in H_{1,\omega}(\Omega)$. We seek a solution to (D) in the form $u = u_0 + w$, where $u_0 \in \overset{\circ}{H}_{1,\omega}(\Omega)$ satisfies

$$b(u_0, v) = \langle q, v \rangle_0 - b(w, u) \quad \forall v \in \overset{\circ}{H}_{1,\omega}(\Omega). \tag{3.5}$$

Since for all $v \in \overset{\circ}{H}_{1,\omega}(\Omega)$

$$\begin{aligned} |b(w, v)| &\leqslant |b_+(w_+, v_+)| + |b_-(w_-, v_-)| \leqslant c(\| w_+ \|_{1;\Omega^+} + \| w_- \|_{1,\omega;\Omega^-}) \| v \|_{1,\omega} \\ &\leqslant c(\| F^+ \|_{1/2;\Gamma} + \| F^- \|_{1/2;\Gamma}) \| v \|_{1,\omega} \\ &\leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| f^- \|_{1/2;\Gamma_0}) \| v \|_{1,\omega} \\ &\leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}) \| v \|_{1,\omega}, \end{aligned}$$

the right-hand side $L(v) =< q, v >_0 - b(w, u)$ in Eq. (3.5) defines the continuous linear functional on $\overset{\circ}{H}_{1,\omega}(\Omega)$ and $\| L \|_{-1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega;\Omega} + \| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma} \right)$; therefore Eq. (3.5) has a unique solution $u_0 \in \overset{\circ}{H}_{1,\omega}(\Omega)$ and

$$\| u_0 \|_{1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega;\Omega} + \| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma} \right).$$

The theorem now follows from this inequality and the estimate $\| w \|_{1,\omega;\Omega} \leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma})$. □

We proceed with problem (3.2). It is clear that, in view of the properties of rigid displacements,

$$\langle q, z \rangle_0 + \left\langle g^+, z \right\rangle_{0;\Gamma_0} - \left\langle g^-, z \right\rangle_{0;\Gamma_0} = 0 \quad \forall z \in \mathcal{F} \tag{3.6}$$

is a necessary solvability condition for (N).

Theorem 3 *Problem (3.2) is solvable for any* $q \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ *and any* $g^+, g^- \in H_{-1/2}(\Gamma_0)$ *such that* $\delta g \in \overset{\circ}{H}_{-1/2}(\Gamma_0)$, *satisfying Eq. (3.6). Any two solutions differ by a rigid displacement, and there is a solution* u_0 *that satisfies the estimate*

$$\| u_0 \|_{1,\omega;\Omega} \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0} \right). \tag{3.7}$$

Proof We notice that the expression

$$\begin{aligned} L(v) &= \left\langle g^+, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} - \left\langle g^-, \gamma_0^- v_- \right\rangle_{0;\Gamma_0} \\ &= \left\langle \delta g, \gamma_0^+ v_+ \right\rangle_{0;\Gamma_0} + \left\langle g^-, \delta v \right\rangle_{0;\Gamma_0}, \quad \forall v \in H_{1,\omega}(\Omega), \end{aligned}$$

where $\delta v = \gamma_0^+ v_+ - \gamma_0^- v_-$, defines a continuous linear functional on $H_{1,\omega}(\Omega)$. Consequently, there is $q_1 \in \overset{\circ}{H}_{-1,\omega}(\Omega)$ such that $L(v) = \langle q_1, v \rangle_0$ for all $v \in H_{1,\omega}(\Omega)$, and

$$\| q_1 \|_{-1,\omega} \leqslant c \left(\| g^- \|_{-1/2;\Gamma_0} + \| \delta g \|_{-1/2;\Gamma} \right). \tag{3.8}$$

We set $q + q_1 = \widetilde{q}$ and write Eq. (3.2) in the form $b(u, v) = \langle \widetilde{q}, v \rangle_0$, $v \in H_{1,\omega}(\Omega)$. We consider the factor space $\mathbb{H}_{1,\omega}(\Omega) = H_{1,\omega}(\Omega) \backslash \mathcal{F}$ with the norm $\| U \|_{\mathbb{H}_{1,\omega}(\Omega)} = \inf_{u \in H_{1,\omega}(\Omega), u \in U} \| u \|_{1,\omega;\Omega}$ and define on it a bilinear form $\mathcal{B}(U, V)$ and a linear functional $\mathcal{L}(V)$ by

$$\mathcal{B}(U, V) = b(u, v), \qquad \mathcal{L}(V) = L(v) = \langle \widetilde{q}, v \rangle_0, \tag{3.9}$$

where u and v are arbitrary representatives of the classes $U, V \in \mathbb{H}_{1,\omega}(\Omega)$. Since $b(z, z) = 0$ and $< \widetilde{q}, z >_0 = 0$ for any $z \in \mathcal{F}$, definitions (3.9) are consistent.

We now consider the problem of finding $U \in \mathbb{H}_{1,\omega}(\Omega)$ such that

$$\mathcal{B}(U, V) = \mathcal{L}(V), \quad \forall V \in \mathbb{H}_{1,\omega}(\Omega). \tag{3.10}$$

We claim that Eq. (3.10) has a unique solution. First, from Eq. (3.8) it follows that

$$|\mathcal{L}(V)| \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0} \right) \| v \|_{1,\omega;\Omega} \quad v \in V,$$

which gives $|\mathcal{L}(V)| \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0} \right) \| V \|_{\mathbb{H}_{1,\omega}(\Omega)}$; this means that $\mathcal{L}(V)$ is continuous. The continuity of $\mathcal{B}$ is clear. In every class U we choose a representative u such that $\left\langle \gamma_0^+ u_+, z \right\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$. By Theorem 7 in [18] and Theorem 4 in [19]

$$\begin{aligned} \| u_- \|^2_{1,\omega;\Omega^-} &\leqslant c \left[b_-(u_-, u_-) + \| \gamma_1^- u_- \|^2_{0;\Gamma_1} \right] \\ &\leqslant c \left[b_-(u_-, u_-) + \| \gamma_1^+ u_+ \|^2_{0;\Gamma_1} \right] \\ &\leqslant c \left[b_-(u_-, u_-) + \| u_+ \|^2_{1;\Omega^+} \right] \end{aligned}$$

and $\| u_+ \|^2_{1;\Omega^+} \leqslant c b_+(u_+, u_+)$, where $\| \cdot \|_{0;\Gamma_1}$ is the norm in $L^2(\Gamma_1)$. Hence, $\| U \|_{\mathbb{H}_{1,\omega}(\Omega)} \leqslant \| u \|^2_{1,\omega;\Omega} \leqslant \mathcal{B}(U, V)$, which proves that $\mathcal{B}$ is coercive on $\mathbb{H}_{1,\omega}(\Omega)$. By the Lax–Milgram lemma (3.10) has a unique solution $U \in \mathbb{H}_{1,\omega}(\Omega)$ and

$$\| U \|_{\mathbb{H}_{1,\omega}(\Omega)} \leqslant c \left(\| q \|_{-1,\omega} + \| \delta g \|_{-1/2;\Gamma} + \| g^- \|_{-1/2;\Gamma_0} \right).$$

Clearly, any element u in U is a solution of Eq. (3.2). If u_1 and u_2 are two solutions of Eq. (3.2), then $w = u_1 - u_2$ satisfies

$$b(w, w) = 0, \quad w \in H_{1,\omega}(\Omega).$$

We conclude that $w \in \mathcal{F}$. To complete the proof, we choose $u_0 \in U$ such that $\| u_0 \|_{1,\omega;\Omega} = \| U \|_{\mathbb{H}_{1,\omega}(\Omega)}$. □

We can describe $H_{-1,\omega}(\Omega)$ explicitly. If we define

$$\begin{aligned} Def\,\overline{u} &= (\partial_1 u_1, \partial_2 u_2, \partial_2 u_1 + \partial_1 u_2)^T, \quad Div\, u = (\partial_1 u_1 + \partial_2 u_3, \partial_2 u_2 + \partial_1 u_3)^T, \\ Grad\, \sigma &= (\partial_2 \sigma, -\partial_1 \sigma)^T, \text{ where } \sigma \in \mathcal{M}_{1\times 1}, \\ R &= \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \end{aligned}$$

then we can write

$$Lu = \begin{pmatrix} Div\,(R\, Def\,\overline{u}) + \alpha\, Grad(2u_3 - (curl\, u)_3) \\ (\gamma + \varepsilon) div \nabla u_3 - 2\alpha(2u_3 - (curl\, u)_3) \end{pmatrix}$$

and

$$\begin{aligned} b\,(u, v) = \langle R\, Def\,\overline{u}, Def\,\overline{v}\rangle_0 + \alpha\langle 2u_3 \\ -(curl\, u)_3, 2v_3 - (curl\, v)_3\rangle_0 + (\gamma + \varepsilon)\langle \nabla u_3, \nabla v_3\rangle_0. \end{aligned}$$

Theorem 4 *$H_{-1,\omega}(\Omega)$ consists of all q of the form*

$$\overline{q} = Div\, P + Grad\, Q, \quad q_3 = div\, V - 2Q, \tag{3.11}$$

where $P \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{3\times 1}$, $Q \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times 1}$, $V \in L^2(\mathbb{R}^2) \cap \mathcal{M}_{1\times 1}$. Also there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \parallel q \parallel_{-1,\omega;\Omega} \leqslant \parallel P \parallel_0 + \parallel Q \parallel_0 + \parallel V \parallel_0 \leqslant c_2 \parallel q \parallel_{-1,\omega;\Omega}.$$

This assertion can be proved using technique described in [17].

We now show that (D) and (N) can be reduced to similar problems for the homogeneous equilibrium equation by means of area potential.

Let $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ be the subspace of $H_{-1,\omega}(\mathbb{R}^2)$ consisting of all q such that $\langle q, z\rangle_0 = 0$ for all $z \in \mathcal{F}$. We choose an $L^2_\omega(\mathbb{R}^2)$-orthonormal basis $\{z^{(i)}\}_{i=1}^3$ for $\mathcal{F}$ and introduce a modified area potential of density $\varphi \in C_0^\infty(\mathbb{R}^2) \cap \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ by

$$(\mathcal{U}\varphi)(x) = (U\varphi)(x) - \left\langle U\varphi, z^{(i)}\right\rangle_{0,\omega} z^{(i)}(x), \quad x \in \mathbb{R}^2,$$

where $\langle\cdot,\cdot\rangle_{0,\omega}$ is the inner product in $L^2_\omega(\mathbb{R}^2)$. It can be shown (see [17] for details) that $\mathcal{U}\varphi \in H_{1,\omega}(\mathbb{R}^2)$ and satisfies

$$b\,(\mathcal{U}q, v) = \langle q, v\rangle_0 \quad \forall v \in H_{1,\omega}(\mathbb{R}^2). \tag{3.12}$$

The defined operator $\mathcal{U}$ can be extended by continuity from $C_0^\infty(\mathbb{R}^2) \cap \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ to $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$. The extended operator $\mathcal{U}$ is continuous from $\mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ to $H_{1,\omega}(\Omega)$. For any $q \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$, $\mathcal{U}(-q)$ is a solution of Eq. (3.12).

We start with (D). By Theorem 4, any $q \in H_{-1,\omega}(\Omega)$ can be represented in the form (3.11), where the equality is understood in $\mathcal{S}'(\Omega)$. Let $\widehat{q} \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$ be defined by the same formula (3.11), in which the equality is understood in $\mathcal{S}'(\mathbb{R}^2)$. We represent the solution of (D) in the form $u = \mathcal{U}(-\widehat{q}) + w$. Since $b(\mathcal{U}(-\widehat{q}), v) = \langle \widehat{q}, v\rangle_0 = \langle q, v\rangle_0$ for $v \in \mathring{H}_{1,\omega}(\Omega)$, we conclude that $w \in H_{1,\omega}(\Omega)$ satisfies

$$b(w, v) = 0 \quad \forall v \in \mathring{H}_{1,\omega}(\Omega),$$
$$\gamma_0^+ w_+ = f^+ - \gamma_0^+ (\mathcal{U}(-\widehat{q}))_+ , \quad \gamma_0^- w_- = f^- - \gamma_0^- (\mathcal{U}(-\widehat{q}))_- .$$

Let γ_0 be the trace operator defined on $H_{1,\omega}(\Omega)$ by $\gamma_0 v = \{\gamma_0^+ v_+, \gamma_0^+ v_+ - \gamma_0^- v_-\}$. It is clear that γ_0 is continuous from $H_{1,\omega}(\Omega)$ to $H_{1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0)$. Consequently, without loss of generality, in what follows we consider the problem (D) that consists in finding $u \in H_{1,\omega}(\Omega)$ such that

$$b(u, v) = 0 \quad \forall v \in \mathring{H}_{1,\omega}(\Omega), \qquad \gamma_0 u = \{f^+, \delta f\}. \tag{3.13}$$

In problem (N) we seek $u \in H_{1,\omega}(\Omega)$ such that

$$b(u, v) = \langle \widetilde{q}, v\rangle_0 , \quad \forall v \in H_{1,\omega}(\Omega), \tag{3.14}$$

where $\widetilde{q} \in \mathring{H}_{-1,\omega}(\Omega)$ satisfies

$$\langle \widetilde{q}, z\rangle_0 = 0, \quad \forall z \in \mathcal{F} . \tag{3.15}$$

Since $H_{1,\omega}(\mathbb{R}^2)$ is a subspace of $H_{1,\omega}(\Omega)$, we may consider $\widetilde{q}$ belonging to $H_{-1,\omega}(\mathbb{R}^2)$; in addition, from Eq. (3.15) it follows that $\widetilde{q} \in \mathcal{H}_{-1,\omega}(\mathbb{R}^2)$. We represent the solution of Eq. (3.14) in the form $u = \mathcal{U}\widehat{q} + w$, then Eq. (3.14) becomes

$$b(w, v) = \langle \widetilde{q}, v\rangle_0 - b(\mathcal{U}\widehat{q}, v), \quad \forall v \in H_{1,\omega}(\Omega).$$

Lemma 5 For all $\widetilde{q} \in \mathring{H}_{-1,\omega}(\Omega)$ satisfying Eq. (3.15), the expression

$$\mathcal{L}(\gamma_0 v) = \langle \widetilde{q}, v\rangle_0 - b(\mathcal{U}\widetilde{q}, v), \quad v \in H_{1,\omega}(\Omega) \tag{3.16}$$

defines a continuous linear functional on $H_{1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0)$; therefore, $\mathcal{L}(\gamma_0 v)$ can be written in the form

$$\langle \widetilde{q}, v\rangle_0 - b(\mathcal{U}\widetilde{q}, v) = \left\langle \delta g, \gamma_0^+ v_+\right\rangle_{0;\Gamma_0} + \left\langle g^-, \delta v\right\rangle_{0;\Gamma_0}, \quad v \in H_{1,\omega}(\Omega),$$

where $\{\delta g, g^-\} \in \mathring{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$.

Proof Let $v_1, v_2 \in H_{1,\omega}(\Omega)$ such that $\gamma_0 v_1 = \gamma_0 v_2$. The difference $v_1 - v_2 \in \mathring{H}_{1,\omega}(\Omega) \subset H_{1,\omega}(\mathbb{R}^2)$, and since $b(\mathcal{U}\widetilde{q}, v_1 - v_2) = \langle \widetilde{q}, v_1 - v_2\rangle_0$, we find that $\mathcal{L}(\gamma_0 v_1) = \mathcal{L}(\gamma_0 v_2)$. This means that definition (3.16) of $\mathcal{L}$ on $H_{1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0)$ is consistent. Let $\{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0)$. Repeating the proof of Theorem 2, we choose $v \in H_{1,\omega}(\Omega)$ so that $\gamma_0 v = \{f^+, \delta f\}$ and $\| v \|_{1,\omega;\Omega} \leqslant c(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma})$. We have

$$\left|\mathcal{L}\left(\{f^+, \delta f\}\right)\right| \leqslant c \,\| \widetilde{q} \|_{-1,\omega} \| v \|_{1,\omega;\Omega} \leqslant c \,\| \widetilde{q} \|_{-1,\omega} (\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}),$$

which shows that $\mathcal{L}$ is continuous on $H_{1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0)$; since $\mathring{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ is the dual of $H_{1/2}(\Gamma_0) \times \mathring{H}_{1/2}(\Gamma_0)$, this completes the proof. □

Lemma 5 implies that, without loss of generality, we may consider (N) only for the homogeneous equilibrium equation; that is, we seek $u \in H_{1,\omega}(\Omega)$ such that

$$b(u, v) = \langle \delta g, \gamma_0^+ v_+ \rangle_{0;\Gamma_0} + \langle g^-, \delta v \rangle_{0;\Gamma_0}, \ \forall v \in H_{1,\omega}(\Omega). \tag{3.17}$$

We remark that Eq. (3.17) is solvable only if

$$\langle z, \delta g \rangle_{0;\Gamma_0} = 0, \ \forall z \in \mathcal{F}. \tag{3.18}$$

4 The Poincaré–Steklov Operator

For $F = \{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ and $G = \{\delta g, g^-\} \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ we use the notation

$$[F, G]_{0;\Gamma_0} = \langle f^+, \delta g \rangle_{0;\Gamma_0} + \langle \delta f, g^- \rangle_{0;\Gamma_0}.$$

We define the Poincaré–Steklov operator $\mathcal{T}$ on $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ by

$$\begin{aligned} [\mathcal{T}F, \Psi]_{0;\Gamma_0} &= b(u, v) \quad \forall \Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0), \\ F &\in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0), \end{aligned} \tag{4.1}$$

where u is the solution of Eq. (3.13) and v is any element in $H_{1,\omega}(\Omega)$ such that $\gamma_0 v = \Psi = \{\psi^+, \delta\psi\}$. The definition is independent of the choice of v. In particular, we may take $v = l\Psi$, where l is an operator extension from Γ_0 to Ω which maps $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ continuously to $H_{1,\omega}(\Omega)$.

We identify $\mathcal{F}$ with the subspace of $H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ consisting of all $Z = \{z, 0\}$, $z \in \mathcal{F}$. We also introduce the spaces

$$\widehat{\mathcal{H}}_{1/2}(\Gamma_0) = \{F \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) : \langle f^+, z \rangle_{0;\Gamma_0} = 0, \forall z \in \mathcal{F}\},$$

$$\widehat{\mathcal{H}}_{-1/2}(\Gamma_0) = \{G \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0) : \langle \delta g, z \rangle_{0;\Gamma_0} = 0, \ \forall z \in \mathcal{F}\}.$$

Theorem 6

(a) $\mathcal{T} : H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) \to \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ *is self-adjoint and continuous.*
(b) *The kernel of* $\mathcal{T}$ *coincides with* $\mathcal{F}$.
(c) *The range of* $\mathcal{T}$ *coincides with* $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$.
(d) *The restriction* $\mathcal{N}$ *of* $\mathcal{T}$ *to* $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ *is a homeomorphism from* $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ *to* $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$.

Proof

(a) If u is the solution of Eq. (3.13) and $v = l\Psi$, then, by definition of $\mathcal{T}$, for $F, \Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$

$$\left|[\mathcal{T}F, \Psi]\right|^2 = |b(u, v)|^2 \le b(u, u) b(v, v) \leqslant c b(u, u) \parallel \Psi \parallel^2_{H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)}.$$

Consequently, $\mathcal{T}F \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ and

$$\begin{aligned}\| \mathcal{T}f \|^2_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} &\leq cb\,(u,u) = c\left[\mathcal{T}F, F\right]_{0;\Gamma_0} \\ &\leq c \,\| \mathcal{T}f \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} \| F \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} .\end{aligned} \tag{4.2}$$

From Eq. (4.2) it follows that

$$\| \mathcal{T}F \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} \leq c \,\| F \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)}, \tag{4.3}$$

which proves the continuity of $\mathcal{T}$. The definition of $\mathcal{T}$ shows that it is self-adjoint in the sense that

$$\left[\mathcal{T}F, \Psi\right]_{0;\Gamma_0} = \left[\Psi, \mathcal{T}F\right]_{0;\Gamma_0} \quad \forall F, \Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0).$$

(b) It is clear that $\mathcal{T}Z = 0$ for $Z \in \mathcal{F}$. If $F \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$, $\mathcal{T}F = 0$ and u is the solution of Eq. (3.13), then $b\,(u,u) = 0$; therefore, $u \in \mathcal{F}$, which implies that $F = \gamma_0 u \in \mathcal{F}$. This also proves that $\mathcal{N}$ is injective.

(c) By Eq. (4.3), the range of $\mathcal{T}$ is a subset of $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$. Let $\{\widetilde{z}^{(i)}\}_{i=1}^3$ be an $L^2(\Gamma_0)$-orthonormal basis for $\mathcal{F}$. From Theorem 7 in [18] and Theorem 4 in [19] it follows that any $u \in H_{1,\omega}(\Omega)$ satisfies

$$\| u \|^2_{1,\omega;\Omega} \leqslant c \left[b\,(u,u) + \sum_{i=1}^{3} \langle \gamma_0^+ u_+, \widetilde{z}^{(i)} \rangle^2_{0;\Gamma_0} \right]. \tag{4.4}$$

Let $F \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$. By the trace theorem and Eq. (4.4)

$$\| F \|^2_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leqslant c \,\| u \|^2_{1,\omega;\Omega} \leqslant cb\,(u,u) = c\left[\mathcal{T}F, F\right]_{0;\Gamma_0};$$

hence,

$$\| F \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leqslant c \,\| \mathcal{T}F \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)},$$

which shows that $\mathcal{N}^{-1}$ is continuous. If the range of $\mathcal{T}$ is not dense in $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ then there is a nonzero $\widehat{F}$ in the dual $\left[H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)\right]\backslash\mathcal{F}$ of $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ such that $\langle \mathcal{T}F, \Psi\rangle_{0;\Gamma_0} = 0$ for all representative F of the class $\widehat{F}$ and all $\Psi \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$. Taking $F \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ and $\Psi = F$, we find that $\left[\mathcal{T}F, F\right]_{0;\Gamma_0} = 0$; therefore, $F \in \mathcal{F}$ and $\widehat{F} = 0$. This contradiction proves the third statement.

(d) This assertion follows from the preceding ones. □

5 Boundary Equations

Let $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ be the subspace of $\overset{\circ}{H}_{-1/2}(\Gamma_0)$ of all g such that $\langle g, z\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$.

We define the modified single layer potential $\mathcal{V}$ of density $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ by

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \left\langle (V\varphi)_0, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}(x), \quad x \in \mathbb{R}^2,$$

where $V\varphi$ is the single layer potential and V_0 is the boundary operator defined by $(V\varphi)_0 = \gamma_0^{\pm}\pi^{\pm}V\varphi$. Let $\mathcal{V}_0\varphi$ be the operator defined on $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ by $\varphi \to (\mathcal{V}\varphi)_0 = \gamma_0^{\pm}\pi^{\pm}\mathcal{V}\varphi$. From the results established in [18] and [19] $\mathcal{V}_0$ is continuous from $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ to the subspace $\mathcal{H}_{1/2}(\Gamma_0)$ of all $f^+ \in H_{1/2}(\Gamma_0)$ such that $\langle f^+, z\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$. Let $\widetilde{\mathcal{V}}$ be the continuous operator from $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ defined by $\widetilde{\mathcal{V}}\varphi = \{\mathcal{V}_0\varphi, 0\}$.

Theorem 7 *The operator $\mathcal{V}_0$ is a homeomorphism from $\overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ to $\mathcal{H}_{1/2}(\Gamma_0)$.*

Proof The continuity of $\mathcal{V}_0$ is proved in [18] and [19]. From the jump formula for the normal boundary stresses and couple stresses of the single laye potential (Theorem 1) it follows that the first component of $\mathcal{N}\widetilde{\mathcal{V}}\varphi \in \widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ is φ. By Theorem 6

$$\| \varphi \|_{-1/2;\Gamma_0} \leqslant \| \mathcal{N}\widetilde{\mathcal{V}}\varphi \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)}$$

$$\leqslant c \, \| \widetilde{\mathcal{V}}\varphi \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} = c \, \| \mathcal{V}_0\varphi \|_{1/2;\Gamma_0},$$

which shows that $\mathcal{V}_0^{-1}$ is continuous. Next, we claim that the range of $\mathcal{V}_0$ is $\mathcal{H}_{1/2}(\Gamma_0)$. Let $f^+ \in \mathcal{H}_{1/2}(\Gamma_0)$, $F = \{f^+, 0\} \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ and let $u \in H_{1,\omega}(\Omega)$ be the solution of Eq. (3.13) with $\delta f = 0$. We take $G = \{\delta g, g^-\} = \mathcal{N}F \in \widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\varphi = \delta g \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$. Then $w = u - \mathcal{V}_0\varphi$ satisfies $\gamma_0 w = \{f^+ - \mathcal{V}_0\varphi, 0\} = \Psi$. By the jump formula the first component of $\mathcal{N}\Psi$ is zero; consequently, $b(w, w) = [\mathcal{N}\Psi, \Psi]_{0;\Gamma_0} = 0$. This means that $w \in \mathcal{F}$ so $\gamma_0^+ w_+$ is a rigid displacement on Γ_0. Since $\gamma_0^+ w_+ = f^+ - \mathcal{V}_0\varphi \in \mathcal{H}_{1/2}(\Gamma_0)$, we have $f^+ = \mathcal{V}_0\varphi$, therefore, the assertion is proved. □

Also we introduce modified double layer potential $\mathcal{W}$ of density $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$

$$(\mathcal{W}\psi)(x) = (W\psi)(x) - \left\langle \pi_0 W^+\psi, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}(x), \quad x \in \Omega.$$

Clearly, if $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ then $\mathcal{W}\psi \in H_{1,\omega}(\Omega)$ and $\| \mathcal{W}\psi \|_{1,\omega;\Omega} \leqslant c \, \| \psi \|_{1/2;\Gamma}$. Hence, for $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ we can define the operators $\mathcal{W}^{\pm}$ of the limiting values of the modified double layer potential on Γ from within $\Omega^{\pm}$ by writing $\mathcal{W}^{\pm}\psi = \gamma^{\pm}\pi^{\pm}\mathcal{W}\psi$. It is obvious that $\mathcal{W}^{\pm}$ are continuous from $\overset{\circ}{H}_{1/2}(\Gamma_0)$ to $H_{1/2}(\Gamma)$ and satisfy the jump formula

$$\mathcal{W}^+\psi - \mathcal{W}^-\psi = -\psi. \tag{5.1}$$

For $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ we now define the operator $\mathcal{W}_0$ of the limiting values of the modified double layer potential on Γ_0 from within Ω by writing

$$\mathcal{W}_0\psi = \left\{\pi_0\mathcal{W}^+\psi, \pi_0(\mathcal{W}^+\psi - \mathcal{W}^-\psi)\right\} = \left\{\pi_0\mathcal{W}^+\psi, -\psi\right\}.$$

Clearly, $\mathcal{W}_0$ is continuous from $\overset{\circ}{H}_{1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$.

Let $\widetilde{\mathcal{G}} = \mathcal{N}\mathcal{W}_0$. From the jump formula for the normal boundary stresses and couple stresses of the double layer potential it follows that the first component of $\widetilde{\mathcal{G}}\psi$ is zero for any $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$; therefore, we can write $\widetilde{\mathcal{G}}\psi = \{0, \mathcal{G}\psi\}$ for all $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$.

Theorem 8 *$\mathcal{G}$ is a homeomorphism from $\overset{\circ}{H}_{1/2}(\Gamma_0)$ to $H_{-1/2}(\Gamma_0)$.*

Proof The continuity of $\mathcal{G}$ follows from the properties of $\mathcal{W}_0$ and $\mathcal{N}$. We claim that $\mathcal{G}^{-1}$ is continuous. Let $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$. By Eq. (5.1) and the trace theorem we conclude that

$$\begin{aligned}\| \psi \|^2_{1/2;\Gamma} &= \| \mathcal{W}^+\psi - \mathcal{W}^-\psi \|^2_{1/2;\Gamma} \leqslant c \| \mathcal{W}\psi \|^2_{1,\omega;\Omega} \\ &\leqslant cb\,(\mathcal{W}\psi, \mathcal{W}\psi) = -c\,\langle \mathcal{G}\psi, \psi\rangle_{0;\Gamma_0} \\ &\leqslant c \| \mathcal{G}\psi \|_{-1/2;\Gamma_0} \| \psi \|_{1/2;\Gamma};\end{aligned}$$

consequently, $\| \psi \|_{1/2;\Gamma} \leqslant c \| \mathcal{G}\psi \|_{-1/2;\Gamma_0}$. If the range of $\mathcal{G}$ is not dense in $H_{-1/2}(\Gamma_0)$ then there is a nonzero ψ in the dual $\overset{\circ}{H}_{1/2}(\Gamma_0)$ such that $\langle \psi, \mathcal{G}\xi\rangle_{0;\Gamma_0} = 0$ for all $\xi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$. We take $\xi = \psi$ and obtain $\langle \psi, \mathcal{G}\psi\rangle_{0;\Gamma_0} = 0$, which means that $\mathcal{W}\psi \in \mathcal{F}$; hence, $\psi = \mathcal{W}^-\psi - \mathcal{W}^+\psi = 0$. This contradiction completes the proof. □

We represent the solution of Eq. (3.13) in the form

$$u = (\mathcal{V}\varphi)_\Omega + W\psi + z, \tag{5.2}$$

where $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ are unknown densities, $(\mathcal{V}\varphi)_\Omega$ is the restriction of $\mathcal{V}\varphi$ to Ω and

$$z = \left\langle f^+ - \pi_0 W^+\psi, \tilde{z}{}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}{}^{(i)}.$$

Representation (5.2) leads to the system of boundary equations

$$\left\{\mathcal{V}_0\varphi + \pi_0 W^+\psi + \gamma_0^+ z, -\psi\right\} = \{f^+, \delta f\}. \tag{5.3}$$

Theorem 9 *For any $\{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$, system (5.3) has a unique solution*

$$\{\varphi, \psi\} \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0),$$

respectively, and

$$\| \{\varphi, \psi\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leq c \| \{f^+, \delta f\} \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} .$$

In this case, Eq. (5.2) is the solution of problem (3.13).

Proof From Eq. (5.3) $\psi = -\delta f \in \overset{\circ}{H}_{1/2}(\Gamma_0)$, and the equation for φ becomes

$$\mathcal{V}_0\varphi = f^+ + \pi_0 W^+\delta f - \left\langle f^+ + \pi_0 W^+\delta f, \tilde{z}^{(i)} \right\rangle_{0;\Gamma_0} \tilde{z}^{(i)}. \tag{5.4}$$

The right-hand side in Eq. (5.4) belongs to $\mathcal{H}_{1/2}(\Gamma_0)$. By Theorem 7, Eq. (5.4) has a unique solution $\varphi \in \overset{\circ}{\mathcal{H}}_{1/2}(\Gamma_0)$ and

$$\begin{aligned}\| \varphi \|_{-1/2;\Gamma} &\leqslant c\left(\| f^+ \|_{1/2;\Gamma_0} + \| \pi_0 W^+\delta f \|_{1/2;\Gamma_0}\right)\\ &\leqslant c\left(\| f^+ \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}\right) = c \, \| \{f^+, \delta f\} \|_{H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} .\end{aligned}$$

The uniqueness of the solution is now obvious. □

We represent the solution of problem (3.17) in the form

$$u = (\mathcal{V}\varphi)_\Omega + \mathcal{W}\psi + z, \tag{5.5}$$

where $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ are unknown densities and $z \in \mathcal{F}$ is arbitrary. Representation (5.5) leads to the systems of boundary equations

$$\mathcal{N}\widetilde{\mathcal{V}}\varphi + \widetilde{\mathcal{G}}\psi = \{\delta g, g^-\}. \tag{5.6}$$

Theorem 10 *For any* $\{\delta g, g^-\} \in \overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$ *satisfying Eq.* (3.18)*, system* (5.6) *has a unique solution* $\{\varphi, \psi\} \in \overset{\circ}{\mathcal{H}}_{1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$ *and*

$$\| \{\varphi, \psi\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leq c \, \| \{\delta g, g^-\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)} .$$

In this case, Eq. (5.5) *is the solution of problem* (3.17)*.*

Proof Comparing first components on both sides of Eq. (5.6), we see that $\varphi = \delta g$; therefore, Eq. (5.6) takes the form

$$\mathcal{G}\psi = g^- - \left(\mathcal{N}\widetilde{\mathcal{V}}\delta g\right)^-, \tag{5.7}$$

where $\left(\mathcal{N}\widetilde{\mathcal{V}}\delta g\right)^-$ is the second component of $\mathcal{N}\widetilde{\mathcal{V}}\delta g$. By Theorems 6, 7 and 8, Eq. (5.7) has a unique solution $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$ and

$$\| \psi \|_{1/2;\Gamma} \leqslant c\left(\| g^- \|_{-1/2;\Gamma_0} + \| \delta g \|_{-1/2;\Gamma}\right).$$ □

6 The Boundary Equations for a Finite Domain

Let ∂S be a simple closed C^2-curve that divides $\mathbb{R}^2$ into interior and exterior domains S^+ and S^-. We assume that S^+ contains inside an auxiliary simple closed C^2-curve $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 is an open arc modeling the crack. We write $\Omega = S^+\backslash\overline{\Gamma}_0$. Let Ω^+ be the interior domain bounded by Γ and let $\Omega^- = S^+\backslash\overline{\Omega}^+$.

If u is defined in Ω then we denote by u_+ and u_- its restrictions to Ω^+ and Ω^-, respectively, and write $u = \{u_+, u_-\}$. The spaces $H_1(\Omega^\pm)$ are introduced in the usual way. The traces of the elements $u_\pm \in H_1(\Omega^\pm)$ on Γ are denoted by γ^+u_+ and γ^-u_-.

We denote by π_i, $i=0,1$, the operators of restrictions from Γ to Γ_i and write $\gamma_i^{\pm}=\pi_i\gamma^{\pm}$, $i=0,1$. The space $H_1(\Omega)$ consists of all $u=\{u_+,u_-\}$ defined in Ω and such that $u_+\in H_1(\Omega^+)$, $u_-\in H_1(\Omega^-)$ and $\gamma_1^+u_+=\gamma_1^-u_-$. The norm in $H_1(\Omega)$ is defined by $\| u \|^2_{1;\Omega}=\| u_+ \|^2_{1;\Omega^+}+\| u_- \|^2_{1;\Omega^-}$. Let γ_0 be the trace operator that acts on $u\in H_1(\Omega)$ according to the formula $\gamma_0 u=\{\gamma_0^+u_+,\gamma_0^+u_+-\gamma_0^-u_-\}$. Clearly, γ_0 is continuous from $H_1(\Omega)$ to $H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)$. The trace of $u\in H_1(\Omega)$ on ∂S is denoted by $\gamma_{\partial S}^+u$. $\overset{\circ}{H}_1(\Omega)$ is the subspace of $H_1(\Omega)$ consisting of all $u\in H_1(\Omega)$ such that $\gamma_0 u=\{0,0\}$ and $\gamma_{\partial S}^+u=0$.

Let $\widehat{\Gamma}=\Gamma_0\cup\partial S$. In what follows we make use of spaces $H_{1/2}(\widehat{\Gamma})=H_{1/2}(\Gamma_0)\times \overset{\circ}{H}_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)$ of all $\widehat{F}=\{F,f_{\partial S}\}$, where $F=\{f^+,\delta f\}$, and $H_{-1/2}(\widehat{\Gamma})=\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)$ of all $\widehat{G}=\{G,g_{\partial S}\}$, where $G=\{\delta g,g^-\}$. It is clear that these spaces are dual with respect to the duality $\left[\widehat{F},\widehat{G}\right]_{0;\widehat{\Gamma}}=\left[F,G\right]_{0;\Gamma_0}+\langle f_{\partial S},g_{\partial S}\rangle_{0;\partial S}$, where $\left[F,G\right]_{0;\Gamma_0}$ is the form defined in Section 4. This duality is generated by the inner product $[\cdot,\cdot]_{0;\widehat{\Gamma}}$ in $L^2(\widehat{\Gamma})=L^2(\Gamma_0)\times L^2(\Gamma_0)\times L^2(\partial S)$.

We consider the following boundary value problems.

Given $\widehat{F}=\{F,f_{\partial S}\}\in H_{1/2}(\widehat{\Gamma})$, we seek $u\in H_1(\Omega)$ such that

$$b_\Omega(u,v)=0\ \ \forall v\in \overset{\circ}{H}_1(\Omega),\qquad \gamma_0 u=F,\ \ \gamma_{\partial S}^+u=f_{\partial S}, \tag{6.1}$$

where $b_\Omega(u,v)=\int_\Omega E(u,v)dx$.

Given $\widehat{G}=\{G,g_{\partial S}\}\in H_{-1/2}(\widehat{\Gamma})$, we seek $u\in H_1(\Omega)$ such that

$$b_\Omega(u,v)=\left[G,\gamma_0 v\right]_{0;\Gamma_0}+\langle g_{\partial S},\gamma_{\partial S}v\rangle_{0;\partial S},\quad \forall v\in H_1(\Omega). \tag{6.2}$$

Clearly, Eq. (6.2) is solvable only if

$$\langle \delta g,z\rangle_{0;\Gamma_0}+\langle g_{\partial S},z\rangle_{0;\partial S}=0,\quad \forall z\in\mathcal{F}. \tag{6.3}$$

In what follows we assume that Eq. (6.3) holds. The proofs of the unique solvability of Eq. (6.1) and Eq. (6.2) repeat those of Theorems 2 and 3 with the obvious changes, so we omit them.

We introduce the Poincaré–Steklov operator $\widehat{\mathcal{T}}$ by $\left[\widehat{\mathcal{T}}\widehat{F},\widehat{\Psi}\right]_{0;\widehat{\Gamma}}=b_\Omega(u,v)$, where $\widehat{F},\widehat{\Psi}\in H_{1/2}(\widehat{\Gamma})$ are arbitrary, u is a solution of Eq. (6.1) and $v\in H_1(\Omega)$ is any extension of $\widehat{\Psi}$ to Ω. Let $\mathcal{F}(\widehat{\Gamma})$ be the space of all $\widehat{Z}=\{Z,z\}$, $Z=\{z,0\}$, where $z\in\mathcal{F}$ is arbitrary. We define the spaces

$$\mathcal{H}_{1/2}(\widehat{\Gamma})=\left\{\widehat{F}\in H_{1/2}(\widehat{\Gamma}):\left[\widehat{F},\widehat{Z}\right]_{0;\widehat{\Gamma}}=0\ \forall\widehat{Z}\in\mathcal{F}(\widehat{\Gamma})\right\},$$

$$\mathcal{H}_{-1/2}(\widehat{\Gamma})=\left\{\widehat{G}\in H_{-1/2}(\widehat{\Gamma}):\left[\widehat{G},\widehat{Z}\right]_{0;\widehat{\Gamma}}=0\ \forall\widehat{Z}\in\mathcal{F}(\widehat{\Gamma})\right\}.$$

Theorem 11

(a) $\widehat{\mathcal{T}}$ is self-adjoint and continuous from $H_{1/2}(\widehat{\Gamma})$ to $H_{-1/2}(\widehat{\Gamma})$.
(b) The kernel of $\widehat{\mathcal{T}}$ coincides with $\mathcal{F}(\widehat{\Gamma})$.
(c) The range of $\widehat{\mathcal{T}}$ coincides with $\mathcal{H}_{-1/2}(\widehat{\Gamma})$.
(d) The restriction $\widehat{\mathcal{N}}$ of $\widehat{\mathcal{T}}$ from $H_{1/2}(\widehat{\Gamma})$ to $\mathcal{H}_{1/2}(\widehat{\Gamma})$ is a homeomorphism from $\mathcal{H}_{1/2}(\widehat{\Gamma})$ to $\mathcal{H}_{-1/2}(\widehat{\Gamma})$.

The proof of this theorem is identical to that of Theorem 6.

Let $\overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$ be the subspace of $\overset{\circ}{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\partial S)$ of all $\varphi = \{\varphi_0, \varphi_{\partial S}\}$ such that $\langle \varphi_0, z\rangle_{0;\Gamma_0} + \langle \varphi_{\partial S}, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$. $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$ is the subspace of $H_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)$ consisting of all $f = \{f^+, f_{\partial S}\}$ such that $\left\langle f^+, z\right\rangle_{0;\Gamma_0} + \langle f_{\partial S}, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$.

We define the single layer potential of density $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$ by

$$(V\varphi)(x) = (V_0\varphi_0)(x) + (V_{\partial S}\varphi_{\partial S})(x), \quad x \in \mathbb{R}^2,$$

where $V_0\varphi_0$ and $V_{\partial S}\varphi_{\partial S}$ are the single layer potentials defined on Γ_0 and ∂S, respectively. Let $\{\widehat{Z}^{(i)}\}_{i=1}^3$ be an $L^2(\widehat{\Gamma})$-orthonormal basis for $\mathcal{F}(\widehat{\Gamma})$, where $\widehat{Z}^{(i)} = \{Z^{(i)}, z^{(i)}\}$ and $Z^{(i)} = \{z^{(i)}, 0\}$. The rigid displacements $z^{(i)}$ satisfy Eq. (6.1) with boundary data $F = Z^{(i)}$, $f_{\partial S} = z^{(i)}$. We introduce the modified single layer potential

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \left[\left\langle (V\varphi)_0\,, z^{(i)}\right\rangle_{0;\Gamma_0} + \left\langle (V\varphi)_{\partial S}\,, z^{(i)}\right\rangle_{0;\partial S}\right] z^{(i)}(x), \;\; x \in \mathbb{R}^2,$$

where $(V\varphi)_0$ and $(V\varphi)_{\partial S}$ are the restrictions of $V\varphi$ to Γ_0 and ∂S. The corresponding boundary operator $\mathcal{V}_{\widehat{\Gamma}}$ is defined by $\mathcal{V}_{\widehat{\Gamma}}\varphi = \left\{\gamma_0^+ (\mathcal{V}\varphi)_+\,, \gamma_{\partial S}^+ (\mathcal{V}\varphi)_\Omega\right\}$, where $(\mathcal{V}\varphi)_\pm$ are the restrictions of $\mathcal{V}\varphi$ to $\Omega^\pm$. We also introduce a boundary operator $\widehat{\mathcal{V}}$ by writing $\widehat{\mathcal{V}}\varphi = \left\{\gamma_0^+ (\mathcal{V}\varphi)_+\,, 0, \gamma_{\partial S}^+ (\mathcal{V}\varphi)_\Omega\right\}$.

Theorem 12 $\mathcal{V}_{\widehat{\Gamma}}$ *is a homeomorphism from* $\overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$ *to* $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$.

Proof The proof of this theorem makes use of already studied properties of the modified single layer potential and the Poincaré–Steklov operator for the exterior region [19] and Theorem 11. □

Let $\mathcal{H}_{1/2}(\partial S)$ be the subspace of $H_{1/2}(\partial S)$ consisting of all f such that $\langle f, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$. $\mathcal{H}_{-1/2}(\partial S)$ is the subspace of $H_{-1/2}(\partial S)$ of all g such that $\langle g, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$.

We define the double layer potential of density $\psi = \{\psi_0, \psi_{\partial S}\} \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ by

$$(W\psi)(x) = (W_0\psi_0)(x) + (W_{\partial S}\psi_{\partial S})(x), \quad x \in \Omega,$$

where $W_0\psi_0$ and $W_{\partial S}\psi_{\partial S}$ are the double layer potentials defined on Γ_0 and ∂S, respectively. We introduce the modified double layer potential

$$(\mathcal{W}\psi)(x) = (W\psi)(x) - \left[\left\langle (W\psi)_0^+\,, z^{(i)}\right\rangle_{0;\Gamma_0} + \left\langle (W\psi)_{\partial S}^+\,, z^{(i)}\right\rangle_{0;\partial S}\right] z^{(i)}(x), \;\; x \in \Omega,$$

where $(W\psi)_0^+$ and $(W\psi)_{\partial S}^+$ are the limiting values of $W\psi$ on Γ_0 and ∂S from within Ω^+ and S^+. We also define the limiting values $\mathcal{W}^\pm$ of the modified double layer potential on Γ from within $\Omega^\pm$ by writing $\mathcal{W}^\pm\psi = \gamma^\pm\pi^\pm\mathcal{W}\psi$. The corresponding boundary operator $\widehat{\mathcal{W}}\psi = \left\{\pi_0\left(\mathcal{W}^+\psi\right), \pi_0(\mathcal{W}^+\psi - \mathcal{W}^-\psi), \gamma_{\partial S}^+ (\mathcal{W}\psi)_\Omega\right\} = \left\{\gamma_0^+\pi^+ (\mathcal{W}\psi)\,, -\psi_0, \gamma_{\partial S}^+ (\mathcal{W}\psi)_\Omega\right\}$.

Let $\widehat{\mathcal{G}} = \widehat{\mathcal{N}}\widehat{\mathcal{W}}$. From the jump formula for the normal boundary stresses and couple stresses of the double layer potential it follows that the first component of $\widehat{\mathcal{G}}\psi$ is 0 for any $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$; therefore, we can write $\widehat{\mathcal{G}}\psi =$

$\{0, (\widehat{\mathcal{G}}\psi)^-, (\widehat{\mathcal{G}}\psi)_{\partial S}\}$ for all $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. We also define boundary operator $\mathcal{G}_{\widehat{\Gamma}}\psi = \{(\widehat{\mathcal{G}}\psi)^-, (\widehat{\mathcal{G}}\psi)_{\partial S}\}$ from $\overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ to $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$.

Theorem 13 $\mathcal{G}_{\widehat{\Gamma}}$ *is a homeomorphism from* $\overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ *to* $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$.

Proof The proof of this assertion makes use of already studied properties of the modified double layer potential and the Poincaré–Steklov operator for the exterior region [19] and Theorem 11. □

We represent the solution of Eq. (6.1) in the form

$$u = (\mathcal{V}\varphi)_\Omega + W_0\psi + z, \tag{6.4}$$

where $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma})$, $W_0\psi$ is the double layer potential of density $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0)$,

$$z = \left[\left\langle f^+ + \gamma_0^+(W_0\delta f)_+, z^{(i)}\right\rangle_{0;\Gamma} + \left\langle f_{\partial S} + \gamma_{\partial S}^+(W_0\delta f)_\Omega, z^{(i)}\right\rangle_{0;\partial S}\right] z^{(i)} \tag{6.5}$$

and $(W_0\delta f)_+$ and $(W_0\delta f)_\Omega$ are the restrictions of $W_0\delta f$ to Ω^+ and Ω. This representation yields the system of boundary equations

$$\widehat{\mathcal{V}}\varphi + \left\{\gamma_0^+(W_0\psi)_+, -\psi, \gamma_{\partial S}^+(W_0\psi)_\Omega\right\} = \widehat{F} - \{z, 0, z\}. \tag{6.6}$$

Theorem 14 *For any* $\widehat{F} \in H_{1/2}(\widehat{\Gamma})$, *system* (6.6) *has a unique solution* $\{\varphi, \psi\} \in \overset{\circ}{\mathcal{H}}_{-1/2}(\widehat{\Gamma}) \times \overset{\circ}{H}_{1/2}(\Gamma_0)$, *which satisfies the estimate*

$$\| \{\varphi, \psi\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)\times \overset{\circ}{H}_{1/2}(\Gamma_0)} \leqslant c \, \| \widehat{F} \|_{H_{1/2}(\widehat{\Gamma})} .$$

In this case, u defined by Eq. (6.4) *is a solution of Eq.* (6.1).

Proof We take $\psi = -\delta f$ and reduce Eq. (6.6) to the system

$$\mathcal{V}_{\widehat{\Gamma}}\varphi = \{f^+, f_{\partial S}\} + \left\{\gamma_0^+(W_0\delta f)_+, \gamma_{\partial S}^+(W_0\delta f)_\Omega\right\} - \{z, z\}. \tag{6.7}$$

By Eq. (6.5), the right-hand side in Eq. (6.7) belongs to $\widehat{\mathcal{H}}_{1/2}(\widehat{\Gamma})$. The assertion now follows from Theorem 12. □

We represent the solution of Eq. (6.2) in the form

$$u = \mathcal{V}_{\Gamma_0}\varphi + \mathcal{W}\psi + z, \tag{6.8}$$

where $\mathcal{V}_{\Gamma_0}\varphi$ is the modified single layer potential of density $\varphi \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0)$, φ and $\psi \in \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ are unknown densities and $z \in \mathcal{F}$ is arbitrary. This representation yields the system of boundary equations

$$\widehat{\mathcal{N}}\left\{\gamma_0^+(\mathcal{V}_{\Gamma_0}\varphi), 0, \gamma_{\partial S}^+(\mathcal{V}_{\Gamma_0}\varphi)_\Omega\right\} + \widehat{\mathcal{G}}\psi = \widehat{G}. \tag{6.9}$$

Theorem 15 *For any* $\widehat{G} \in H_{-1/2}(\widehat{\Gamma})$, *system* (6.9) *has a unique solution* $\{\varphi, \psi\} \in \overset{\circ}{\mathcal{H}}_{-1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$, *which satisfies the estimate*

$$\| \{\varphi, \psi\} \|_{\overset{\circ}{H}_{-1/2}(\Gamma_0) \times \overset{\circ}{H}_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)} \leqslant c \, \| \widehat{G} \|_{H_{-1/2}(\widehat{\Gamma})} .$$

In this case, u defined by Eq. (6.8) *is a solution of Eq.* (6.2).

Proof From the jump formula for normal boundary stresses and couple stresses of the single layer potential (Theorem 1) the first component of $\widehat{\mathcal{N}} \left\{ \gamma_0^+ (\mathcal{V}_{\Gamma_0}\varphi), 0, \gamma_{\partial S}^+ (\mathcal{V}_{\Gamma_0}\varphi)_\Omega \right\}$ is equal to φ. Comparing the first components on the both sides of Eq. (6.9) we see that $\varphi = \delta g$. The assertion now follows from Theorems 11, 12 and 13. □

Remark 16 In this paper we have assumed that Γ and ∂S are C^2 -curves. It can be shown that all the above results remain valid for piecewise-smooth $C^{0,1}$-curves that consist of finitely many C^2-arcs [20].

7 Summary

In this paper we have formulated Dirichlet and Neumann boundary value problems of plane Cosserat elasticity for a domain weakened by a crack in Sobolev spaces and showed these problems to be well-posed and depend continuously (in a suitable Sobolev-type norm) on the data. This result is important for practical purposes, since it validates further applications of numerical procedures. We have also shown that the corresponding weak solutions can be represented in terms of modified integral potentials with unknown distributional densities, which facilitate the construction of appropriate boundary element methods for finding these distributional densities and solving the problem numerically.

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