Multiple Scattering of Radiation by an Arbitrary Configuration of Parallel Cylinders*†

VIC TWERSKY
Mathematics Research Group, Washington Square College, New York University, New York, New York
(Received June 6, 1951)

A formal solution in terms of cylindrical wave functions is obtained for the scattering of a plane acoustic or electromagnetic wave by an arbitrary configuration of parallel cylinders which takes into account all possible contributions to the excitation of a particular cylinder by the radiation scattered by the remaining cylinders. The solution, satisfying any of the usual prescribed boundary conditions simultaneously at the surface of each cylinder, is expressed as the incident wave plus a sum of various orders of scattering. The first order of scattering (the usual single scattering approximation) results from the excitation of each cylinder by only the plane wave or primary excitation. The second order results from the excitation of each cylinder by the first order of scattering from the remaining cylinders, and so on to an infinite order of scattering. The first order therefore consists of waves scattered by one cylinder; the second order of waves scattered by two cylinders, etc. The scattering coefficients of the m'th order of scattering are expressed recursively in terms of the previous orders, and finally as sums of products of m scattering coefficients of the single cylinder and Hankel and trigonometric functions depending on the geometry of the configuration.

INTRODUCTION

THE scattering of radiation by configurations of elementary scatterers is too well known a problem to require much discussion of its physical significance. It arises whenever acoustic or electromagnetic radiation is propogated through a physical medium (which is basically merely a particular distribution of certain elementary scatterers), and is obviated only by the assumption of homogeneity. When attempts are made to deal with macro fluctuations of such media as caused by turbulence, rain, fog, or suspensions of other relatively large scatterers; or even with the relatively simpler problems of non-specular reflection from striated or rough surfaces; or finite regular configurations such as lattices and gratings, one usually falls back on the "single scattering" hypothesis. This hypothesis or approximation, no doubt due to Rayleigh,[1] is based on the assumptions that the individual scatterers are sufficiently far apart so that their excitation can be considered as only the incident wave, and/or that there are no coherent phase relations between the waves they scatter so that the scattered energy is proportional to their concentration.

The shortcomings of this hypothesis, as evidenced by experimental "anomalies," result from neglecting the contributions to the excitations of a particular element by the radiation scattered by the remaining elements; i.e., it is only the configuration as a unit, rather than its elements, which is excited by the incident or primary radiation. While it would be esthetically preferable to treat the configuration as a unit, this approach seems limited to certain special problems.[2] On the other hand, if we attempt to synthesize a solution for the configuration, say, for a plane wave incident on an arbitrary configuration of parallel cylinders, it is not sufficient to consider each cylinder as excited only by the incident radiation (except for certain limiting conditions), but rather we must take account of the various multiple excitations. Thus each cylinder in the field scatters waves which excite the remaining cylinders, which in turn respond to this excitation and scatter waves to contribute to the excitation of the original cylinder, etc.[3]

Some theoretical work has been done, although not directly connected with the present analysis, which takes into account these "interactions" explicitly. Thus Reiche obtained an iterated solution for scattering by a random distribution of Herzian dipoles to treat dispersion in gases and fogs;[4] the results of which were later applied to the problem of opalescence by Buchwald.[5] More recently scattering by a random distribution of isotropic scatterers was treated by Foldy,[6] and also approximate solutions for a random distribution of spheres which take into account some of the effects of their mutual influence have been obtained by Ament[7] and Theissing.[8] The technique of "multiple scattering" that we will employ is essentially an extension of that

* This work was supported in part by Contract AF-19(122)42 with the U. S. Air Force through sponsorship of the Geophysical Research Directorate of the Air Force Cambridge Research Laboratories, Air Materiel Command. This material is part of Section 1, "Multiple Scattering of Radiation, Part I," Research Report EM-34, Math. Res. Gys., NYU, WSC, 1951.

† J. Acoust. Soc. Am. **23**, 631 (1951).

[1] See for example Rayleigh, Scientific Papers IV (Cambridge, 1903), p. 397; or Phil. Mag. **47**, 375 (1899).

[2] See for example the work on gratings and periodic surfaces by Rayleigh, Proc. Roy. Soc. (London) **A79**, 399 (1907); S. O. Rice, Comm. on Pure and Appl. Math. **IV**, 351 (1951); C. T. Tai, Harvard Tech. Report No. 28 (Cruft Laboratory, 1948); W. V. Ignatowsky, Ann. Physik **XX 44**, 369 (1914); J. Shmoys. Research Report EM-18, Math. Research Group, NYU, 1950; N. Marcuvitz, *Waveguide Handbook* (McGraw-Hill Book Company, Inc., New York, 1951).

[3] See O. Heaviside, *Electromagnetic Theory* (Dover Publications, New York, 1950), Sec. 182, for a more lucid and more interesting account of "multiple scattering" which indicates that this concept goes back to at least 1893, when this section was originally published in the Electrician.

[4] F. Reiche, Ann. Physik **50**, 1 (1916); **50**, 121 (1916).

[5] E. Buchwald, *Zur Theorie der Opaleszenzstrahlung* (Habilitationsschrift, Breslau, 1917).

[6] L. L. Foldy, Phys. Rev. **67**, 107 (1949).

[7] W. S. Ament, NRL Report R-3238, 1948; also a great deal of unpublished material.

[8] H. H. Theissing, J. Opt. Soc. Am. **40**, 232 (1950).

employed previously by the writer[9] to take into account some of the effects of the nearest neighbors in a grating of parallel cylinders.

FORMULATION OF THE PROBLEM

The scheme of the analysis and its physical interpretation are as follows. A plane wave ψ_i (velocity potential for acoustics, or the scalars of the Hertz potentials for electromagnetics) is incident on $N+1$ parallel cylinders of arbitrary physical parameters. Initially, we consider the s'th cylinder as excited only by the incident plane wave in response to which it scatters its "first order of scattering," ${}^s\psi^1$, such that $\psi_i+{}^s\psi^1=\Psi^1$ is a solution of the wave equation satisfying the radiation condition in the coordinate frame of cylinder s, and any of the usual prescribed boundary conditions at its surface. (We note that $\psi_i+\sum_s {}^s\psi^1$ is the usual single scattering approximation). Next, in response to all the waves of the first order of scattering from the remaining cylinders, $\sum_{s'}{}' {}^{s'}\psi^1$, the s'th cylinder scatters a wave of the "second order of scattering," ${}^s\psi^2$, such that $\sum_{s'}{}' {}^{s'}\psi^1 + {}^s\psi^2=\Psi^2$ satisfies the required conditions in the coordinate frame of s. We proceed in this fashion to the m'th order of scattering, and letting m approach infinity, sum the various orders of scattering to obtain

$$ {}^s\psi=\sum_{m=1}^{\infty} {}^s\psi^m $$

as the total wave scattered by the s'th cylinder.

Each order of scattering is written in terms of the usual cylindrical wave functions or

$$ {}^s\psi^m=\sum_{n=-\infty}^{\infty} {}^sA_n{}^m\, H_n(Kr_s)e^{in\theta_s}, \tag{1} $$

where the Hankel functions are of the first kind to satisfy the radiation condition for an incident wave with time dependence $\exp(-i\omega t)$. Once we have determined the scattering coefficients ${}^sA_n{}^m$ (which are different for each order, and may be different for various cylinders as well), from the prescribed boundary conditions, we write the total wave scattered by the entire configuration as

$$ \psi=\sum_{s=0}^{N}\sum_m {}^s\psi^m=\sum_s\sum_n H_n(Kr_s)e^{in\theta_s}\sum_m {}^sA_n{}^m. \tag{2} $$

The total field, satisfying the required boundary conditions simultaneously at the surface of each cylinder is then

$$ \Psi=\psi_i+\psi=\sum_m \Psi^m; \tag{3} $$

since $\Psi^1, \Psi^2, \cdots \Psi^m$, all satisfy the required conditions, so does their sum $\sum_m\Psi^m$.

[9] V. Twersky, J. Acoust. Soc. Am. 22, 539 (1950).

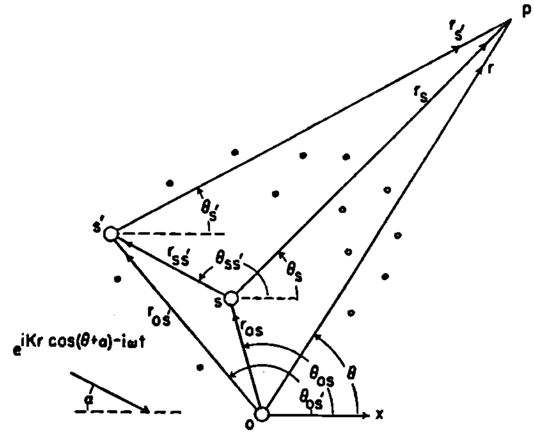

FIG. 1. Plane acoustic or electromagnetic wave incident on an arbitrary configuration of parallel cylinders of different radii and physical parameters.

DETERMINATION OF THE SCATTERING COEFFICIENTS

The geometry of the situation is as in Fig. 1. The incident propagation vector, $\mathbf{K}=K\mathbf{n}=K(\cos\alpha\mathbf{i}-\sin\alpha\mathbf{j})$, is perpendicular to the axes of the cylinders, one of which, say $s=0$, is chosen as the reference origin of coordinates x, y, or r, θ, at which the phase of the incident wave is zero. The axis of each cylinder is the z-axis of its own cylindrical coordinate frame, the polar axis of all frames being parallel to the x-axis. The polar coordinates of a field point P with respect to the axis of the s'th cylinder are r_s, $\theta_s (r_0=r,\ \theta_0=\theta)$, and the polar coordinates that locate cylinder s' with respect to the axis of the s'th cylinder are $r_{ss'}$, $\theta_{ss'}$. Thus $\mathbf{r}=r(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})$, $\mathbf{r}_{ss'}=r_{ss'}(\cos\theta_{ss'}\mathbf{i}+\sin\theta_{ss'}\mathbf{j})$, $\mathbf{r}=\mathbf{r}_{0s}+\mathbf{r}_s$, $\mathbf{r}_{ss'}=\mathbf{r}_s-\mathbf{r}_{s'}=\mathbf{r}_{0s'}-\mathbf{r}_{0s}$, etc.

The incident wave is expressed with reference to the coordinate frame of the s'th cylinder as

$$ \psi_i=e^{i\mathbf{K}\cdot\mathbf{r}}=e^{i\mathbf{K}\cdot(\mathbf{r}_{0s}+\mathbf{r}_s)} $$
$$ =e^{iKr_{0s}\cos(\theta_{0s}+\alpha)}\sum_{n=-\infty}^{\infty} J_n(Kr_s)e^{in(\theta_s+\alpha+\pi/2)}. \tag{4} $$

In response to this primary or first order of excitation each cylinder scatters a wave as in Eq. (1), with $m=1$, the coefficients ${}^sA_n{}^1$, or first-order scattering coefficients, being determined from the prescribed boundary conditions at $r_s=a_s$. For our present purpose we will assume for simplicity that the total field satisfies $\Psi|_{r_s=a_s}=0$. On invoking this boundary condition we obtain

$$ (\psi_i+{}^s\psi^1)|_{r_s=a_s}=(\textstyle\sum_n\{e^{i\mathbf{K}\cdot\mathbf{r}_{0s}+in(\alpha+\pi/2)}J_n(Ka_s) $$
$$ +{}^sA_n{}^1H_n(Ka_s)\}e^{in\theta_s})=0. \tag{5} $$

Equating each term of the expansion to zero yields

$$ {}^sA_n{}^1={}^sA_n\exp[in(\alpha+\pi/2)+i\mathbf{K}\cdot\mathbf{r}_{0s}], \tag{6} $$

where for brevity we have set

$$ {}^sA_n=-J_n(Ka_s)/H_n(Ka_s), \tag{7} $$

which we will refer to as the scattering coefficient of the single cylinder. The result we would obtain by summing the contributions of the various ${}^s\psi^1$ at the field point P and adding ψ_i would constitute the usual single scattering approximation.

To proceed further we now consider the s'th cylinder as excited by the sum of the first-order scattered waves from the remaining cylinders (the second order of excitation for the s'th cylinder) to yield the second-order scattered wave as in Eq. (1) with $m=2$. We determine the second-order scattering coefficients, ${}^sA_n{}^2$, from

$$(\sum_{s'}{}' \, {}^{s'}\psi^1 + {}^s\psi^2)|_{r_s=a_s} = (\sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'}{}^1 H_{n'}(Kr_{s'})e^{in'\theta_{s'}} + \sum_n {}^sA_n{}^2 H_n(Kr_s)e^{in\theta_s})|_{r_s=a_s}=0, \quad (8)$$

where $\sum_{s'}{}'$ indicates that $s=s'$ is to be excluded from the sum. In order to impose the boundary condition at $r_s=a_s$ we must expand the ${}^{s'}\psi^1$ in terms of cylindrical waves referred to the coordinates of the s'th cylinder. From Watson[10] we obtain the expansion formula or "addition theorem" valid for all $r_s=a_s<r_{ss'}$,

$$H_{n'}(Kr_{s'})e^{in'\theta_{s'}} = \sum_{m=-\infty}^{\infty} H_m(Kr_{ss'})J_{n'+m}(Kr_s)e^{i(n'+m)\theta_s - im\theta_{ss'}}. \quad (9)$$

The second order of excitation of the s'th cylinder as in (3.7) can therefore be expressed as

$$\sum_{s'}{}' \, {}^{s'}\psi^1 = \sum_{s'}{}' \sum_{n'} \sum_n {}^{s'}A_{n'}{}^1 H_{n-n'}(Kr_{ss'}) \times J_n(Kr_s)e^{in\theta_s - i(n-n')\theta_{ss'}}, \quad (10)$$

where we have replaced $n'+m$ by n. Substituting into (8) and equating the coefficients of $\exp(in\theta)$ to zero yields

$$ {}^sA_n{}^2 = -[J_n(Ka_s)/H_n(Ka_s)] \times \sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'}{}^1 H_{n-n'}(Kr_{ss'})e^{-i(n-n')\theta_{ss'}} \quad (11)$$

which we rewrite in terms of the scattering coefficients of the single cylinder by means of (6) and (7) as

$$ {}^sA_n{}^2 = {}^sA_n \sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'} H_{n-n'}(Kr_{ss'}) \times e^{i\mathbf{K}\cdot\mathbf{r}_{0s'} - i(n-n')\theta_{ss'} + in'(\alpha+\pi/2)}. \quad (12)$$

The result we would obtain by summing ψ_i and the various ${}^s\psi^1$ and ${}^s\psi^2$ would constitute a solution to the problem correct to the second order of scattering. Physically this represents a solution which treats each cylinder as excited by the incident plane wave and by the cylindrical waves scattered by all the remaining cylinders in the configuration as a result of the plane wave excitation. Such a solution is a better approximation than that obtained by single scattering and is significant in that it possesses a parametric behavior clearly indicating the role of the multiply scattered terms as governed by the geometry of the configuration. It can be seen that the second-order scattering coefficients consist of a sum of N terms of various products of two coefficients of the single cylinder, the amplitude of each term of the sum $\sum_{s'}{}'$ depending on the distance between the corresponding cylinders. Hence, the second order of scattering is readily interpreted as consisting of waves which have undergone two scattering processes, or as arising from the plane wave being successively scattered by two different cylinders.

Even if ${}^sA_n \ll 1$ and $Kr_{ss'} \gg 1$, this does not preclude the second order of scattering from contributing appreciably to the scattered field for certain geometries or ranges of the parameters, because of the marked increase in the numbers of terms that must be retained. If the ${}^sA_n$ are vanishingly small, however, the multiply scattered terms vanish to a higher order regardless of the value of the other parameters. The multiply scattered terms also vanish for infinitely large separation between the cylinders compared to wavelength (as was expected from physical considerations), because of the behaviour of the Hankel functions which approach zero as $Kr_{ss'}$ approaches infinity. We also note that the smallness of the ratio $|{}^s\psi^2/{}^s\psi^1|$ provides an analytic criterion for the range of validity of the single scattering hypothesis.

From (11) the general expression for the scattering coefficients of the m'th order of scattering can be readily obtained in terms of the $(m-1)$'th. Hence we have a recursive relation by means of which any order of scattering can be expressed in terms of the coefficients of the preceding orders, and ultimately in terms of the scattering coefficients of the single cylinder. Thus

$$\begin{aligned} {}^sA_n{}^m &= {}^sA_n \sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'}{}^{m-1} H_{n-n'}(Kr_{ss'})e^{-i(n-n')\theta_{ss'}} \\ &= {}^sA_n \sum_{s'}{}' \sum_{n'} H_{n-n'}(Kr_{ss'})e^{-i(n-n')\theta_{ss'}} \\ &\quad \times \{ {}^{s'}A_{n'} \sum_{s''}{}' \sum_{n''} {}^{s''}A_{n''}{}^{m-2} H_{n'-n''} \\ &\quad \times (Kr_{s's''})e^{i(n'-n'')\theta_{s's''}} \} \\ &= {}^sA_n \sum_{s'}{}' \sum_{n'} {}^{s'}A_{n'} H_{n-n'}(Kr_{ss'})e^{-i(n-n')\theta_{ss'}} \\ &\quad \times \sum_{s''}{}' \sum_{n''} \times \cdots \sum_{s(m-1)}{}' \sum_{n(m-1)} \\ &\quad \times H_{n(m-2)-n(m-1)}(Kr_{s(m-2)s(m-1)}) \\ &\quad \times \exp[i(n^{(m-2)}-n^{(m-1)})\theta_{s(m-2)s(m-1)}] \\ &\quad \times s^{(m-1)}A_{n(m-1)}\exp[in^{(m-1)}(\alpha+\pi/2) \\ &\qquad + i\mathbf{K}\cdot\mathbf{r}_{0s(m-1)}], \end{aligned} \quad (13)$$

where $\sum_{s''}{}'$, etc., means that $s''=s'$ is to be excluded from the sum, and where the parenthetical label on the indices n and s refers to the number of primes. We can

[10] G. N. Watson, *Bessel Functions*, (Cambridge University Press, London, 1948), pp. 359–61.

rewrite (13) as the formal product

$${}^{s}A_n{}^{m}={}^{s}A_n\{\prod_{\mu=(1)}^{(m-1)}\sum_{s^\mu}{}'\sum_{n^\mu}{}^{s^\mu}A_{n^\mu}H_{n^{\mu-1}-n^\mu}(Kr_{s^{\mu-1}s^\mu})$$
$$\times\exp[-i(n^{\mu-1}-n^\mu)\theta_{s^{\mu-1}s^\mu}]\}$$
$$\times\exp[in^{(m-1)}(\alpha+\pi/2)+i\mathbf{K}\cdot\mathbf{r}_{0s^{(m-1)}}].\quad(14)$$

We see then that the scattering coefficients of the m'th order of scattering are rather complicated sums of the products of m scattering coefficient of the single cylinder and Hankel and trigonometric functions depending on the geometry of the configuration. The interpretation of the m'th order of scattering as consisting of waves which have undergone m scattering processes follows immediately from the above as well as from the structure of the analysis. We also note that we need not alter our criterion for the use of the single scattering approximation, since if $|\psi^2|\ll|\psi^1|$ it follows immediately that $|\psi^3|\ll|\psi^2|$, etc., and all ψ^m negligibly small compared to ψ^1.

Noting that

$$\mathbf{r}_{0s^{(m-1)}}=\mathbf{r}_{0s}+\mathbf{r}_{ss'}+\mathbf{r}_{s's''}+\cdots+\mathbf{r}_{s^{(m-2)}s^{(m-1)}}$$
$$=\mathbf{r}_{0s}+\sum_{\mu=(1)}^{(m-1)}\mathbf{r}_{s^{\mu-1}s^\mu},$$

and that

$$n^{(m-1)}=n-(n-n')-(n'-n'')-\cdots-(n^{(m-2)}-n^{(m-1)})$$
$$=n-\sum_\mu(n^{\mu-1}-n^\mu),$$

we rewrite (14) as

$${}^{s}A_n{}^{m}={}^{s}A_ne^{in(\alpha+\pi/2)+i\mathbf{K}\cdot\mathbf{r}_{0s}}$$
$$\times\prod_\mu\{\sum_{s^\mu}{}'\sum_{n^\mu}{}^{s^\mu}A_{n^\mu}H_{n^{\mu-1}-n^\mu}(Kr_{s^{\mu-1}s^\mu})$$
$$\times\exp[-i(n^{\mu-1}-n^\mu)\varphi_{s^{\mu-1}s^\mu}+i\mathbf{K}\cdot\mathbf{r}_{s^{\mu-1}s^\mu}]\},\quad(15)$$

where for brevity $\varphi_{s^{\mu-1}s^\mu}=\theta_{s^{\mu-1}s^\mu}+\alpha+\pi/2$. The wave scattered by the configuration, as in (2), can then be written as

$$\psi=\sum_s\sum_m{}^{s}A_nH_n(Kr_s)e^{in\varphi_s+i\mathbf{K}\cdot\mathbf{r}_{0s}}\sum_m{}^{s}P_n{}^{m},\quad(16)$$

where $\varphi_s=\theta_s+\alpha+\pi/2$, and

$${}^{s}P_n{}^{m}=\prod_\mu\{\quad\},\quad{}^{s}P_n{}^{1}=1,\quad(17)$$

the curley bracket being as in (15).

This expression for ψ when added to the incident wave ψ_i constitutes a formal solution to the problem of the scattering of a plane wave by an arbitrary configuration of parallel cylinders satisfying the boundary conditions simultaneously at the surface of each cylinder and the radiation condition (provided that all cylinders are in the finite domain). To stress the method of approach and technique of solution we will more or less retrace the previous strips to show that the boundary conditions are satisfied simultaneously at the surface of each cylinder.

We wish to show that $\Psi|_{r_s=a_s}=0$. We have at $r_s=a_s$ that

$$\Psi=\psi_i+\psi=\psi_i+{}^{s}\psi+\sum_{s'}{}'\,{}^{s'}\psi,\quad(18)$$

or the total wave consists of the incident wave plus the wave scattered by the s'th cylinder plus the waves scattered by the remaining cylinders. We decompose the scattered waves into their multiple orders of scattering. Thus

$$\Psi=\psi_i+{}^{s}\psi^1+\sum_{m=2}^{\infty}{}^{s}\psi^m+\sum_{s'}{}'\sum_{m=1}^{\infty}{}^{s'}\psi^m.\quad(19)$$

By virtue of (5) the first two terms cancel to leave

$$\Psi={}^{s}\psi^2+\sum_{m=3}^{\infty}{}^{s}\psi^m+\sum_{s'}{}'\,{}^{s'}\psi^1+\sum_{s'}{}'\sum_{m=2}^{\infty}{}^{s'}\psi^m.\quad(20)$$

By virtue of (8) the first and third terms cancel, and similarly for all orders of scattering, since the scattered waves from the remaining cylinders constitute the excitation for any individual cylinder in response to which it scatters the next order of scattering. Hence for the n'th order of excitation

$${}^{s}\psi^{n+1}+\sum_{s'}{}'\,{}^{s'}\psi^n=0\quad(21)$$

and the total field vanishes simultaneously at the surface of each cylinder or

$$\Psi|_{r_s=a_s}=0\quad(22)$$

as is required.

Although the problem was formulated subject to the boundary condition that the field vanish at the surface of the cylinders, the solution can be readily extended to more general boundary conditions, and expressions can also be obtained for the field inside any cylinder when it exists. The solutions we obtain for the scattered field are formally identical with those derived, but the ${}^{s}A_n$ are to be replaced by the appropriate scattering coefficients of the single cylinder for impedance or continuity boundary conditions.[11] For the "transmitted" wave in the interior of the s'th cylinder we obtain

$${}^{s}\psi_t=\sum_{m=1}^{\infty}{}^{s}\psi_t{}^{m}$$
$$=\sum_n{}^{s}B_nJ_n(Kr_s)e^{in\varphi_s+i\mathbf{K}\cdot\mathbf{r}_{0s}}\sum_m{}^{s}P_n{}^{m},\quad(23)$$

where the ${}^{s}B_n$ are the usual "transmission coefficients" for the single cylinder subject to continuity boundary conditions, and where the ${}^{s}P_n{}^{m}$ are as in (17).

It should be noted that the solution does not require that the cylinders have identical radii or identical physical properties so that the boundary conditions may differ from cylinder to cylinder. It is also needless to mention that the derived solution, while completely

[11] See, for example, Lord Rayleigh, *Theory of Sound*, (Dover Publications, New York, 1945), Section 343; R. K. Cook and P. Chrzanowski, J. Acoust. Soc. Am. 17, 315 (1946) for the acoustic case: and C. Schaefer, *Einfürhrung in die Theoretische Physik*, (de Gruyter, Berlin, 1949), Volume 3, Part 1, Section 104, for the electromagnetic case.

general, is practically useless for purposes of calculation without the introduction of qualifying assumptions and approximations as to the character of the configuration and its components. In future articles we will concern ourselves with applying this solution to specific problems of general interest, introducing the usual restrictions required for comparison with experimental results and further restrictions, wherever necessary, to bring our results to as an explicit form as possible.

Certain configurations which have already been treated, subject to the restrictions that the distance between neighboring cylinders is large compared to their radii and to the wavelength, are those of two cylinders, the grating, a uniform random planar distribution, and the analogous problems of semicylindrical bosses on an infinite rigid or perfectly conducting plane. It is interesting to note that the multiple scattering solution for the grating predicts the same sort of "anomalies" first discovered by Wood[12] and allows for a simple and physically plausible interpretation of the observed maxima and minima in white light spectra on the basis of favorable and unfavorable phase relations among the various excitations and orders of scattering. Also the multiple scattering solution for the model of a striated surface previously proposed[13] is no longer marred by the presence of infinities at grazing incidence; rather the total reflected wave reduces to the specular wave which is observed experimentally.

The theory of multiple scattering here presented, although completely general, is restricted in its applications to elementary scatterers for which known solutions exist, and to coordinate frames for which the appropriate addition or expansion theorems (required to transfer the origin of the elementary wave functions so as to facilitate satisfying boundary conditions) have been developed. Another large class of problems which can be treated by essentially the identical procedure here employed is that involving scattering by arbitrary configurations of spheres. Calculations are in progress for such configurations, the required addition theorems having been derived by Professor Bernard Friedman.

The writer is grateful to Professor Fritz Reiche for suggesting the research, to Professor Joseph B. Keller for his aid and many valuable suggestions, and to Dr. Jerry Shmoys for his interest.

[12] R. W. Wood, Proc. Phys. Soc. (London) 18, 396 (1902); Phil. Mag. 23, 310 (1912); Phys. Rev. 48, 928 (1935); also J. Strong, Phys. Rev. 49, 291 (1936).

[13] V. Twersky, J. Acoust. Soc. Am. 23, 336 (1951); J. Appl. Phys. (1951) and reference 9.

THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA VOLUME 24, NUMBER 1 JANUARY, 1952

Directionality Patterns for Acoustic Radiation from a Source on a Rigid Cylinder

DONALD T. LAIRD AND HIRSH COHEN*
Ordnance Research Laboratory, The Pennsylvania State College, State College, Pennsylvania
(Received September 20, 1951)

The directionality patterns produced by an acoustic source located on a rigid cylinder of infinite length have been investigated for the case in which the source strength may be represented as a separable function of azimuth angle and axial dimension. It is observed that the pattern in a plane orthogonal to the axis of the cylinder is independent of the axial distribution of the source, and is, in fact, identical with the pattern given by Morse[1] for a source of infinite axial extent. For the particular case in which the ratio of circumference to wavelength is 14, patterns in this plane have been computed numerically using an IBM Card-Programmed Calculator. Patterns in a plane containing the axis of the cylinder have also been investigated.

GENERAL CONSIDERATIONS

THIS paper will consider the patterns of radiation from a source located on a rigid cylinder of infinite length. For the case where the source is a vibrating strip extending indefinitely along the cylinder, the pattern has been given by P. M. Morse.[1] The present work extends the theory to the case where the source is any separable function of the azimuth and axial dimensions, e.g., a rectangular piston.

A cylindrical coordinate system, as shown in Fig. 1, is used. For purposes of describing the far zone field, spherical coordinates R, φ, θ, as shown, will eventually be introduced. The radiating source is assumed to vibrate radially in such a way that its velocity distribution may be represented as a separable function of φ and z. The boundary conditions at $r=a$, where a is the radius of the cylinder, is then given by the expression,

$$u_r\Big|_{r=a} = U_0 e^{-i\omega t}\Big(\sum_{m=0}^{\infty} a_m \cos m\varphi\Big)\left(\int_{-\infty}^{\infty} F(k_z)e^{ik_z z}dk_z\right). \quad (1)$$

The fourier series, which for simplicity has been taken to be a cosine series, gives the dependence on φ. The fourier integral represents the dependence on z, with $F(k_z)$, the transform of the velocity distribution in the z-direction, being independent of φ. It is observed that

* Now at Department of Aeronautical Engineering, Hebrew Institute of Technology, Haifa, Israel.

[1] Philip M. Morse, *Vibration and Sound* (McGraw-Hill Book Company, Inc., New York, 1948).